

## LEGENDRE CONDITIONS FOR A VARIATIONAL PROBLEM WITH ONE-SIDED PHASE CONSTRAINTS

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**Abstract** This paper is concerned with a variational problem with inequality phase constraints. We present second-order necessary conditions (Legendre condition) for weak minimal solutions of the variational problem. Our optimality conditions give an information about the Hessian of the integrand at not only inactive points but also active points. Since the present constraints do not include  $\dot{x}$ , our conditions differ from the Legendre-Clebsch condition.

### 1. Introduction

This paper is concerned with a variational problem with one-sided phase constraints:

$$\begin{aligned} & \text{minimize} && \int_0^1 f(t, x, \dot{x}) dt \\ & \text{subject to} && x(0) = x_0, \quad x(1) = x_1, \quad x \in X_0, \\ (P_0) &&& a(t) \leq x(t) \quad \forall t \in [0, 1], \end{aligned}$$

where  $f(t, x, \dot{x})$  is a continuous function defined on  $R^{1+n+n}$ ,  $X_0$  is the space of all  $n$ -dimensional vector-valued absolutely continuous functions  $x : [0, 1] \rightarrow R^n$  equipped with the norm

$$\|x\|_{X_0} = \max_{t \in [0, 1]} \|x(t)\| + \text{esssup}_{t \in [0, 1]} \|\dot{x}(t)\|, \quad (1.1)$$

where  $\text{esssup}$  denotes the smallest number  $M$  such that  $\|\dot{x}(t)\| \leq M$  for almost everywhere  $[0, 1]$ . The end-points  $x_0$  and  $x_1$  are given points in  $R^n$ ,  $a : [0, 1] \rightarrow R^n$  is a given continuous function and  $a(t) \leq x(t)$  means the component wise inequalities. We assume that  $f(t, x, \dot{x})$  is twice continuously differentiable w.r.t.  $x$  and  $\dot{x}$ . We give second-order necessary conditions (Legendre condition) for weak minimal solutions of  $(P_0)$ . A local solution  $\bar{x}$  in the sense of the norm  $\|\cdot\|_{X_0}$  is said to be weak, and  $\bar{x}$  in the sense of the norm  $\max_{t \in [0, 1]} \|\bar{x}(t)\|$  is said to be strong. We encounter the one-sided phase constraint, for example, production planning and planning mathematics of the employment, see pp. 234, 253 in [18].

In the literature, we can find many second-order necessary optimality conditions (Weierstrass condition, Legendre-Clebsch condition, Legendre condition, maximum principle) for variational problems or optimal control problems with various types of constraints: equality, inequality or set constraints, [1] [2] [4] [5] [6] [7] [8] [9] [10] [12] [13] [18] [19] [20] [21] [22] [23] [25]. All of them except [23] dealt with strong minimal solutions or mixed constraints  $g(t, x, \dot{x}) \leq 0$ , where  $g_x$  is of full rank for the active constraints, which is never satisfied by the present constraint  $a(t) - x(t) \leq 0$ . We note that Páles and Zeidan [23] dealt with optimal control problem where the objective function is a supremum-type function.

Our optimality conditions give an information about  $f_{\bar{x}\bar{x}}$  at not only inactive points ( $a(t) < x(t)$ ) but also active points ( $a_i(t) = x_i(t)$  for some  $i$ ), where  $a_i(t)$  denotes the  $i$ -th component of  $a(t)$ .

## 2 Preliminary results

Let  $X$ ,  $V$  and  $W$  be Banach spaces. Let  $K$  be a closed convex cone of  $V$  with non-empty interior  $\text{int } K$ . We denote by  $V^*$  the topological dual space of  $V$ . The polar cone of  $K$  is defined by  $K^\circ = \{v^* \in V^*; \langle v^*, v \rangle \leq 0 \ \forall v \in K\}$ .

