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ANALYSIS OF A DISCRETE-TIME QUEUE WITH GEOMETRICALLY DISTRIBUTED GATE OPENING INTERVALS

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Abstract This paper considers a discrete-time BBP/G/1 queue with geometrically distributed gate opening intervals. The system has two queues and a gate. Customers arriving at the system are accommodated in the first queue at the gate. When the gate opens, all the customers who are waiting in the first queue move to the second queue at the server. The gate closes immediately after all the customers in the first queue move to the second queue. The server serves only the customers present in the second queue. For this system, we derive the probability generating functions for the queue length, the amount of work and the waiting time. We also provide some numerical examples in order to show the computational feasibility of the analytical results.

1. Introduction

This paper considers a discrete-time BBP/G/1 queue with a gate, where BBP denotes a batch Bernoulli process. Customers arrive at the system in a batch and service times of customers are independent and identically distributed (i.i.d.) according to a general distribution function. The system has two queues and a gate (see Fig. 1). Customers arriving at the system are accommodated in the first queue at the gate. When the gate opens, all the customers who are waiting in the first queue move to the second queue at the server. The travel times from the first queue to the second queue are assumed to be zero. The gate closes immediately after all the customers in the first queue move to the second queue. We assume that the intervals between successive openings of the gate are geometrically distributed. The server serves only the customers present in the second queue. The purpose of this paper is to provide a complete set of the analytical results for various performance measures.

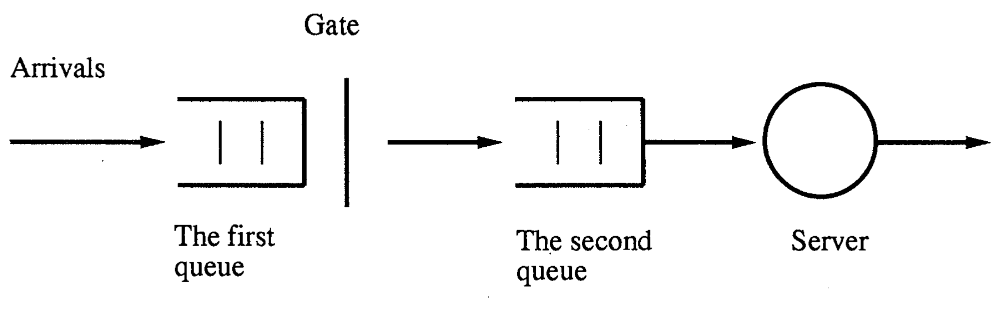

Figure 1: A Queue with a Gate

The queue with a gate is considered as a mathematical abstraction on many occasions.

For example, let us consider a shuttle bus in an airport which comes occasionally to pick up customers and take them to a hotel. All customers who arrive to the hotel forms a queue at the registration counter. In computer communications, [1, 2] show an application of the model to the performance evaluation of the Cyclic-Reservation Multiple-Access (CRMA) taking account of a backpressure mechanism. Other examples could be found, e.g., mail pickup and so on. Thus, the queue with a gate has rich applications in wide areas.

The queue with a gate falls into the category of the queue with generalized vacations [4, 7]. In the queue with generalized vacations, a server takes vacations even when waiting customers are present in the system. Note that, in the queue with a gate, there is a possibility that a server becomes idle, while customers are waiting outside the gate. Thus the idle periods of the server when waiting customers exist outside the gate are considered as vacations of the server. It is well known that in the queue with generalized vacations, the queue length, the amount of work in the system and the waiting time under the FCFS discipline have the so-called decomposition properties (see [6] and references therein). Note that the queue with generalized vacations has been studied mainly in the continuous-time model. However, very similar decomposition properties hold for the discrete-time counterpart, too. See, for example, [3].

Another interesting feature of the queue with a gate is the correlation between the gate opening interval and the number of customers who move to the second queue when the gate opens. It is easy to see that, when the gate opening interval is long, (relatively) many customers are likely to wait in the first queue, while (relatively) few customers are likely to wait in the second queue. Thus, the waiting times in the first queue and the second queue would be negatively correlated (i.e., a long waiting in the first queue leads to a short waiting in the second queue). Also, if we consider the second queue as an isolated system, the interarrival time of batches (i.e., the gate opening interval) and the number of customers in each batch (i.e., the number of customers who move to the second queue at the same time) are positively correlated. Yet another view of this feature is that there exists the correlation between the interarrival time and the service time if each batch moving to the second queue is considered as a supercustomer.

Takahashi has studied continuous-time queues with gates [13], where the service times of customers are exponentially distributed and the gate opening intervals are deterministic or exponentially distributed. Borst et al. have studied the continuous-time queue with exponential gate opening intervals [1, 2], where the service times of customers are generally distributed. They were mainly concerned with the second queue, and discussed the effect of the correlation between the interarrival time and the number of customers in batches on the performance of the second queue. Boxma and Combé have studied an M/G/1 queue with a rather general dependency between the interarrival time and the service time [5]. Kawata has studied a discrete-time queue with geometrically distributed gate opening intervals and derived the probability generating function (PGF) for the sojourn times of supercustomers [9]. Ishizaki et al. have studied a discrete-time queue with bounded gate opening intervals [8].

The rest of the paper is organized as follows. In section 2, we describe the mathematical model in detail. Our model is considered as a discrete-time version of the model of [2]. Note that the model in this paper allows batch arrivals, while [2] considers only single arrivals. Many communication systems are operated based on a time-slot basis, which are naturally modeled by discrete-time queues. In [2], they have referred to the applications of their model to the performance evaluation of the CRMA taking account of a backpressure mechanism. Note that CRMA is a transfer protocol in high-speed local and metropolitan area networks which use slot-base transmission. Thus, our model is more suitable to evaluate the performance measures of CRMA. From a theoretical viewpoint, a discrete-time model is more general than the continuous-time counterpart in a sense that the discrete-time model is reduced to the continuous-time counterpart by letting the slot length to zero (see [3], for example).

In the next three sections, we provide various formulas of the performance measures of interest. In section 3, we study the number of customers in the system. We first derive the joint PGF for the number of customers in the first queue and that in the second queue immediately after departures of customers. The PGF is given in terms of a function which is represented by an infinite product. Next we derive the joint PGF for the number of customers in the first queue and that in the second queue at the beginning of a randomly chosen slot. Note that [2] did not provide any results on the joint queue length distribution at a random point in time. Furthermore, we analytically show the decomposition properties for the total number of customers in the system at departures and at a randomly chosen slot. In section 4, we analyze the amount of work in the system. Using the joint distribution of the queue lengths and the remaining service time, we first derive the joint PGF for the amount of work in the first queue and that in the second queue at the beginning of a randomly chosen slot. Next we derive the PGF for the amount of total work in the system. Furthermore, we show the decomposition property for the amount of total work in the system. Note that the PGF for the amount of work in the system is identical to the PGF for the sojourn times of supercustomers [9]. In section 5, we consider the waiting times of customers. We derive the joint PGF for the waiting times of individual customers in the first queue and the second queue and the PGF for the waiting time of supercustomers. Also we analytically show the decomposition property for the total waiting time of individual customers. Finally, in section 6, we provide some numerical examples, where we discuss three kinds of correlations in the model: the effect of the correlation between the interarrival time and the service time of supercustomers on the mean waiting time of supercustomers, the effect of the correlation between the interarrival time of each batch composed of customers who move to the second queue at the same time and the number of the customers in the batch on the mean waiting time of individual customers in the second queue, and the correlation between the waiting times in the first queue and the second queue.

2. Mathematical Model

We consider a discrete-time queueing model with the following characteristics:

- Time is slotted.
- Customers arrive at the system in a batch immediately before slot boundaries. The batch sizes and the service times of individual customers are independent and identically distributed. Customers arriving at the system are accommodated in the first queue at the gate.
- The gate opens immediately before slot boundaries. When the gate opens, all the customers waiting in the first queue move to the second queue at the server. The travel times of customers to the second queue are assumed to be zero. We assume that customers arriving in a slot also move to the second queue when the gate opens in the slot, so that the waiting times of such customers in the first queue become zero. The gate closes immediately after all the customers in the first queue move to the second queue. The intervals between successive openings of the gate are geometrically distributed.

- There is a single server who serves the customers only in the second queue. When the server finds some amount of work in the second queue immediately after a slot boundary, he serves exactly one unit of work in the current slot. We assume that customers are served on an FCFS basis. Furthermore, as for customers who arrive in the same slot, the next customer for service is randomly chosen among those customers.

We now introduce random variables and notations to describe the above model. Let B and C denote random variables representing the number of individual customers who arrive at the system in a slot and the service time of an individual customer, respectively. Further, let A denote a random variable representing the amount of work brought into the system in a slot (i.e., the sum of the service times of customers arriving in a slot). We define the following PGFs:

$$A(z) \triangleq E\left[z^A\right], \qquad B(z) \triangleq E\left[z^B\right], \qquad C(z) \triangleq E\left[z^C\right]. \tag{2.1}$$

By definition, we have

$$A(z) = B(C(z)). \tag{2.2}$$

Let G denote a random variable representing the length of an interval between successive openings of the gate. Let $g(n) = \Pr(G = n)$ $(n \geq 1)$. We then have for a parameter γ

$$g(n) = (1-\gamma)\gamma^{n-1} \qquad (0 \leq \gamma \leq 1). \tag{2.3}$$

We denote the PGF of the $g(n)$ by $G(z)$:

$$G(z) = \sum_{n=1}^{\infty} g(n) z^n = \frac{(1-\gamma)z}{1-\gamma z}. \tag{2.4}$$

We assume that B, C and G are independent, identically distributed random variables, and those are independent each other. Throughout the paper, for any PGF $f(z)$, we use the symbol $f'(1)$ to denote $\lim_{z\to 1-} df(z)/dz$. Furthermore, we assume $A'(1) = B'(1)C'(1) < 1$ and the system is in equilibrium.

