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TIME-DEPENDENT DISTRIBUTION OF THE WORKLOAD IN $M/G/1$ QUEUES WITH VACATIONS

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Abstract This paper studies a large class of $M/G/1$ queues with vacations. By means of a probabilistic interpretation we obtain a functional equation which gives a unified approach to derive time-dependent results for the workload distribution. We also discuss application of our result to a number vacation models.

1. Introduction

Recently, analysis of the workload for single-server queues with vacations of the server is a subject of interest. Analysis of the stationary workload has been conducted, among others, by Boxma [2], Boxma and Groenendijk [3] and Takagi et al. [18]. An interesting result in this direction is the so-called decomposition property for the workload established by Boxma [2] for an $M/G/1$ system with generalized vacations. Here the service of a customer can be preemptive. However, the preemption process should not affect the amount of service time given to a customer or the arrival time of any customer.

Besides the stationary analysis, there are few treatments of the time-dependent distribution of the workload conducted by Keilson and Ramaswamy [10], Takagi [17] and Takine and Hasegawa [19]. However, most of the known derivations of time-dependent results for the workload are not quite satisfactory because they involve very lengthy and complicated calculations. Moreover, those derivations are restricted to special vacation models.

In the present paper, we give a unified approach to the study of the time-dependent distribution of the workload for a general class of $M/G/1$ queues with vacations. Using the method of collective marks we derive an expression for the Laplace-Stieltjes transform of the workload at time t. The new formula allows us to rederive and unify some existing results. It also allows us to study the limiting behaviour of the workload process when the traffic intensity is equal to or greater than one. For information about the method of collective marks and the use of it in queueing theory we refer the reader to Cong [7] and the references therein. The present paper can be read without any special knowledge of this method.

2. The main results

Following Boxma [2] we consider a modified $M/G/1$ queueing system which satisfies the following assumptions.

Assumption 1. Customers arrive at the system according to a stationary Poisson process. The service times of customers are i.i.d. non-negative random variables. The sequence of interarrival times and the sequence of service times are independent stochastic processes.
Assumption 2. The service discipline does not affect the amount of service time given to a customer or the arrival time of any customer.
Assumption 3. The state of the server can be {*serving*} or {*non-serving*}. A non-serving state can also be either *free* or *interrupted* (i.e. the server is not serving when there is at least one customer present in the system). The interruption process does not affect the amount of service time given to a customer or the arrival time of any customer.

The above assumptions are slightly different from Assumption 2.1 of Boxma [2]. Here we do not require the existence of the equilibrium distribution of the workload process.

Throughout this paper λ is the arrival rate and S is the random service time with LSt (Laplace-Stieltjes transform) $\beta(\xi)$. The traffic intensity $\rho = \lambda E[S]$ is finite. The state of the server at time t is denoted by $I(t)$, where $I(t) = 1$ if at time t the server is serving and $I(t) = 0$ otherwise.

We obtain the following time-dependent result for the workload.
Theorem 1. The LSt of $U(t)$, for $t \geq 0$, satisfies the equation

$$E[\mathrm{e}^{-\xi U(t)}] = \mathrm{e}^{\phi(\xi)t}\left[E[\mathrm{e}^{-\xi U(0)}] - \xi \int_0^t \mathrm{e}^{-\phi(\xi)s} E[\mathrm{e}^{-\xi U(s)} \mathbf{1}_{\{I(s)=0\}}]ds\right], \qquad (2.1)$$

where ξ is a complex number with $\Re\xi \geq 0$, $\mathbf{1}_{\mathrm{A}}$ is the indicator function of the set A and

$$\phi(\xi) \stackrel{\text{def}}{=} \xi - \lambda + \lambda\beta(\xi). \qquad (2.2)$$

Proof. It suffices to prove (2.1) for $\xi > 0$ because if this is done, then by analytic continuation (2.1) holds for complex ξ with $\Re\xi \geq 0$. Let us consider an additional Poisson process with parameter ξ producing *catastrophes*. This Poisson process does not depend on the original queueing process. We introduce the following random events which depend on t

A $\stackrel{\text{def}}{=}$ a catastrophe does not occur either during the remaining service time of the customers who are present at time 0 or during the service time of the customers who arrive in the time interval $(0, t)$,

B $\stackrel{\text{def}}{=}$ the first catastrophe occurs after time t and a catastrophe does not occur during the remaining service time of the customers who are present at time t,

C $\stackrel{\text{def}}{=}$ the first catastrophe occurs at a time $s < t$ when the server is not serving and after that a catastrophe does not occur either during the remaining service time of the customers who are present at time s or during the service time of the customers who arrive in (s, t).