In this section, we consider the following abstract optimization problem with generalized equality and inequality constraints:

$$(P) \quad \begin{aligned} &\text{minimize } F(x) \\ &\text{subject to } G(x) \in K, \quad H(x) = 0. \end{aligned}$$

where  $F : X \rightarrow R$ ,  $G : X \rightarrow V$  and  $H : X \rightarrow W$  are twice continuously Fréchet differentiable mappings. We denote by  $F'(x)$  and  $F''(x)$  the first and second Fréchet differential of  $F(x)$ , respectively.

Now, let  $\bar{x}$  be a weak minimal solution of (P). A direction  $y \in X$  is said to be critical at  $\bar{x}$  if

$$F'(\bar{x})y = 0, \quad G'(\bar{x})y \in \text{clcone}(K - G(\bar{x})), \quad H'(\bar{x})y = 0,$$

where  $\text{clcone } A$  denotes the closure of the conical hull of  $A$ .

A feasible solution  $\bar{x}$  is said to satisfy the Mangasarian-Fromovitz condition (regular) if

$$(i) \quad H'(\bar{x}) : X \rightarrow W \text{ is onto}$$

$$(ii) \quad \exists z \in X \text{ s. t. } H'(\bar{x})z = 0, \quad G(\bar{x}) + G'(\bar{x})z \in \text{int } K.$$

The following theorem can be found in e. g. Ben-Tal and Zowe [3], Kawasaki [14].

**Theorem 2.1** *Let  $\bar{x}$  be a regular minimal solution of (P). Then, for each critical direction  $y$  at  $\bar{x}$  satisfying*

$$G'(\bar{x})y \in \text{cone}(K - G(\bar{x})), \tag{2.1}$$

*there exist  $v^* \in K^\circ$  and  $w^* \in W^*$  such that*

$$L'(\bar{x}) = 0, \tag{2.2}$$

$$L''(\bar{x})(y, y) \geq 0, \tag{2.3}$$

$$\langle v^*, G(\bar{x}) \rangle = 0, \quad \langle v^*, G'(\bar{x})y \rangle = 0, \tag{2.4}$$

*where*

$$L(x) = F(x) + \langle v^*, G(x) \rangle + \langle w^*, H(x) \rangle. \tag{2.5}$$

## 3 Main theorems

Let  $\bar{x}(t)$  be a weak minimal solution satisfying  $a(0) < x_0$  and  $a(1) < x_1$  for  $(P_0)$ . We use the following abbreviated notation:

$$\bar{f}(t) = f(t, \bar{x}(t), \dot{\bar{x}}(t)), \quad \bar{f}_{\bar{x}}(t, \bar{x}(t), \dot{\bar{x}}(t)), \quad \text{etc.}$$

All vectors except gradient vectors are column vectors. For each  $a \in R^n$ , we denote  $a^T$  the transpose of  $a$  and by  $a_i$  denotes the  $i$ -th component of  $a$ . For each  $t \in [0, 1]$ , we define

$$J_R(t) = \{i; \exists \delta > 0 \text{ s.t. } \bar{x}_i > a_i \text{ on } (t, t + \delta)\},$$

$$J_L(t) = \{i; \exists \delta > 0 \text{ s.t. } \bar{x}_i > a_i \text{ on } (t - \delta, t)\}.$$

A function  $\bar{x}$  is said to be piecewise smooth if  $[0, 1]$  is divided into a finite number of subintervals and the function has continuous derivative on each subintervals.

**Theorem 3.1** *Let  $\bar{x}(t)$  be a piecewise smooth weak minimal solution satisfying  $a(0) < \bar{x}(0)$  and  $a(1) < \bar{x}(1)$  for  $(P_0)$ . Put*

$$\lambda(t) = \int_0^t \bar{f}_x(\tau)^T d\tau - \bar{f}_x(t)^T. \quad (3.1)$$

Then

- (i)  $\lambda_i(t)$  is nondecreasing and increases only on  $\{t; a_i(t) = \bar{x}_i(t)\}$ ,
- (ii)  $\xi^T \bar{f}_{\bar{x}\bar{x}}(t-0)\xi \geq 0 \quad \forall \xi \text{ satisfying } \xi_i = 0 \text{ for } i \notin J_L(t)$ ,
- (iii)  $\xi^T \bar{f}_{\bar{x}\bar{x}}(t+0)\xi \geq 0 \quad \forall \xi \text{ satisfying } \xi_i = 0 \text{ for } i \notin J_R(t)$ .