3. Number of Individual Customers

In this section, we consider the number of individual customers in the first queue and that in the second queue. First we observe an imbedded Markov chain which is composed of two types of imbedded points. Next we derive the PGF for the number of customers immediately after departures of customers. Finally, we obtain the PGF for the number of customers at the beginning of a randomly chosen slot in terms of the PGF for the number of customers immediately after departures of customers.

3.1. Number of customers immediately after departures

In this subsection, we derive the formula for the number of customers immediately after departures. To do so, we introduce an imbedded Markov chain which is composed of two types of imbedded points:

- type 1: immediately after departures of individual customers,
- type 2: immediately after gate opening instants during idle periods.

Let $X^{(1)}$ and $X^{(2)}$ denote random variables representing the numbers of individual customers in the first queue and in the second queue, respectively, at a randomly chosen imbedded point. Note here that $X^{(1)}$ and $X^{(2)}$ are dependent. Moreover, let T_P denote a random variable representing the type of a randomly chosen imbedded point, i.e., $T_P = i$ $(i = 1, 2)$ when a randomly chosen imbedded point is of type i. We define $P(z_1, z_2)$ and $Q(z_2)$ as

$$P(z_1, z_2) \triangleq E\left[z_1^{X^{(1)}} z_2^{X^{(2)}} \mathbf{1}_{\{T_P=1\}}\right], \tag{3.1}$$

$$Q(z_2) \triangleq E\left[z_2^{X^{(2)}} \mathbf{1}_{\{T_P=2\}}\right], \tag{3.2}$$

where $\mathbf{1}_T$ denotes the indicator function of a set T. Note here that $X^{(1)} = 0$ if $T_P = 2$ because all the customers waiting in the first queue move to the second queue when the gate opens.

Let $X_D^{(1)}$ and $X_D^{(2)}$ denote random variables representing the numbers of individual customers in the first queue and in the second queue, respectively, immediately after the departure of a randomly chosen customer. Note here that $X_D^{(1)}$ and $X_D^{(2)}$ are dependent. We define $Q_D(z_1, z_2)$ as the joint PGF associated with $X_D^{(1)}$ and $X_D^{(2)}$:

$$Q_D(z_1, z_2) \triangleq E\left[z_1^{X_D^{(1)}} z_2^{X_D^{(2)}}\right]. \tag{3.3}$$

We then have the following theorem. (The proof is given in Appendix A.1.)

Theorem 3.1. *$Q_D(z_1, z_2)$ satisfies*

$$\begin{aligned} [z_2 - C(\gamma B(z_1))]Q_D(z_1, z_2) = {} & C(\gamma B(z_1))[\Psi(z_2) - \Psi(z_1)] \\ & + \left[\frac{z_2}{C(B(z_2))}\theta(z_1, z_2) - C(\gamma B(z_1))\right] \\ & \cdot \frac{C(B(z_2))}{z_2 - C(B(z_2))} \frac{B(z_2) - 1}{1 - \gamma B(z_2)} \Psi(z_2), \end{aligned} \tag{3.4}$$

where

$$\Psi(z) = \frac{P(z, 0) + Q(0)}{P(1, 1)}, \tag{3.5}$$

$$\theta(z_1, z_2) = C(\gamma B(z_1)) + H(z_1, z_2) \tag{3.6}$$

with

$$H(z_1, z_2) = \frac{(1-\gamma)B(z_2)}{B(z_2) - \gamma B(z_1)} [C(B(z_2)) - C(\gamma B(z_1))]. \tag{3.7}$$

Note that (3.4) is rewritten to be

$$\begin{aligned} Q_D(z_1, z_2) = \frac{1}{z_2 - C(\gamma B(z_1))} & \Big[C(\gamma B(z_1))\{Q_D(z_2, 0) - Q_D(z_1, 0)\} \\ & + \left\{\frac{z_2}{C(B(z_2))}\theta(z_1, z_2) - C(\gamma B(z_1))\right\} Q_D(z_2, z_2)\Big], \end{aligned} \tag{3.8}$$

The equation (3.8) may be useful because it can replace $Q_D(z_1, z_2)$ by $Q_D(z_1, 0)$, $Q_D(z_2, 0)$ and $Q_D(z_2, z_2)$. Indeed, we will use (3.8) in Proofs.

(3.4) shows that $Q_D(z_1, z_2)$ is expressed in terms of $\Psi(z)$ $(z = z_1, z_2)$. We shall therefore consider $\Psi(z)$. Define for $|z_1| \leq 1$:

$$\delta(z_1) \triangleq C(\gamma B(z_1)), \tag{3.9}$$

$$\delta^{(0)}(z_1) \triangleq z_1, \tag{3.10}$$

$$\delta^{(i)}(z_1) \triangleq \delta(\delta^{(i-1)}(z_1)) \qquad (i \geq 1). \tag{3.11}$$

Since the system is stable, $Q_D(z_1, z_2)$ is bounded and analytic for $|z_1| \leq 1$ and $|z_2| \leq 1$. Thus, for $z_2 = \delta(z_1)$, the left-hand side of (3.4) becomes zero, so that the right-hand side of (3.4) must become zero, too. We then have

$$\Psi(z_1) = \frac{(1-\gamma)B(\delta(z_1))(1-\gamma B(z_1))}{(B(\delta(z_1)) - \gamma B(z_1))(1-\gamma B(\delta(z_1)))}\Psi(\delta(z_1)). \tag{3.12}$$

To simplify notations, we introduce for $|z_1| \leq 1$,

$$\phi(z_1) \triangleq \frac{(1-\gamma)B(\delta(z_1))(1-\gamma B(z_1))}{(B(\delta(z_1)) - \gamma B(z_1))(1-\gamma B(\delta(z_1)))}. \tag{3.13}$$

Then (3.12) becomes

$$\Psi(z_1) = \phi(z_1)\Psi(\delta(z_1)). \tag{3.14}$$

Using (3.14), we have the following theorem. (The proof is given in Appendix A.2.)

Theorem 3.2. *$\Psi(z) = (P(z,0) + Q(0))/P(1,1)$ is given by*

$$\Psi(z) = \frac{1 - B'(1)C'(1)}{B'(1)}(1-\gamma)\frac{\alpha(z)}{\alpha(1)}, \tag{3.15}$$

where for $|z_1| \leq 1$,

$$\alpha(z_1) = \prod_{h=0}^{\infty} \phi(\delta^{(h)}(z_1)). \tag{3.16}$$

Next we present a corollary which immediately follows from Theorem 3.1. Let $X_A^{(1)}$ and $X_A^{(2)}$ denote random variables representing the numbers of individual customers in the first queue and in the second queue, respectively, at the beginning of a randomly chosen slot. Note here that $X_A^{(1)}$ and $X_A^{(2)}$ are dependent. We define $Q_{A|busy}(z_1, z_2)$ as the joint PGF for the numbers of individual customers in the first queue and in the second queue at the beginning of a randomly chosen slot given that the server is busy. Also, we define $Q_{A|idle}(z_1)$ as the PGF for the number of individual customers in the first queue at the beginning of a randomly chosen slot given that the server is idle:

$$Q_{A|busy}(z_1, z_2) \triangleq E\left[z_1^{X_A^{(1)}} z_2^{X_A^{(2)}} \middle| T_S = 1\right], \quad Q_{A|idle}(z_1) \triangleq E\left[z_1^{X_A^{(1)}} \middle| T_S = 0\right], \tag{3.17}$$

where T_S denotes a random variable defined as

$$T_S \triangleq \begin{cases} 1 & \text{if the server is busy,} \\ 0 & \text{if the server is idle,} \end{cases} \tag{3.18}$$

at a randomly chosen slot. Note that, by definition, $Q_{A|idle}(z)$ is given by

$$Q_{A|idle}(z) = \frac{1-\gamma}{1-\gamma B(z)} \frac{P(z,0)+Q(0)}{P(1,0)+Q(0)}. \tag{3.19}$$

Corollary 3.1. *The PGF $Q_D(z,z)$ for the total number of customers in the system immediately after departures are given by*

$$Q_D(z,z) = Q_{Db}(z) Q_{A|idle}(z), \tag{3.20}$$

where

$$Q_{Db}(z) = (1 - B'(1)C'(1)) \frac{(z-1)C(B(z))}{z - C(B(z))} \cdot \frac{B(z)-1}{B'(1)(z-1)} \tag{3.21}$$

and $Q_{A|idle}(z)$ is given in (3.19).

The proof is given in Appendix A.4.

Remark 3.1. Note that $Q_{Db}(z)$ denotes the PGF for the number of customers immediately after departures of customers corresponding BBP/G/1 queue without gates and $Q_{A|idle}(z)$ denotes the PGF for the number of individual customers in the system given that the server is idle. Thus, the total number of customers in the system immediately after departures are decomposed into the two independent factors.

3.2. Number of customers at the beginning of a randomly chosen slot

In this subsection, we derive the formula for the number of customers at the beginning of a randomly chosen slot. To do so, we first consider the number of customers in the first queue and that in the second queue at the start of the service of a randomly chosen customer. Let $\hat{X}^{(1)}$ and $\hat{X}^{(2)}$ denote random variables representing the numbers of individual customers in the first queue and in the second queue, respectively, at the start of the service of a randomly chosen customer. We define $\hat{Q}(z_1, z_2)$ as the joint PGF for the numbers of individual customers in the first queue and in the second queue at the start of the service of a randomly chosen customer:

$$\hat{Q}(z_1, z_2) \triangleq E\left[z_1^{\hat{X}^{(1)}} z_2^{\hat{X}^{(2)}}\right]. \tag{3.22}$$

We then have the following lemma. (The proof is given in Appendix A.5.)