We can find (i) in [18]. This condition can be regarded as an inequality version of the Euler-Lagrange equation. So we call it the Euler-Lagrange condition for the one-sided phase constraints.

For the sake of better understanding, let us consider the case of  $n = 1$ . Then (ii) and (iii) amount, respectively, to

$$\bar{f}_{\bar{x}\bar{x}}(t-0) \geq 0 \text{ if } \exists \delta > 0, ; x > a \text{ on } (t - \delta, t), \quad (3.2)$$

$$\bar{f}_{\bar{x}\bar{x}}(t+0) \geq 0 \text{ if } \exists \delta > 0, ; x > a \text{ on } (t, t + \delta). \quad (3.3)$$

Fig. 3.1 is a standard picture of  $\bar{x}(t)$ . In Fig. 3.1, the above conditions assert that  $\bar{f}_{\bar{x}\bar{x}} \geq 0$  on  $[0, t_2] \cup [t_3, 1]$ . The following theorem gives an information about  $\bar{f}_{\bar{x}\bar{x}}$  on  $[t_2, t_3]$ .

**Theorem 3.2** *Let  $\bar{x}(t)$  be a piecewise smooth weak minimal solution satisfying  $a(0) < x_0$  and  $a(1) < x_1$  for  $(P_0)$ . Let  $E_L(t)$  denote the set of all indices  $i \notin J_L(t)$  for which the Euler equation w. r. t.  $x_i$*

$$\frac{d}{dt} \bar{f}_{\bar{x}_i} = \bar{f}_{x_i} \quad (3.4)$$

*holds a. e. on  $(t - \delta, t]$  for some  $\delta > 0$ . Then we may replace, in (ii) of the previous theorem,  $\xi_i = 0$  by*

$$\xi_i \geq 0 \text{ for } i \in E_L(t), \quad \xi_i = 0 \text{ for } i \in J_L(t) \cup E_L(t).$$

*Similarly, let  $E_R(t)$  denote the set of all indices  $i \notin J_R(t)$  for which the Euler equation w. r. t.  $x_i$  holds a. e. on  $[t, t + \delta)$  for some  $\delta > 0$ , then we may replace, in (iii) of the previous theorem,  $\xi_i = 0$  by  $\xi_i \geq 0$  for  $i \in E_R(t)$ ,  $\xi_i = 0$  for  $i \in J_R(t) \cup E_R(t)$ .*

It is clear that we may replace  $[0, 1]$  with any bounded closed interval  $[c, d]$  in the above theory.

**Example 3.1** Let us consider the following problem on  $[0, 2]$ :

$$\begin{aligned} & \text{minimize } \int_0^2 \{4x - \dot{x}^2\} dt \\ & \text{subject to } x(0) = x(2) = 1, \\ & \quad x(t) \geq 0 \quad \forall t \in [0, 2]. \end{aligned}$$

Take

$$\bar{x}(t) = \begin{cases} -t^2 - t + 1 & \text{on } [0, \frac{\sqrt{5}-1}{2}], \\ 0 & \text{on } [\frac{\sqrt{5}-1}{2}, \frac{5-\sqrt{5}}{2}], \\ -t^2 + 5t - 5 & \text{on } [\frac{5-\sqrt{5}}{2}, 2], \end{cases}$$

see Fig 3. 2. Then  $\lambda(t)$  defined by (3.1) is

$$\lambda(t) = \begin{cases} -2 & \text{on } [0, \frac{\sqrt{5}-1}{2}], \\ 4t & \text{on } [\frac{\sqrt{5}-1}{2}, \frac{5-\sqrt{5}}{2}], \\ 10 & \text{on } [\frac{5-\sqrt{5}}{2}, 2], \end{cases}$$

which is nondecreasing, see Fig 3.3. Hence  $\bar{x}(t)$  satisfies the Euler-Lagrange condition. But it follows from Theorem 3.1 that  $\bar{x}(t)$  is not a weak minimal solution, since  $\bar{f}_{\bar{x}\bar{x}}(t) \equiv -2 < 0$ .

Theorem 3.1 can not exclude the following non-minimal solution.

**Example 3.2** Let us consider the following problem on  $[-2, 2]$ .

$$\begin{aligned} & \text{minimize } \int_{-2}^2 (t^2 - 1)\dot{x}^2 dt \\ & \text{subject to } x(-2) = x(2) = 1, \quad a(t) \leq x(t) \quad \forall t \in [-2, 2], \end{aligned}$$

where  $a(t)$  is given by Fig. 3.4. Take  $\bar{x}(t) \equiv 1$ . Then  $\lambda(t)$  defined by (3.1) is identically zero. Thus  $\bar{x}(t)$  satisfies the Euler equation (3.4). Moreover,  $\bar{x}(t)$  satisfies all conditions of Theorem 3.1. But it follows from Theorem 3.2 that  $\bar{x}(t)$  is not a weak minimal solution, since  $\bar{f}_{\bar{x}\bar{x}}(t) = 2(t^2 - 1) < 0$  on  $(-1, 1)$ .

#### 4 Proofs of the theorems

Define  $F : X_0 \rightarrow R$ ,  $G : X_0 \rightarrow (C[0, 1])^n$ ,  $H : X_0 \rightarrow R^{2n}$  and  $K \subset (C[0, 1])^n$  by

$$F(x) = \int_0^1 f(t, x, \dot{x}) dt,$$

$$G(x) = a - x,$$

$$H(x) = (x(0) - x_0, x(1) - x_1),$$

$$K = \{v \in (C[0, 1])^n; v(t) \leq 0 \forall t \in [0, 1]\}.$$

Then the problem  $(P_0)$  is expressed as (P). Furthermore,  $F$ ,  $G$  and  $H$  are twice continuously Fréchet differentiable and their first and second Fréchet differentials are given by

$$F'(\bar{x})y = \int_0^1 \{\bar{f}_x y + \bar{f}_{\dot{x}} \dot{y}\} dt, \quad (4.1)$$

$$F''(\bar{x})(y, y) = \int_0^1 \{y^T \bar{f}_{xx} y + 2y^T \bar{f}_{x\dot{x}} \dot{y} + \dot{y}^T \bar{f}_{\dot{x}\dot{x}} \dot{y}\} dt, \quad (4.2)$$

$$G'(\bar{x})y = -y, \quad G''(\bar{x})(y, y) = 0, \quad (4.3)$$

$$H'(\bar{x})y = (y(0), y(1)), \quad H''(\bar{x})(y, y) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad (4.4)$$

see e. g. Girsanov [11] and Ioffe and Tihomirov [13]. It is easily seen that the Mangasarian-Fromovitz condition is satisfied at  $\bar{x}$  if  $a(0) < x_0$  and  $a(1) < x_1$ . Hence we may apply Theorem 2.1 to  $(P_0)$ . By Riesz's representation theorem, the Lagrange function  $L(x)$  is represented as

$$L(x) = \int_0^1 f dt + \int_0^1 d\lambda^T(a - x) + \sum_{k=0}^1 \mu_k^T(x(k) - x_k), \quad (4.5)$$

where  $\mu_0, \mu_1 \in R^n$  and  $\lambda : [0, 1] \rightarrow R^n$  is a component wise nondecreasing function and increases only on  $\{t; a_i(t) = \bar{x}_i(t)\}$ , see e. g. Rudin [24], Luenberger [18]. It follows from (2.2), (4.1), (4.3) and (4.4) that

$$\int_0^1 \{\bar{f}_x y + \bar{f}_{\dot{x}} \dot{y}\} dt - \int_0^1 d\lambda^T y + \sum_{k=0}^1 \mu_k^T y(k) = 0$$