Lemma 3.1. *$\hat{Q}(z_1, z_2)$ is given by*

$$\begin{aligned} \hat{Q}(z_1, z_2) = & \frac{z_2}{z_2 - C(\gamma B(z_1))} \{Q_D(z_2, 0) - Q_D(z_1, 0)\} \\ & + \frac{z_2}{C(B(z_2))} S(z_1, z_2) Q_D(z_2, z_2), \end{aligned} \tag{3.23}$$

where

$$S(z_1, z_2) = \frac{z_2 + H(z_1, z_2) - C(B(z_2))}{z_2 - C(\gamma B(z_1))} \tag{3.24}$$

and $H(z_1, z_2)$ is given in (3.7).

Taking $z_1 = z_2 = z$ in (3.23) and noting

$$S(z,z) = 1, \tag{3.25}$$

we can easily confirm that the following equation holds:

$$\hat{Q}(z,z) = \frac{z}{C(B(z))} Q_D(z,z). \tag{3.26}$$

The equation (3.26) implies that the number of customers in the system left behind by a randomly chosen customer is equal to the number of customers in the system at the start of his service (including himself) plus the number of customers arriving to the system during his service time minus one (himself).

We now consider the number of customers in the first queue and that in the second queue at the beginning of a randomly chosen slot. We define $Q_A(z_1, z_2)$ as the joint PGF associated with $X_A^{(1)}$ and $X_A^{(2)}$:

$$Q_A(z_1,z_2) \triangleq E\left[z_1^{X_A^{(1)}} z_2^{X_A^{(2)}}\right]. \tag{3.27}$$

We then have the following theorem. (The proof, in which Lemma 3.1 is used, is given in Appendix A.6.)

Theorem 3.3. *$Q_A(z_1, z_2)$ is given by*

$$\begin{aligned} Q_A(z_1,z_2) = {} & B'(1)C'(1)\hat{\theta}(z_1,z_2)\frac{z_2}{C(B(z_2))}Q_D(z_2,z_2) \\ & + B'(1)C'(1)\frac{z_2}{z_2 - C(\gamma B(z_1))}\frac{1 - C(\gamma B(z_1))}{C'(1)(1-\gamma B(z_1))} \\ & \cdot\{Q_D(z_2,0) - Q_D(z_1,0)\} + (1 - B'(1)C'(1))\frac{Q_D(z_1,z_1)}{Q_{Db}(z_1)}, \end{aligned} \tag{3.28}$$

where

$$\hat{\theta}(z_1,z_2) = \hat{H}(z_1,z_2) + \frac{1 - C(\gamma B(z_1))}{C'(1)(1-\gamma B(z_1))} S(z_1,z_2) \tag{3.29}$$

with

$$\hat{H}(z_1,z_2) = \frac{(1-\gamma)B(z_2)}{C'(1)(B(z_2) - \gamma B(z_1))}\left[\frac{B(z_2) - C(B(z_2))}{1 - B(z_2)} - \frac{\gamma B(z_1) - C(\gamma B(z_1))}{1 - \gamma B(z_1)}\right] \tag{3.30}$$

and $S(z_1, z_2)$ is given in (3.24).

We present a corollary which immediately follows from Theorem 3.3.

Corollary 3.2. *The PGF $Q_A(z,z)$ for the total number of customers at the beginning of a randomly chosen slot is given by*

$$Q_A(z,z) = Q_{Ab}(z) Q_{A|idle}(z), \tag{3.31}$$

where

$$Q_{Ab}(z) = (1 - B'(1)C'(1))\frac{(z-1)C(B(z))}{z - C(B(z))} \tag{3.32}$$

and $Q_{A|idle}(z)$ is given in (3.19).

The proof is given in Appendix A.7

Remark 3.2. Note that $Q_{Ab}(z)$ denotes the PGF for the number of customers at the beginning of a randomly chosen slot in the corresponding BBP/G/1 queue without gates. This decomposition result is a discrete-time example of the general result for the continuous-time queue given in [7].

4. Work in the System

In this section, we consider the amount of work in the first queue and that in the second queue. To obtain the formulas for the amount of work in the system, we first derive the formula for the joint PGF for the numbers of customers and the remaining service time at the beginning of a randomly chosen slot.

Let $\tilde{X}^{(1)}$ (resp. $\tilde{X}^{(2)}$) denote a random variable representing the number of individual customers who arrive and remain in the first queue (resp. arrive and move to the second queue) during the backward recurrence time of the service time of a customer who is served in a randomly chosen slot. Also, let $\hat{C}$ denote a random variable representing the forward recurrence time of the service time of a customer who is served in a randomly chosen slot. We define $Q_{A|busy}(z_1, z_2, w)$ as the joint PGF for the numbers of customers who arrive and remain in the first queue and customers who arrive and move to the second queue during the backward recurrence time of the service time of a customer who is served in a randomly chosen slot, and the forward recurrence time of the service time of the customer given that the server is busy:

$$Q_{A|busy}(z_1, z_2, w) \triangleq E\left[z_1^{\tilde{X}^{(1)}} z_2^{\tilde{X}^{(2)}} w^{\hat{C}} \middle| T_S = 1\right]. \tag{4.1}$$

We then have the following lemma. (The proof is given in Appendix A.8.)

Lemma 4.1. *The joint PGF $Q_{A|busy}(z_1, z_2, w)$ is given by*

$$\begin{aligned} Q_{A|busy}(z_1, z_2, w) &= \hat{\theta}(z_1, z_2, w)\frac{z_2}{C(B(z_2))}Q_D(z_2, z_2) \\ &+ \frac{z_2}{z_2 - C(\gamma B(z_1))}\frac{w(C(w) - C(\gamma B(z_1)))}{C'(1)(w - \gamma B(z_1))}\{Q_D(z_2, 0) - Q_D(z_1, 0)\}, \end{aligned} \tag{4.2}$$

where

$$\hat{\theta}(z_1, z_2, w) = \hat{H}(z_1, z_2, w) + S(z_1, z_2)\frac{w(C(w) - C(\gamma B(z_1)))}{C'(1)(w - \gamma B(z_1))} \tag{4.3}$$

with

$$\begin{aligned} \hat{H}(z_1, z_2, w) = \frac{(1-\gamma)B(z_2)}{C'(1)(B(z_2) - \gamma B(z_1))}\Bigg[&\frac{B(z_2)C(w) - wC(B(z_2))}{w - B(z_2)} \\ &- \frac{\gamma B(z_1)C(w) - wC(\gamma B(z_1))}{w - \gamma B(z_1)}\Bigg] \end{aligned} \tag{4.4}$$

and $S(z_1, z_2)$ is given in (3.24).

We now consider the amount of work at the beginning of a randomly chosen slot. Let $U^{(1)}$ (resp. $U^{(2)}$) denote a random variable representing the amount of work in the first queue (resp. in the second queue) at the beginning of a randomly chosen slot. Note here that $U^{(1)}$ and $U^{(2)}$ are dependent. We define the joint PGF $U(z_1, z_2)$ associated with $U^{(1)}$ and $U^{(2)}$:

$$U(z_1, z_2) \triangleq E\left[z_1^{U^{(1)}} z_2^{U^{(2)}}\right]. \tag{4.5}$$

We then have the following theorem. (The proof, in which Lemma 4.1 is used, is given in Appendix A.9.)

Theorem 4.1. *The joint PGF $U(z_1, z_2)$ is given by*

$$U(z_1, z_2) = (1 - B'(1)C'(1))\frac{z_2(A(z_2) - 1)}{z_2 - A(z_2)}\frac{z_2 - \gamma A(z_2)}{z_2 - \gamma A(z_1)}\frac{Q_D(C(z_2), C(z_2))}{Q_{Db}(C(z_2))} \tag{4.6}$$

$$+(1-B'(1)C'(1))\frac{Q_D(C(z_1),C(z_1))}{Q_{Db}(C(z_1))}$$
$$+B'(1)\frac{z_2}{z_2-\gamma A(z_1)}\{Q_D(C(z_2),0)-Q_D(C(z_1),0)\}.$$

We present a corollary which immediately follows from Theorem 4.1. Let $U=U^{(1)}+U^{(2)}$ denote the amount of the total work in the system and we define $U(z)$ as the PGF for U.

Corollary 4.1. *The PGF $U(z)$ is given by*

$$U(z)=U_b(z)U_{idle}(z), \tag{4.7}$$

where

$$U_b(z)=(1-B'(1)C'(1))\frac{(z-1)A(z)}{z-A(z)}, \tag{4.8}$$

$$U_{idle}(z)=Q_{A|idle}(C(z)) \tag{4.9}$$
$$=\frac{1-\gamma}{1-\gamma A(z)}\frac{P(C(z),0)+Q(0)}{P(1,0)+Q(0)}.$$

The proof of the above is given in Appendix A.10.

Remark 4.1.

1. Note that $U(z)$ is identical to the PGF for the sojourn time of supercustomers and coincides with the result in [9].
2. $U_b(z)$ denotes the PGF for the amount of work in the corresponding BBP/G/1 queue without gates, and $U_{idle}(z)$ denotes the PGF for the amount of work in the first queue given that the server is idle. Thus (4.7) shows that the amount of the total work in the system is decomposed into two independent factors. This is a discrete-time example for the work decomposition property in the queue with the generalized vacations [4].