for all  $y \in X$ . By integration by parts, we get

$$\int_0^1 \{\bar{f}_{\dot{x}} - \int_0^t \bar{f}_x d\tau + \int_0^t d\lambda^T\} \dot{y} dt + \{\int_0^1 \bar{f}_x d\tau - \int_0^1 d\lambda^T + \mu_1\} y(1) = 0$$

for all  $y$  with  $y(0) = 0$ ; see [18], [13]. Hence we have

$$\bar{f}_{\dot{x}}(t) - \int_0^t \bar{f}_x(\tau) d\tau + \lambda^T(t) = \text{constant} = c^T \quad a.e. \quad (4.6)$$

Now, let  $\tau$  be an arbitrary point of  $(0, 1]$ . For any  $s < \tau$  sufficiently close to  $\tau$  and sufficiently small  $\sigma > 0$ , put

$$h_\sigma(t) = \begin{cases} 2\sqrt{\sigma}^{-1}(t - s + \sigma) & \text{on } [s - \sigma, s - \frac{\sigma}{2}] \\ -2\sqrt{\sigma}^{-1}(t - s) & \text{on } [s - \frac{\sigma}{2}, s] \\ 0 & \text{otherwise,} \end{cases} \quad (4.7)$$

and

$$y_\sigma(t) = h_\sigma(t)\xi, \quad (4.8)$$

where  $\xi \in R^n$  is an arbitrary vector which satisfies

$$\xi_i = 0 \quad \forall i \notin J_L(\tau). \quad (4.9)$$

Here we note that

$$J_L(\tau) \subset J_L(s). \quad (4.10)$$

Then we see that

$$G'(\bar{x})y_\sigma = -y_\sigma \in \text{cone}(K - G(\bar{x})). \quad (4.11)$$

Indeed, (4.11) is equivalent to

$$\exists \alpha > 0 \quad s.t. \quad \alpha y_\sigma(t) - a(t) + \bar{x}(t) \geq 0 \quad \forall t \in [0, 1]. \quad (4.12)$$

For any  $i \notin J_L(\tau)$ , since  $y_{\sigma i}(t) \equiv 0$ , the  $i$ -th inequality of (4.12) holds. For any  $i \in J_L(\tau)$ , since  $|y_{\sigma i}(t)| \leq \sqrt{\sigma} |\xi_i|$  and  $a_i(t) < \bar{x}_i(t)$  on  $[s - \sigma, s]$ , the  $i$ -th inequality of (4.12) holds. Thus we get (4.11).

Moreover, since  $y_{\sigma}(0) = y_{\sigma}(1) = 0$ , we have

$$\begin{aligned} F'(\bar{x})y_{\sigma} &= \int_0^1 (\bar{f}_x y_{\sigma} + \bar{f}_{\dot{x}} \dot{y}_{\sigma}) dt = \int_0^1 (\bar{f}_{\dot{x}} - \int_0^t \bar{f}_{xx} d\tau) \dot{y}_{\sigma} dt \\ &= \int_0^1 (c - \lambda)^T \dot{y}_{\sigma} dt = \int_0^1 d\lambda^T y_{\sigma} = 0, \end{aligned}$$

where the last equality follows from the complementary condition (2.4). Therefore  $y_{\sigma}$  is a critical direction. Hence, by Theorem 2.1, we have

$$\begin{aligned} 0 &\leq L''(\bar{x})(y_{\sigma}, y_{\sigma}) \\ &= \int_{s-\sigma}^s (\xi^T \bar{f}_{xx} \xi h_{\sigma}^2 + 2\xi^T \bar{f}_{x\dot{x}} \xi h_{\sigma} \dot{h}_{\sigma} + \xi^T \bar{f}_{\dot{x}\dot{x}} \xi h_{\sigma}^2) d\tau. \end{aligned} \quad (4.13)$$