Note that, with (4.7) and noting $(P(1,0)+Q(0))/P(1,1)=(1-\gamma)(1-B'(1)C'(1))/B'(1)$ from Theorem 3.2, (4.6) is rewritten to be

$$U(z_1,z_2)=\frac{z_2}{z_2-\gamma A(z_1)}(1-\gamma)U(z_2)+\frac{\gamma(z_1-A(z_1))}{z_2-\gamma A(z_1)}\frac{z_2-1}{z_1-1}U(z_1). \tag{4.10}$$

The equation (4.10) may be useful because it can replace $U(z_1,z_2)$ by $U(z_1)$ and $U(z_2)$. Indeed, we will use (4.10) in order to derive the PGFs for the waiting times in the next section.

5. Waiting Times

In this section, we consider the waiting times of supercustomers and individual customers. We first derive the PGF for the waiting time of a randomly chosen supercustomer in terms of the PGF for the amount of work in the system. Next we obtain the PGFs for the waiting times of a randomly chosen individual customer in terms of the PGF for the amount of work in the system.

5.1. Waiting Time of a Randomly Chosen Supercustomer

In this subsection, we consider the waiting time of a randomly chosen supercustomer. We define a supercustomer as a batch composed of individual customers moving to the second queue at the same time when the gate opens. Note here that it is possible that there is no individual customer in the first queue when the gate opens. We regard such a case as an arrival of a supercustomer with zero service time at the second queue. Let W_s denote a random variable representing the waiting time of a randomly chosen supercustomer. We define $W_s(z)$ as the PGF for W_s. We then have the following theorem. (The proof is given in Appendix A.11.)

Theorem 5.1. *The PGF $W_s(z)$ is given by*

$$W_s(z) = \frac{1-\gamma}{z-\gamma}U(z) + \frac{z-1}{z-\gamma}(1 - B'(1)C'(1)), \tag{5.1}$$

where $U(z)$ is given in (4.7).

Remark 5.1. After some algebra with (4.10) and (5.1), we have the following relationship between the work in the second queue and the waiting time and the sojourn time of a supercustomer:

$$U(1,z) = 1 - A'(1) + A'(1)z\frac{W_s(z) - U(z)}{G'(1)A'(1)(1-z)} \tag{5.2}$$

Note that (5.2) can also be derived from the equality of the virtual delay and attained waiting time distribution (See, for example, [10, 11, 12]).

5.2. Waiting Time of a Randomly Chosen Customer

In this subsection, we consider the waiting time of a randomly chosen individual customer. Let $W_c^{(1)}$ (resp. $W_c^{(2)}$) denote a random variable representing the waiting time of a randomly chosen customer in the first queue (resp. in the second queue). Note here that $W_c^{(1)}$ and $W_c^{(2)}$ are dependent. We define $W_c(z_1, z_2)$ as the joint PGF for $W_c^{(1)}$ and $W_c^{(2)}$:

$$W_c(z_1, z_2) \triangleq E\left[z_1^{W_c^{(1)}} z_2^{W_c^{(2)}}\right]. \tag{5.3}$$

We then have the following theorem. (The proof is given in Appendix A.12.)

Theorem 5.2. *The joint PGF $W_c(z_1, z_2)$ is given by*

$$W_c(z_1, z_2) = \frac{1-\gamma}{z_2 - \gamma z_1}\frac{1}{z_1 - A(z_2)}\left[(z_1 - z_2)U(z_2) + \frac{1-\gamma}{1-\gamma z_1}z_1(z_2 - 1)U(\gamma z_1)\right] \cdot \frac{1 - A(z_2)}{B'(1)(1 - C(z_2))}, \tag{5.4}$$

where $U(z)$ is given in (4.7).

Let $W_c = W_c^{(1)} + W_c^{(2)}$ denote the total waiting time of a randomly chosen customer in the system. We then define $W_c(z)$, $W_{c1}(z)$ and $W_{c2}(z)$ as the PGFs for W_c, $W_c^{(1)}$ and $W_c^{(2)}$, respectively. Now we present a corollary which immediately follows from Theorem 5.2.

Corollary 5.1. *The PGFs $W_{c1}(z)$,$W_{c2}(z)$ and $W_c(z)$ are given by*

$$W_{c1}(z) = \frac{1-\gamma}{1-\gamma z}, \tag{5.5}$$

$$W_{c2}(z) = \frac{1-\gamma}{z-\gamma}\frac{1-z}{1-A(z)}\left[U(z) - (1 - B'(1)C'(1))\right]\frac{1-A(z)}{B'(1)(1-C(z))} \tag{5.6}$$

$$W_c(z) = W_b(z)U_{back}(z), \tag{5.7}$$

respectively, where

$$W_b(z) = (1 - B'(1)C'(1))\frac{z-1}{z-A(z)} \cdot \frac{1-A(z)}{B'(1)(1-C(z))}, \tag{5.8}$$

$$U_{back}(z) = \frac{1-\gamma}{1-\gamma z}\frac{U(\gamma z)}{1-B'(1)C'(1)}, \tag{5.9}$$

and $U(z)$ is given in (4.7).

Remark 5.2.

1. $W_b(z)$ denotes the PGF for the waiting times of customers in the corresponding BBP/G/1 queue without gates, and $U_{back}(z)$ denotes the PGF for the backward recurrence time of the gate opening interval given that the server is idle. This is a discrete-time example of the waiting-time decomposition property in the queue with generalized vacations [7].

2. After some algebra with (4.10) and (5.6), we have the following relationship between the work in the second queue and the waiting time and the sojourn time of an individual customer in the second queue:

   $$U(1, z) = 1 - A'(1) + A'(1)zW_{c2}(z)\frac{1-C(z)}{C'(1)(1-z)} \tag{5.10}$$

   Note that (5.10) can also be derived from the equality of the virtual delay and attained waiting time distribution (See, for example, [10, 11, 12]).

6. Numerical Examples

In this section, we provide some numerical examples. First we regard the second queue as an isolated system and observe the effect of the gate opening interval on the mean waiting time. More precisely, we consider the gate opening interval in terms of the covariances and the correlation coefficients. In introduction, we mentioned two types of correlations:

- type 1: correlation between the interarrival time G and the service time C_S of each supercustomer,
- type 2: correlation between the interarrival time G of each batch composed of customers who move to the second queue at the same time and the number B_G of the customers.

Then, the covariances and the correlation coefficients for the two types of correlation are given by

$$Cov[G, C_S] = \frac{\gamma}{(1-\gamma)^2}A'(1), \tag{6.1}$$

$$correl[G, C_S] = \left[\frac{\gamma(A'(1))^2}{(1-\gamma)(A'(1)+A''(1)) + (2\gamma-1)(A'(1))^2}\right]^{1/2}, \tag{6.2}$$

(6.3) $$Cov[G, B_G] = \frac{\gamma}{(1-\gamma)^2} B'(1),$$

(6.4) $$correl[G, B_G] = \left[\frac{\gamma (B'(1))^2}{(1-\gamma)(B'(1)+B''(1)) + (2\gamma - 1)(B'(1))^2} \right]^{1/2},$$

respectively. Note here that the correlation coefficients are increasing functions of the mean gate opening interval.

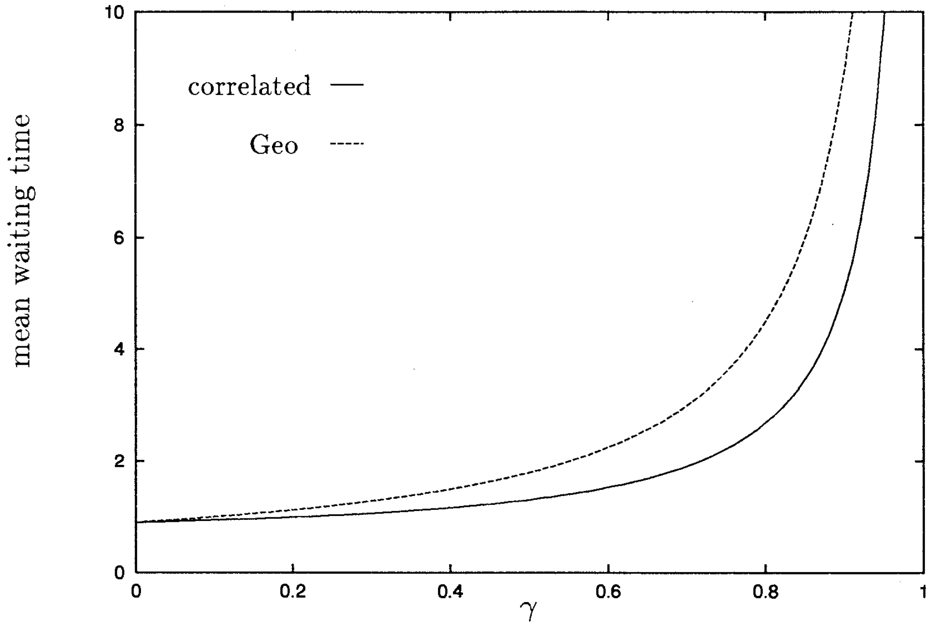

Figure 2: Effect of correlation (1)

Now we observe the effect of the gate opening interval on the mean waiting time of supercustomers. We show the formula for the mean waiting time of supercustomers in Appendix A.14. To compare the result, we also consider a corresponding Geo/G/1 queue where the PGF for the service time of a customer is $G(A(z))$ and the PGF for the interarrival time of customers is $G(z)$. Fig. 2 shows the mean waiting time of (super)customers in the second queue as a function of the parameter γ in the following settings: (1) the number of individual customers arriving to the system in a slot is geometrically distributed with mean 0.6, (2) the service times of individual customers are deterministic and equal to one slot. Note here that the increase of the parameter γ implies the increase of the correlation coefficient between the interarrival time and the service time of each supercustomer. In Fig. 2, we observe that the positive correlation leads to the reduction of the mean waiting time of supercustomers, whereas the mean waiting time increases with the increase of the correlation coefficient. A similar observation has been shown in [2].