Putting  $M = \max\{|\xi^T \bar{f}_{xx}(t)\xi|; t \in [s - \sigma, s]\}$ , we have  $|\int_{s-\sigma}^s \xi^T \bar{f}_{xx} \xi h_{\sigma}^2 dt| \leq \int_{s-\sigma}^s M \sigma dt = O(\sigma^2)$ . Similarly the second term of (4.13) is  $O(\sigma)$ . By mean-value theorem, the third term is equal to

$$4\sigma^{-1} \int_{s-\sigma}^s \xi^T \bar{f}_{\dot{x}\dot{x}} \xi dt = 4\xi^T \bar{f}_{\dot{x}\dot{x}}(t_{\sigma})\xi$$

for some  $t_{\sigma} \in (s - \sigma, s)$ . Hence, from (4.13),

$$0 \leq O(\sigma^2) + O(\sigma) + 4\xi^T \bar{f}_{\dot{x}\dot{x}}(t_{\sigma})\xi.$$

Since  $t_{\sigma} \rightarrow s$  as  $\sigma \rightarrow 0$ , we get

$$\xi^T \bar{f}_{\dot{x}\dot{x}}(s)\xi \geq 0.$$

Taking  $s \rightarrow \tau$ , we have

$$\xi^T \bar{f}_{\dot{x}\dot{x}}(\tau - 0)\xi \geq 0.$$

The assertion (iii) is similarly obtained. This completes the proof of Theorem 3.1.

Next, we prove Theorem 3.2. We define  $y_{\sigma}$  by (4.8), where  $\xi \in R^n$  is an arbitrary vector which satisfies

$$\begin{cases} \xi_i \geq 0 & \forall i \in E_L(\tau) \\ \xi_i = 0 & \forall i \in J_L(\tau) \cup E_L(\tau). \end{cases} \quad (4.14)$$

Then we see that

$$-y_{\sigma} \in \text{cone}(K - G(\bar{x})). \quad (4.15)$$

Indeed, for  $i \in E_L(\tau)$ , the  $i$ -th component of the left-hand sided of (4.12) is

$$\alpha y_{\sigma i}(t) - a_i(t) + \bar{x}_i(t) \geq \bar{x}_i(t) - a_i(t) \geq 0 \quad \forall t.$$

So the  $i$ -th inequality holds. It follows from the Euler equation (3.4) that

$$F'(\bar{x})y_{\sigma} = \int_0^1 \left\{ \bar{f}_{\dot{x}} - \int_0^t \bar{f}_{xx} d\tau \right\} \dot{y}_{\sigma} dt = 0.$$

Thus  $y_{\sigma}$  is critical. The rest is as in the previous proof. This completes the proof of Theorem 3.2.

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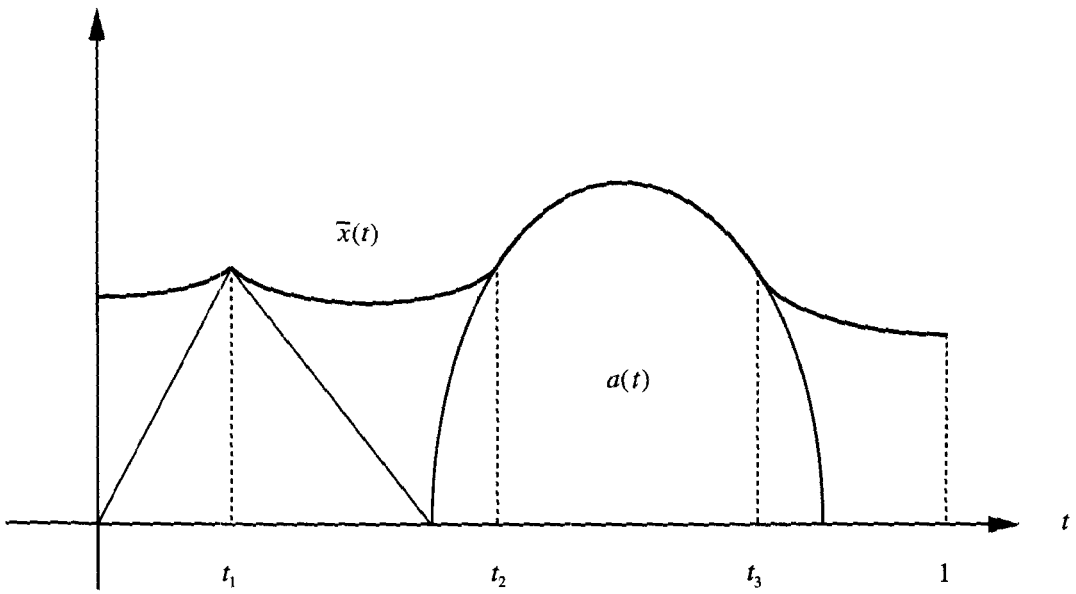


Figure 3.1

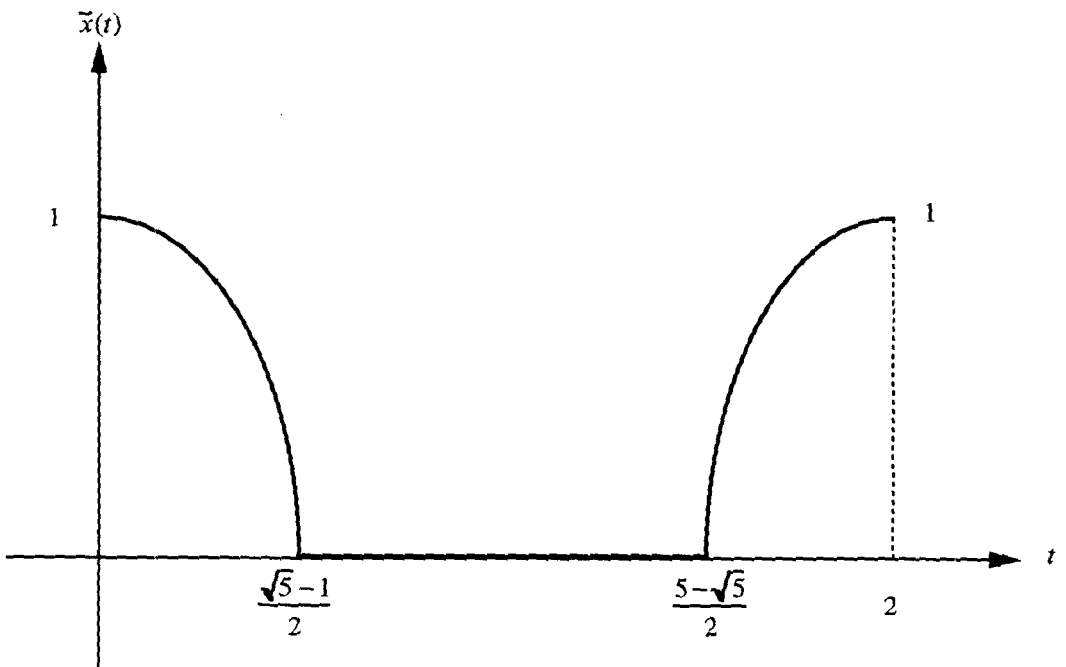


Figure 3.2

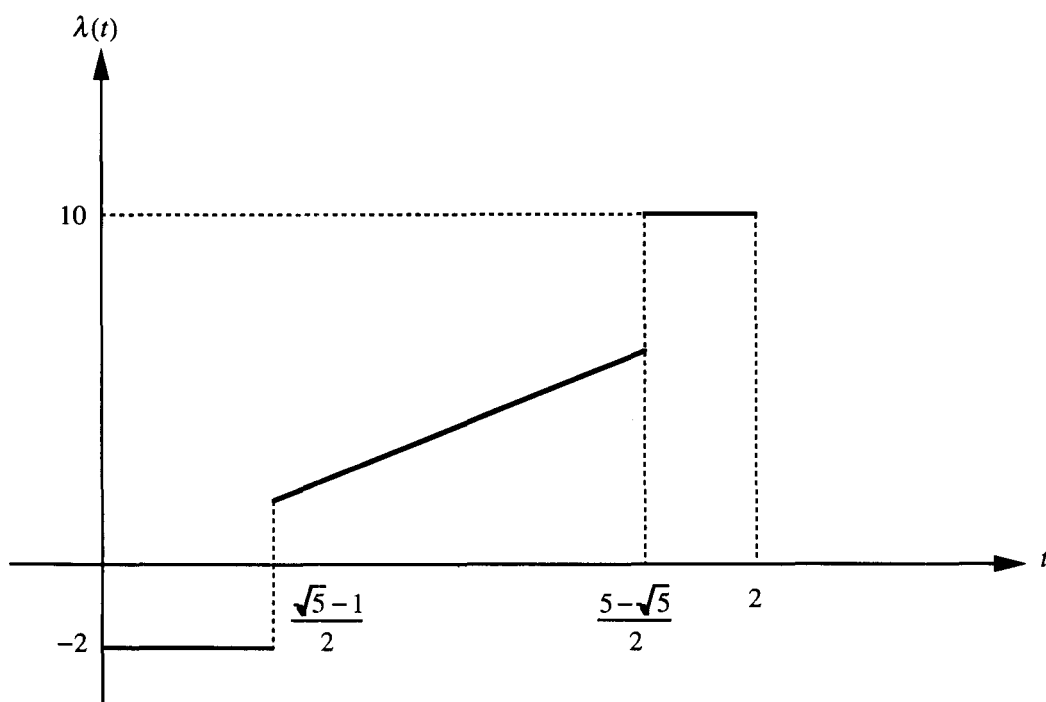