Next, we observe the effect of the gate opening interval on the mean waiting time of individual customers in the second queue. We show the formula for the mean waiting time of individual customers in Appendix A.15. To compare the result, we also consider a corresponding BBP/G/1 queue where the PGF for the service time of a customer is $C(z)$ and the PGF for the batch size (arriving to the second queue) is $(1-\gamma)G(B(z))+\gamma$. Fig.

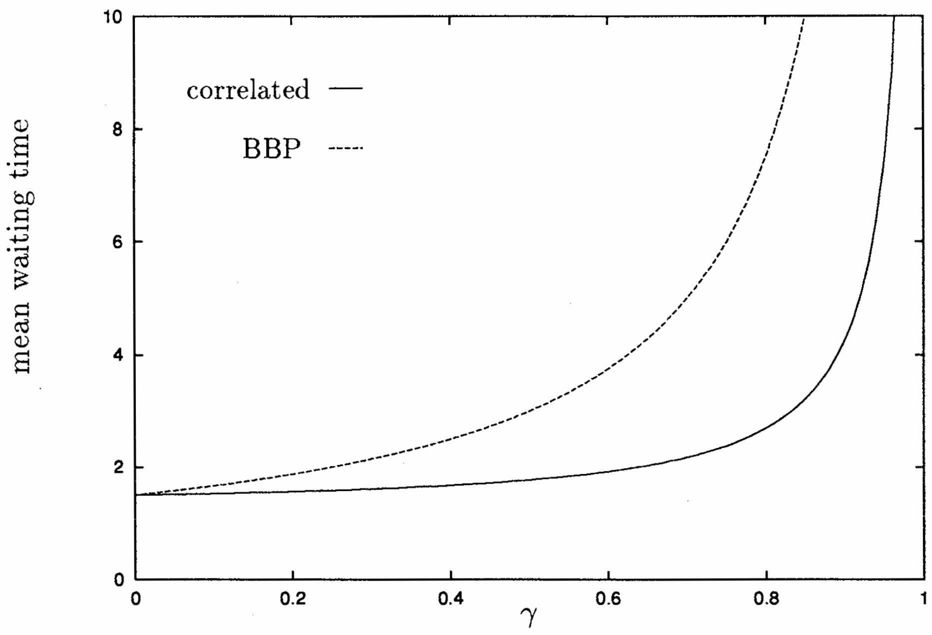

Figure 3: Effect of correlation (2)

3 shows the mean waiting time of individual customers in the second queue as a function of the parameter γ in the same settings as those in Fig. 2. Note here that the increase of the parameter γ implies the increase of the correlation coefficient between the interarrival time of the batches and the number of customers in each batch. In Fig. 3, we also observe that the positive correlation leads to the reduction of the mean waiting time of individual customers in the second queue, whereas the mean waiting time increases with the increase of the correlation coefficient.

Finally we observe the correlation between the waiting times of a randomly chosen individual customer in the first queue and in the second queue. Fig. 4 shows the correlation coefficients between the waiting times of a randomly chosen individual customer in the first queue and in the second queue, which are obtained by using numerical differentiation as a function of the parameter γ in the following settings: (1) the number of individual customers arriving to the system in a slot is geometrically distributed with mean 0.4, 0.6 and 0.8, (2) the service times of individual customers are deterministic and equal to one slot. In Fig. 4, we observe that, as expected, the correlation is negative and the correlation coefficient decreases with the increase of the parameter γ. Further, we observe that the increase of the traffic intensity leads to the increase of the correlation coefficient.

Appendix: Proofs

A.1. Proof of Theorem 3.1

By definition, $Q_D(z_1, z_2)$ satisfies

$$Q_D(z_1, z_2) = \frac{P(z_1, z_2)}{P(1, 1)}. \tag{A.1}$$

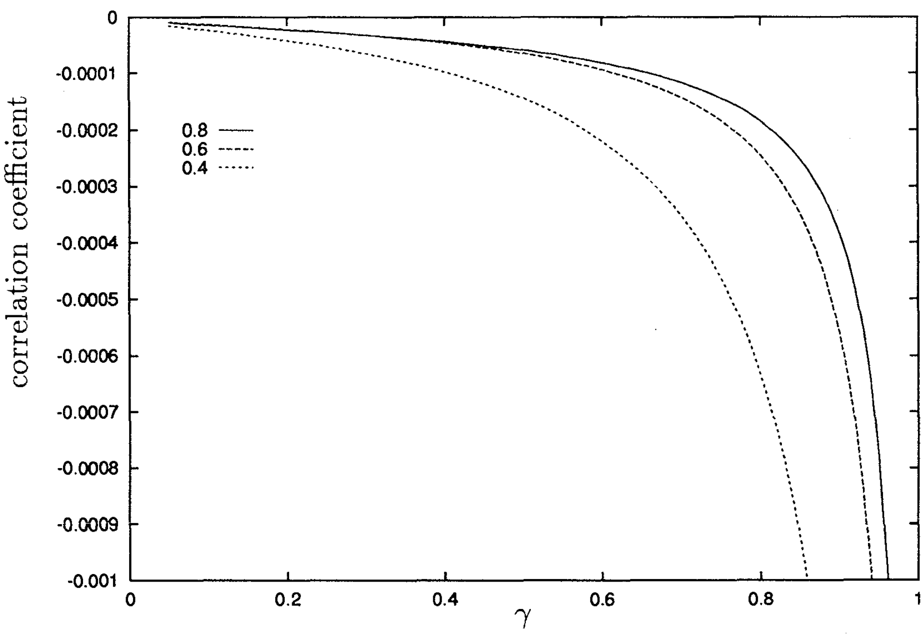

Figure 4: Correlation between the waiting times in each queue

Thus $Q_D(z_1, z_2)$ is obtained once we have $P(z_1, z_2)$. Using the memoryless property of the gate opening intervals, we have

$$Q(z_2) = G(B(z_2))\{P(z_2, 0) + Q(0)\}. \tag{A.2}$$

Let $Y^{(1)}$ denote a random variable representing the number of individual customers who arrive and remain in the first queue during the service time of a customer whose service starts immediately after the randomly chosen imbedded point. Also, let $Y^{(2)}$ denote a random variable representing the number of individual customers who arrive and move to the second queue during the service time. Note here that $Y^{(1)}$ and $Y^{(2)}$ are dependent. We define a random variable T_G as

$$T_G \triangleq \begin{cases} 1 & \text{if the gate opens at least once during the service time,} \\ 0 & \text{if the gate does not open during the service time.} \end{cases} \tag{A.3}$$

Note here that $Y^{(2)} = 0$ if $T_G = 0$. We define $H(z_1, z_2)$ as

$$H(z_1, z_2) \triangleq E\left[z_1^{Y^{(1)}} z_2^{Y^{(2)}} \mathbf{1}_{\{T_G=1\}}\right]. \tag{A.4}$$

To derive an expression for $H(z_1, z_2)$, we suppose that the service time of a randomly chosen customer is τ $(1 \le \tau < \infty)$ and the gate last opens in the kth $(1 \le k \le \tau)$ slot during the service time of the customer. It follows that

$$\begin{aligned} H(z_1, z_2) &= \sum_{\tau=1}^{\infty} \Pr(C = \tau) \sum_{k=1}^{\tau} (1-\gamma)\gamma^{k-1}(B(z_1))^{k-1}(B(z_2))^{\tau-k+1} \\ &= \frac{(1-\gamma)B(z_2)}{B(z_2) - \gamma B(z_1)}\Big[C(B(z_2)) - C(\gamma B(z_1))\Big]. \end{aligned} \tag{A.5}$$

Furthermore, we define $\theta(z_1, z_2)$ as

$$\text{(A.6)} \quad \theta(z_1, z_2) \triangleq E\left[z_1^{Y^{(1)}} z_2^{Y^{(2)}} \mathbf{1}_{\{T_G=0\}}\right] + E\left[z_1^{Y^{(1)}} z_2^{Y^{(2)}} \mathbf{1}_{\{T_G=1\}}\right] = C(\gamma B(z_1)) + H(z_1, z_2).$$

To obtain $P(z_1, z_2)$, we now consider three exclusive events:

- The preceding imbedded point is of type 1, there exists at least one customer in the second queue at the preceding imbedded point and the gate does not open during the service time of the customer who is served immediately after the imbedded point, i.e., $\{T_P = 1, X^{(2)} > 0, T_G = 0\}$,
- The preceding imbedded point is of type 1, there exists at least one customer in the second queue at the preceding imbedded point and the gate opens during the service time of the customer who is served immediately after the imbedded point, i.e., $\{T_P = 1, X^{(2)} > 0, T_G = 1\}$,
- The preceding imbedded point is of type 2 and there exists at least one customer in the second queue, i.e., $\{T_P = 2, X^{(2)} > 0\}$.

From the above observation, we obtain

$$\text{(A.7)} \quad P(z_1, z_2) = \left[P(z_1, z_2) - P(z_1, 0)\right]\frac{1}{z_2}C(\gamma B(z_1)) + \left[P(z_2, z_2) - P(z_2, 0)\right]\frac{1}{z_2}H(z_1, z_2) + \left[Q(z_2) - Q(0)\right]\frac{1}{z_2}\theta(z_1, z_2).$$

Setting $z_1 = z_2$ in (A.7) and using (A.2), $H(z_2, z_2) = C(B(z_2)) - C(\gamma B(z_2))$ and $\theta(z_2, z_2) = C(B(z_2))$, we have

$$\text{(A.8)} \quad P(z_2, z_2) = \frac{C(B(z_2))}{z_2 - C(B(z_2))}\left[G(B(z_2)) - 1\right]\left[P(z_2, 0) + Q(0)\right].$$

Using (2.4), (A.2) and (A.8) in (A.7), we obtain

$$\text{(A.9)} \quad \left[z_2 - C(\gamma B(z_1))\right]P(z_1, z_2) = C(\gamma B(z_1))\left[P(z_2, 0) - P(z_1, 0)\right] + \left[\frac{z_2}{C(B(z_2))}\theta(z_1, z_2) - C(\gamma B(z_1))\right] \cdot \frac{C(B(z_2))}{z_2 - C(B(z_2))}\frac{B(z_2) - 1}{1 - \gamma B(z_2)}\left[P(z_2, 0) + Q(0)\right],$$

from which and (A.1), (3.4) immediately follows.

A.2. Proof of Theorem 3.2

Iterating (3.14), we obtain for $|z_1| \le 1$ and $M \ge 0$

$$\text{(A.10)} \quad \Psi(z_1) = \prod_{h=0}^{M} \phi(\delta^{(h)}(z_1))\Psi(\delta^{(M+1)}(z_1)).$$

We now need the following lemma.

Lemma A.1.

1. *The equation* $\delta(z_1) = z_1$ $(|z_1| \leq 1)$ *has a unique solution* z_1^* *and* z_1^* *is real.*

2. $\lim_{M\to\infty} \delta^{(M)}(z_1) = z_1^*$ *for all* z_1 *with* $|z_1| \leq 1$.

3. $\prod_{h=0}^{\infty} \phi(\delta^{(h)}(z_1))$ *converges for all* z_1 *with* $|z_1| \leq 1$.

The proof of Lemma A.1 is given in Appendix A.3. Letting $M \to \infty$ in (A.10), then Lemma A.1 leads to the following expression for $\Psi(z_1)$:

$$\Psi(z_1) = \prod_{h=0}^{\infty} \phi(\delta^{(h)}(z_1))\Psi(z_1^*). \tag{A.11}$$

Thus (A.11) becomes

$$\Psi(z_1) = \alpha(z_1)\Psi(z_1^*). \tag{A.12}$$

Letting $z_1 = 1$ in (A.12), we obtain

$$\Psi(z_1^*) = \frac{1}{\alpha(1)}\Psi(1). \tag{A.13}$$

Substituting (A.13) into (A.12) leads to

$$\Psi(z_1) = \frac{\alpha(z_1)}{\alpha(1)}\Psi(1). \tag{A.14}$$

Also, letting $z_2 = 1$ in (A.8), we have

$$\Psi(1) = \frac{1 - B'(1)C'(1)}{B'(1)}(1-\gamma). \tag{A.15}$$

(3.15) immediately follows from (A.14) and (A.15).

A.3. Proof of Lemma A.1

Using (3.9), we then find that $\delta(z_1) = z_1$ if and only if $\xi(\gamma B(z_1)) = \gamma B(z_1)$ and that $\xi^{(h)}(\gamma B(z_1)) = \gamma B(\delta^{(h)}(z_1))$ $(|z_1| \leq 1, h = 1, 2, \ldots)$, where

$$\xi(z) \triangleq \gamma A(z), \tag{A.16}$$

$$\xi^{(0)}(z) \triangleq z, \tag{A.17}$$

$$\xi^{(h)}(z) \triangleq \xi(\xi^{(h-1)}(z)) \quad (h = 1, 2, \ldots). \tag{A.18}$$

From the results in [9], we know that the equation $\xi(w) = w$, $|w| \leq 1$ has a unique solution w^*, that $0 \leq w^* \leq 1$ and that $\lim_{M\to\infty} \xi^{(M)}(w) = w^*$ for all w with $|w| \leq 1$. As $|\gamma B(z_1)| \leq 1$ for all z_1 with $|z_1| \leq 1$, we conclude that the equation $\delta(z_1) = z_1$, $|z_1| \leq 1$ has a unique solution $z_1^* = C(w^*)$, z_1^* is real, and that $\lim_{M\to\infty} \delta^{(M)}(z_1) = z_1^*$ for all z_1 with $|z_1| \leq 1$.

Using (3.13), we have

$$\prod_{h=0}^{M} \phi(\delta^{(h)}(z_1)) = \frac{1 - \gamma B(z_1)}{1 - \gamma B(\delta^{(M+1)}(z_1))} \prod_{h=0}^{M} \frac{(1-\gamma)B(\delta^{(h+1)}(z_1))}{B(\delta^{(h+1)}(z_1)) - \gamma B(\delta^{(h)}(z_1))}. \tag{A.19}$$

Thus we obtain

$$\text{(A.20)} \qquad \prod_{h=0}^{\infty} \phi(\delta^{(h)}(z_1)) = \frac{1-\gamma B(z_1)}{1-\gamma B(z_1^*)} \prod_{h=0}^{\infty} \frac{(1-\gamma)B(\delta^{(h+1)}(z_1))}{B(\delta^{(h+1)}(z_1)) - \gamma B(\delta^{(h)}(z_1))}.$$

From the theory of infinite products, the infinite product

$$\text{(A.21)} \qquad \prod_{h=0}^{\infty} \frac{(1-\gamma)B(\delta^{(h+1)}(z_1))}{B(\delta^{(h+1)}(z_1)) - \gamma B(\delta^{(h)}(z_1))}$$

converges if and only if the infinite sum

$$\text{(A.22)} \quad \sum_{h=0}^{\infty}\left[1 - \frac{(1-\gamma)B(\delta^{(h+1)}(z_1))}{B(\delta^{(h+1)}(z_1)) - \gamma B(\delta^{(h)}(z_1))}\right] = \gamma \sum_{h=0}^{\infty} \frac{B(\delta^{(h+1)}(z_1)) - B(\delta^{(h)}(z_1))}{B(\delta^{(h+1)}(z_1)) - \gamma B(\delta^{(h)}(z_1))}$$

converges. For some real η_1 $(0 \le \eta_1 \le 1)$, we have

$$\text{(A.23)} \qquad \left|\delta^{(h+1)}(z_1) - \delta^{(h)}(z_1)\right| = \left|\delta^{(h)}(z_1) - \delta^{(h-1)}(z_1)\right|\left|\delta'(\eta_1)\right|.$$

Since $|\delta'(\eta_1)| \le |\delta'(1)| = \gamma B'(1)C'(\gamma)$, we obtain

$$\text{(A.24)} \qquad \left|\delta^{(h+1)}(z_1) - \delta^{(h)}(z_1)\right| \le \gamma B'(1)C'(\gamma)\left|\delta^{(h)}(z_1) - \delta^{(h-1)}(z_1)\right|.$$

Similarly, we have for some real η_2 $(0 \le \eta_2 \le 1)$,

$$\text{(A.25)} \qquad \left|B(\delta^{(h+1)}(z_1)) - B(\delta^{(h)}(z_1))\right| = \left|\delta^{(h+1)}(z_1) - \delta^{(h)}(z_1)\right|\left|B'(\eta_2)\right|.$$

Since $|B'(\eta_2)| < 1$, we obtain

$$\text{(A.26)} \qquad \left|B(\delta^{(h+1)}(z_1)) - B(\delta^{(h)}(z_1))\right| < \left|\delta^{(h+1)}(z_1) - \delta^{(h)}(z_1)\right|.$$

From (A.26), it follows that

$$\text{(A.27)} \qquad \left|\frac{B(\delta^{(h+1)}(z_1)) - B(\delta^{(h)}(z_1))}{B(\delta^{(h+1)}(z_1)) - \gamma B(\delta^{(h)}(z_1))}\right| < \left|\frac{\delta^{(h+1)}(z_1) - \delta^{(h)}(z_1)}{B(\delta^{(h+1)}(z_1)) - \gamma B(\delta^{(h)}(z_1))}\right|.$$

We define x_n as

$$\text{(A.28)} \qquad x_n \triangleq \frac{\delta^{(n)}(z_1) - \delta^{(n-1)}(z_1)}{B(\delta^{(n)}(z_1)) - \gamma B(\delta^{(n-1)}(z_1))}.$$

Using (A.24), we have

$$\text{(A.29)} \qquad \left|\frac{x_{n+1}}{x_n}\right| = \left|\frac{\delta^{(n+1)}(z_1) - \delta^{(n)}(z_1)}{\delta^{(n)}(z_1) - \delta^{(n-1)}(z_1)} \frac{B(\delta^{(n)}(z_1)) - \gamma B(\delta^{(n-1)}(z_1))}{B(\delta^{(n+1)}(z_1)) - \gamma B(\delta^{(n+1)}(z_1))}\right|$$
$$\le \left|\frac{B(\delta^{(n+1)}(z_1)) - \gamma B(\delta^{(n)}(z_1))}{B(\delta^{(n+2)}(z_1)) - \gamma B(\delta^{(n+1)}(z_1))}\right| .$$

Since $\gamma B'(1)C'(\gamma) < 1$ and

$$\text{(A.30)} \qquad \lim_{n\to\infty}\left|\frac{B(\delta^{(n+1)}(z_1)) - B(\delta^{(n)}(z_1))}{B(\delta^{(n+2)}(z_1)) - B(\delta^{(n+1)}(z_1))}\right| = 1,$$

the infinite sum

$$\text{(A.31)} \qquad \sum_{h=0}^{\infty}\left|\frac{\delta^{(h+1)}(z_1) - \delta^{(h)}(z_1)}{B(\delta^{(h+1)}(z_1)) - \gamma B(\delta^{(h)}(z_1))}\right|$$

converges, and therefore the infinite product

$$\text{(A.32)} \qquad \prod_{h=0}^{\infty} \phi(\delta^{(h)}(z_1))$$

also converges.