Figure 3.3

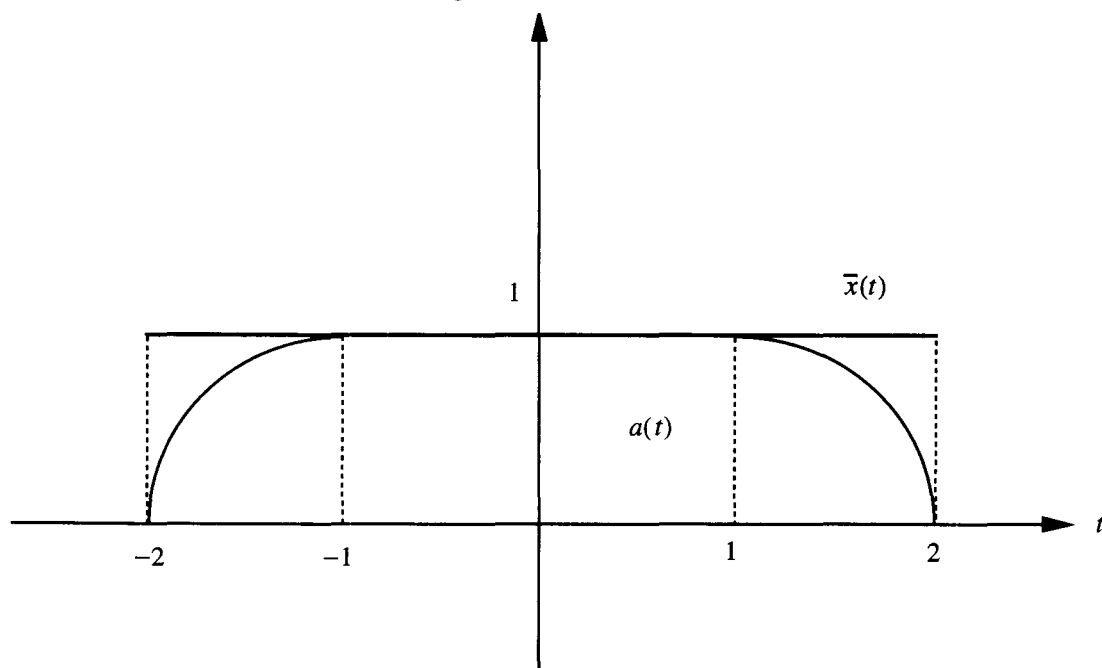


Figure 3.4