A.4. Proof of Corollary 3.1

First we note that

$$\theta(z,z) = C(B(z)). \tag{A.33}$$

Furthermore, from (3.5) and (A.15), we have

$$\frac{P(1,0)+Q(0)}{P(1,1)} = \frac{1-B'(1)C'(1)}{B'(1)}(1-\gamma). \tag{A.34}$$

Letting $z = z_1 = z_2$, and using (A.33) and (A.34) in (3.4), we obtain

$$\begin{aligned} Q_D(z,z) = {} & \frac{(z-1)C(B(z))}{z-C(B(z))}(1-B'(1)C'(1))\frac{B(z)-1}{B'(1)(z-1)} \\ & \cdot \frac{1-\gamma}{1-\gamma B(z)} \frac{P(z,0)+Q(0)}{P(1,0)+Q(0)}, \end{aligned} \tag{A.35}$$

from which, (3.20) immediately follows.

A.5. Proof of Lemma 3.1

We observe the imbedded point preceding the service time of a randomly chosen customer. We then consider two events:

- The imbedded point is of type 1 and there exists at least one customer in the second queue at the imbedded point, i.e., $\{T_P = 1, X^{(2)} > 0\}$,
- The imbedded point is of type 2 and there exists at least one customer in the second queue at the imbedded point, i.e., $\{T_P = 2, X^{(2)} > 0\}$.

From the above observation and using (3.8), (A.1), (A.2) and (A.8), it follows that

$$\begin{aligned} \hat{Q}(z_1,z_2) &= \left[P(z_1,z_2) - P(z_1,0) + Q(z_2) - Q(0)\right]/P(1,1) \\ &= \Big[P(z_1,z_2) + P(z_2,0) - P(z_1,0) \\ &\qquad + \{G(B(z_2)) - 1\}\{P(z_2,0) + Q(0)\}\Big]/P(1,1) \\ &= \frac{z_2}{z_2 - C(\gamma B(z_1))}\left[Q_D(z_2,0) - Q_D(z_1,0) + \{\frac{\theta(z_1,z_2)}{C(B(z_2))} - 1\}Q_D(z_2,z_2)\right] \\ &\quad + \frac{z_2}{C(B(z_2))}Q_D(z_2,z_2), \end{aligned} \tag{A.36}$$

from which and (A.6), (3.23) immediately follows.

A.6. Proof of Theorem 3.3

Since the server is busy with probability $B'(1)C'(1)$, we have

$$Q_A(z_1,z_2) = B'(1)C'(1)Q_{A|busy}(z_1,z_2) + (1-B'(1)C'(1))Q_{A|idle}(z_1). \tag{A.37}$$

We relate $Q_{A|busy}(z_1,z_2)$ with $\hat{Q}(z_1,z_2)$. To do so, we define $\tilde{C}$ and $\tilde{G}$ as random variables which represent the backward recurrence time of the service time of an individual customer and that of the gate opening interval, respectively. We then consider two events:

- The server is busy and the gate opened at least once during the backward recurrence time of the current service, i.e., $\{T_S = 1, \tilde{C} > \tilde{G}\}$,

- The server is busy and the gate did not open during the backward recurrence time of the current service, i.e., $\{T_S = 1, \tilde{C} \le \tilde{G}\}$.

Let $\tilde{X}^{(1)}$ denote a random variable representing the number of individual customers who arrive and remain in the first queue during the backward recurrence time of the service time of a customer who is served in a randomly chosen slot. Also, let $\tilde{X}^{(2)}$ denote a random variable representing the number of individual customers who arrive and move to the second queue during the backward recurrence time of the service time. We define $\hat{H}(z_1, z_2)$ as

$$\hat{H}(z_1, z_2) \triangleq E\left[z_1^{\tilde{X}^{(1)}} z_2^{\tilde{X}^{(2)}} \mathbf{1}_{\{\tilde{C}>\tilde{G}\}} \middle| T_S = 1 \right]. \tag{A.38}$$

It then follows from an argument similar to the one for (A.5) that

$$\hat{H}(z_1, z_2) = \sum_{\tau=1}^{\infty} \frac{1}{C'(1)} \sum_{n=\tau+1}^{\infty} \Pr(C = n) \sum_{k=0}^{\tau-1} (1-\gamma)\gamma^k (B(z_1))^k (B(z_2))^{\tau-k}, \tag{A.39}$$

from which, we obtain (3.30). We then have

$$Q_{A|busy}(z_1, z_2) = \hat{Q}(z_2, z_2)\hat{H}(z_1, z_2) + \hat{Q}(z_1, z_2)\frac{1 - C(\gamma B(z_1))}{C'(1)(1 - \gamma B(z_1))}. \tag{A.40}$$

Finally, using Corollary 3.1, Lemma 3.1 and (3.26), we obtain

$$\begin{aligned} Q_A(z_1, z_2) &= B'(1)C'(1)Q_{A|busy}(z_1, z_2) + (1 - B'(1)C'(1))Q_{A|idle}(z_1) \\ &= B'(1)C'(1)\left[\hat{H}(z_1, z_2) + \frac{1 - C(\gamma B(z_1))}{C'(1)(1 - \gamma B(z_1))} S(z_1, z_2) \right] \\ &\quad \cdot \frac{z_2}{C(B(z_2))} Q_D(z_2, z_2) \\ &\quad + B'(1)C'(1) \frac{z_2}{z_2 - C(\gamma B(z_1))} \frac{1 - C(\gamma B(z_1))}{C'(1)(1 - \gamma B(z_1))} \\ &\quad \cdot \{Q_D(z_2, 0) - Q_D(z_1, 0)\} + (1 - B'(1)C'(1)) \frac{Q_D(z_1, z_1)}{Q_{Db}(z_1)}, \end{aligned} \tag{A.41}$$

from which, (3.28) follows.

A.7. Proof of Corollary 3.2

Setting $z_1 = z_2 = z$ in (3.28), noting

$$\hat{\theta}(z, z) = \frac{C(B(z)) - 1}{C'(1)(B(z) - 1)}, \tag{A.42}$$

and using Corollary 3.1, it follows that

$$\begin{aligned} Q_A(z, z) &= B'(1)C'(1) \frac{z}{C(B(z))} \frac{C(B(z)) - 1}{C'(1)(B(z) - 1)} Q_D(z, z) \\ &\quad + (1 - B'(1)C'(1)) \frac{Q_D(z, z)}{Q_{Db}(z)} \\ &= \frac{B'(1)(z - 1)}{B(z) - 1} Q_D(z, z) \\ &= \frac{(z - 1)C(B(z))}{z - C(B(z))} (1 - B'(1)C'(1)) \frac{1 - \gamma}{1 - \gamma B(z)} \frac{P(z, 0) + Q(0)}{P(1, 0) + Q(0)}, \end{aligned} \tag{A.43}$$

from which, (3.31) follows.

A.8. Proof of Lemma 4.1

We define $\hat{H}(z_1, z_2, w)$:

$$\hat{H}(z_1, z_2, w) \triangleq E\left[z_1^{\tilde{X}^{(1)}} z_2^{\tilde{X}^{(2)}} w^{\hat{C}} \mathbf{1}_{\{\tilde{C} > \tilde{G}\}} \middle| T_S = 1 \right]. \tag{A.44}$$

From an argument similar to the one for (A.5), it follows that

$$\hat{H}(z_1, z_2, w) = \sum_{n=0}^{\infty} \sum_{\tau=1}^{n} \sum_{k=0}^{\tau-1} \frac{\Pr(C = n+1)}{C'(1)} w^{n-\tau} (1-\gamma)\gamma^k (B(z_1))^k (B(z_2))^{\tau-k}, \tag{A.45}$$

which yields (4.4). By definition and using (3.26), we have

$$\begin{aligned} Q_{A|busy}(z_1, z_2, w) &= \hat{Q}(z_2, z_2)\hat{H}(z_1, z_2, w) + \hat{Q}(z_1, z_2)\frac{w(C(w) - C(\gamma B(z_1)))}{C'(1)(w - \gamma B(z_1))} \\ &= \left[\hat{H}(z_1, z_2, w) + S(z_1, z_2)\frac{w(C(w) - C(\gamma B(z_1)))}{C'(1)(w - \gamma B(z_1))}\right]\hat{Q}(z_2, z_2) \\ &\quad + \frac{z_2}{z_2 - C(\gamma B(z_1))}\frac{w(C(w) - C(\gamma B(z_1)))}{C'(1)(w - \gamma B(z_1))} \\ &\quad \cdot\{Q_D(z_2, 0) - Q_D(z_1, 0)\}, \end{aligned} \tag{A.46}$$

from which, (4.2) immediately follows.

A.9. Proof of Theorem 4.1

By definition and using (3.19) and Lemma 4.1, we have

$$\begin{aligned} U(z_1, z_2) &= B'(1)C'(1)\frac{1}{C(z_2)}Q_{A|busy}(C(z_1), C(z_2), z_2) \\ &\quad +(1 - B'(1)C'(1))Q_{A|idle}(C(z_1)) \\ &= B'(1)C'(1)\frac{1}{C(z_2)}\left[\hat{Q}(C(z_2), C(z_2))\hat{H}(C(z_1), C(z_2), z_2)\right. \\ &\quad \left. +\hat{Q}(C(z_1), C(z_2))\frac{z_2(C(z_2) - C(\gamma A(z_1)))}{C'(1)(z_2 - \gamma A(z_1))}\right] \\ &\quad +(1 - B'(1)C'(1))\frac{P(C(z_1), 0) + Q(0)}{P(1, 0) + Q(0)}\frac{1-\gamma}{1 - \gamma A(z_1)} \\ &= B'(1)C'(1)\frac{1}{C(z_2)}\left[\hat{H}(C(z_1), C(z_2), z_2)\right. \\ &\quad \left. + \frac{z_2(C(z_2) - C(\gamma A(z_1)))}{C'(1)(z_2 - \gamma A(z_1))}S(C(z_1), C(z_2))\right]\hat{Q}((C(z_2), C(z_2)) \\ &\quad +B'(1)\frac{z_2}{z_2 - \gamma A(z_1)}\{Q_D(C(z_2), 0) - Q_D(C(z_1), 0)\} \\ &\quad +(1 - B'(1)C'(1))\frac{Q_D(C(z_1), C(z_1))}{Q_{Db}(C(z_1))} \\ &= B'(1)C'(1)\frac{1}{C(z_2)}\hat{\theta}(C(z_1), C(z_2), z_2)\hat{Q}((C(z_2), C(z_2)) \\ &\quad +B'(1)\frac{z_2}{z_2 - \gamma A(z_1)}\{Q_D(C(z_2), 0) - Q_D(C(z_1), 0)\} \\ &\quad +(1 - B'(1)C'(1))\frac{Q_D(C(z_1), C(z_1))}{Q_{Db}(C(z_1))}. \end{aligned} \tag{A.47}$$

Note that

$$\text{(A.48)} \quad \hat{\theta}(C(z_1), C(z_2), z_2) = \frac{1}{C'(1)} \frac{z_2}{z_2 - A(z_2)} \frac{z_2 - \gamma A(z_2)}{z_2 - \gamma A(z_1)} \{C(z_2) - C(A(z_2))\}.$$

(4.6) immediately follows from the above two equations.

A.10. Proof of Corollary 4.1

Using (A.35) in (4.6), it follows that

$$\text{(A.49)} \quad \begin{aligned} U(z) &= U(z, z) \\ &= \frac{(z-1)A(z)}{z - A(z)}(1 - B'(1)C'(1)) \frac{1-\gamma}{1-\gamma A(z)} \frac{P(C(z),0) + Q(0)}{P(1,0) + Q(0)}, \end{aligned}$$

from which, (4.7) immediately follows.

A.11. Proof of Theorem 5.1

It follows that

$$\text{(A.50)} \quad W_s(z) = \frac{U(1,z) - U(1,0)}{z} + U(1,0).$$

Using (4.10) in (A.50), (5.1) immediately follows.

A.12. Proof of Theorem 5.2

We divide the waiting time of a randomly chosen customer into three parts:

$$\text{(A.51)} \quad W_c^{(1)} = F^{(1)}, \quad W_c^{(2)} = F^{(2)} + D,$$

where $F^{(1)}$ (resp. $F^{(2)}$) denotes a random variable representing the waiting time of the batch which includes the randomly chosen customer in the first queue (resp. in the second queue), and D denotes a random variable representing the sum of the service times of customers who arrive in the same batch as the randomly chosen customer and are served before the randomly chosen customer. Note here that $F^{(1)} + F^{(2)}$ and D are independent.

Now we define the following PGFs:

$$\text{(A.52)} \quad F(z_1, z_2) \triangleq E\left[z_1^{F^{(1)}} z_2^{F^{(2)}}\right], \qquad D(z) \triangleq E\left[z^D\right].$$

First we consider $F(z_1, z_2)$. Let $\hat{G}_A$ denote a random variable representing the remaining gate opening interval. Also, let $W_A^{(1)}$ and $W_A^{(2)}$ denote random variables representing the amounts of work in the first queue and in the second queue, respectively, immediately before the arrival of a randomly chosen customer. Note that the joint distribution of the amount of work immediately before arrivals is identical to that at the beginning of a randomly chosen slot, since customers arrive to the system according to a batch Bernoulli process [3]. Thus, it follows that

$$\text{(A.53)} \quad \begin{aligned} F(z_1, z_2) &= E\left[z_1^{\hat{G}_A} z_2^{W_A^{(1)} + W_A^{(2)} - \hat{G}_A} \mathbf{1}_{\{W_A^{(2)} \geq \hat{G}_A\}}\right] + E\left[z_1^{\hat{G}_A} z_2^{W_A^{(1)}} \mathbf{1}_{\{W_A^{(2)} < \hat{G}_A\}}\right] \\ &= \sum_{k=0}^{\infty} \sum_{n=1}^{\infty} \sum_{m=n}^{\infty} (1-\gamma)\gamma^{n-1} z_1^{n-1} z_2^{k+m-1-(n-1)} \Pr(U^{(1)} = k, U^{(2)} = m) \\ &\quad + \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=m+1}^{\infty} (1-\gamma)\gamma^{n-1} z_1^{n-1} z_2^{k} \Pr(U^{(1)} = k, U^{(2)} = m) \\ &= \frac{1-\gamma}{z_2 - \gamma z_1}\{U(z_2, z_2) - U(z_2, \gamma z_1)\} + \frac{1-\gamma}{1-\gamma z_1} U(z_2, \gamma z_1). \end{aligned}$$

On the other hand, $D(z)$ is given by

(A.54)
$$D(z) = \frac{1-A(z)}{B'(1)(1-C(z))}.$$

We then have

(A.55)
$$W_c(z_1, z_2) = F(z_1, z_2)D(z_2).$$

Using (4.10) in (A.55), (5.4) immediately follows.

A.13. Proof of Theorem 5.1

Letting $z_2 = 1$, $z_1 = 1$ and $z_1 = z_2 = z$ in (5.4), we obtain (5.5), (5.6) and (5.7), respectively.

A.14. Mean Waiting Time of Supercustomers

We consider the mean waiting time of supercustomers $E[W_s]$. From (5.1), we have

(A.56)
$$E[W_s] = \frac{d}{dz}W_s(z)\bigg|_{z=1} = U'(1) - \frac{1}{1-\gamma}A'(1)$$

Using Theorem 3.2, Corollary 4.1 and (A.34), we obtain

(A.57)
$$U'(1) = A'(1) + \frac{A''(1)}{2(1-A'(1))} + C'(1)\sum_{n=0}^{\infty}\frac{\phi'(\delta^{(n)}(1))\delta^{(n)\prime}(1)}{\phi(\delta^{(n)}(1))} + \frac{\gamma}{1-\gamma}A'(1).$$

Thus, we have

(A.58)
$$E[W_s] = \frac{A''(1)}{2(1-A'(1))} + C'(1)\sum_{n=0}^{\infty}\frac{\phi'(\delta^{(n)}(1))\delta^{(n)\prime}(1)}{\phi(\delta^{(n)}(1))}.$$

A.15. Mean Waiting Times of Individual Customers

We consider the mean waiting times of individual customers $E[W_c^{(1)}]$, $E[W_c^{(2)}]$ and $E[W_c]$. From (5.5), it follows that

(A.59)
$$E[W_c^{(1)}] = \frac{d}{dz}W_{c1}(z)\bigg|_{z=1} = \frac{\gamma}{1-\gamma}.$$

From (5.6), we obtain

(A.60)
$$E[W_c^{(2)}] = \frac{d}{dz}W_{c2}(z)\bigg|_{z=1} = \frac{1}{A'(1)}U'(1) - \frac{C''(1)}{2C'(1)} - \frac{1}{1-\gamma},$$

where $U'(1)$ is given in (A.57). Moreover, we have

(A.61)
$$E[W_c] = E[W_c^{(1)}] + E[W_c^{(2)}] = \frac{1}{A'(1)}U'(1) - \frac{C''(1)}{2C'(1)} - 1.$$

References

[1] Borst, S.C., Boxma, O.J. and Combé, M.B.: Collection of customers: a correlated $M/G/1$ queue. *Perfor. Eval. Rev.*, vol.20 (1992) 47–59.

[2] Borst, S.C., Boxma, O.J. and Combé, M.B.: An $M/G/1$ queue with customer collection. *Stochastic Models*, vol.9 (1993) 341–371.

[3] Boxma, O.J. and Groenendijk, W.P.: Waiting times in discrete-time cyclic-service systems. *IEEE Trans. on Commun.*, vol.36 (1988) 164–170.

[4] Boxma,O.J.: Workloads and waiting times in single-server systems with multiple customer classes. *Queueing Sys.*, vol.5 (1989) 185–214.

[5] Boxma, O.J. and Combé, M.B.: The correlated $M/G/1$ queue. *AEÜ*, vol.47 (1993) 330–335.

[6] Doshi, B.T.: Single server queues with vacations. *Stochastic Analysis of Computer and Communication Systems* (ed. H.Takagi), North-Holland, Amsterdam (1990) 217–266.

[7] Fuhrmann, S.W. and Cooper, R.B.: Stochastic decompositions in the $M/G/1$ queue with generalized vacations. *Oper. Res.*, vol.33 (1985) 1117–1129.

[8] Ishizaki, F., Takine, T. and Hasegawa, T.: Analysis of a discrete-time queue with a gate. *Proc. of ITC 14* (ed. J.Labetoulle and J.W.Roberts) Antibe Juan-les-Pins, France, (1994) 169–178.

[9] Kawata, T.: Analysis of a discrete-time queue with dependence between interarrival and service times. Bachelor thesis, Dept. of Appl. Math. and Phys., Kyoto Univ. (1993) (in Japanese).

[10] Miyazawa, M. and Takahashi, Y.: Rate conservation principle for discrete-time queues. *Queueing Sys.*, vol.12 (1992) 215–230.

[11] Sakasegawa, H. and Wolff, R.W.: The equality of the virtual delay and attained waiting time distributions. *Adv. Appl. Prob.*, vol.22 (1990) 257–259.

[12] Sengupta, B.: An invariance relationship for the $G/G/1$ queue. *Adv. Appl. Prob.*, vol.21 (1989) 956–957.

[13] Takahashi, Y.: Queueing systems with gates. *J. Oper. Res. Soc. J.*, vol.14 (1971) 103–126.

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