It is clear that the event A is the union of the disjoint events B and C. Therefore

$$P(\mathrm{A}) = P(\mathrm{B}) + P(\mathrm{C}). \qquad (2.3)$$

The probability that a catastrophe does not occur during the service time of the customers who arrive in the time interval $(0,t)$ is

$$\sum_{n=0}^{\infty} \mathrm{e}^{-\lambda t}\frac{(\lambda t)^n}{n!}\beta(\xi)^n = \mathrm{e}^{-\lambda(1-\beta(\xi))t}, \tag{2.4}$$

because during a service time with probability $\beta(\xi)$ no catastrophe occurs. The probability that a catastrophe does not occur during the remaining service time of the customers who are present at time s is nothing but $E[\mathrm{e}^{-\xi U(s)}]$. Hence

$$P(\mathrm{A}) = \mathrm{e}^{-\lambda(1-\beta(\xi))t}E[\mathrm{e}^{-\xi U(0)}]. \tag{2.5}$$

For the event B we have

$$P(\mathrm{B}) = \mathrm{e}^{-\xi t}E[\mathrm{e}^{-\xi U(t)}]. \tag{2.6}$$

The probability that the server is not serving at time s and no catastrophe occurs during the remaining service time of the customers who are present at that time is $E[\mathrm{e}^{-\xi U(s)}\mathbf{1}_{\{I(s)=0\}}]$, while the event no catastrophe occurs during the service time of the customers who arrive in the time interval (s,t) has probability $\mathrm{e}^{-\lambda(1-\beta(\xi))(t-s)}$. Therefore

$$P(\mathrm{C}) = \int_0^t E[\mathrm{e}^{-\xi U(s)}\mathbf{1}_{\{I(s)=0\}}]\mathrm{e}^{-\lambda(1-\beta(\xi))(t-s)}\xi\mathrm{e}^{-\xi s}ds. \tag{2.7}$$

Equation (2.1) now follows from (2.3), (2.5)-(2.7) and simple algebra. □

Corollary 1. We have for $t \geq 0$

$$E[U(t)] = E[U(0)] + \int_0^t P(I(s)=0)ds - (1-\rho)t. \tag{2.8}$$

Proof. Equation (2.8) follows either from (2.1) by differentiating in ξ and then letting $\xi \downarrow 0$ or from the following observation

$$U(0) + S(t) + \int_0^t \mathbf{1}_{\{I(s)=0\}}ds = U(t) + t,$$

where $S(t)$ is the sum of the service times of all customers who arrive in the interval $(0,t)$. Taking expectations on both sides of the last equation results in

$$E[U(0)] + \rho t + \int_0^t P(I(s)=0)ds = E[U(t)] + t,$$

so that (2.8) follows. □

Corollary 2. Let $U^*(\xi,\omega) = \int_0^\infty \mathrm{e}^{-\xi t}E[\mathrm{e}^{-\omega U(t)}]dt$. Then

$$U^*(\xi,\omega) = \frac{E[\mathrm{e}^{-\omega U(0)}] - \omega U_0^*(\xi,\omega)}{\xi - \omega + \lambda - \lambda\beta(\omega)}, \tag{2.9}$$

where

$$U_0^*(\xi,\omega) \stackrel{\text{def}}{=} \int_0^\infty e^{-\xi t} E[e^{-\omega U(t)} \mathbf{1}_{\{I(t)=0\}}]dt. \tag{2.10}$$

Proof. Formula (2.9) immediately follows from (2.1) by taking Laplace transforms on both sides. □

Corollary 3. Let $\rho < 1$. Assume that $(I(t), U(t))$ converges in distribution as $t \to \infty$. Let (I, U) be a random vector with as joint distribution the joint limiting distribution of $(I(t), U(t))$. We have the following decomposition result

$$E[e^{-\omega U}] = \frac{(1-\rho)\omega}{\omega - \lambda + \lambda\beta(\omega)} E[e^{-\omega U} | I = 0]. \tag{2.11}$$

Proof. We obtain from (2.9) and (2.10) by applying an Abelian theorem

$$\begin{aligned} E[e^{-\omega U}] &= \lim_{\xi \downarrow 0} \xi U^*(\xi,\omega) \\ &= \frac{\omega}{\omega - \lambda + \lambda\beta(\omega)} E[e^{-\omega U} \mathbf{1}_{\{I=0\}}] = \frac{P(I=0)\omega}{\omega - \lambda + \lambda\beta(\omega)} E[e^{-\omega U} | I = 0]. \end{aligned}$$

Letting $\omega \downarrow 0$ we have $P(I = 0) = 1 - \rho$. □

Some comments about the above results may be helpful.

Remark 1. If we restrict to the standard $M/G/1$ queue, i.e. there is no interruption at all, then (2.1) simplifies to

$$E[e^{-\xi U(t)}] = e^{\phi(\xi)t} \left[E[e^{-\xi U(0)}] - \xi \int_0^t e^{-\phi(\xi)s} p(s) ds \right],$$

where $p(t) = P(\text{server is idle at time } t)$. This is the well-known Takács equation for the workload or the virtual waiting time under the FIFO discipline (see Takács [14], p. 51). This equation was first derived by means of an integro-differential equation (see Takács [13] and Remark 2 below). A derivation by means of the method of collective marks was given by Runnenburg [12]. In the proof of (2.1) we follow the line of reasoning used in [6] and [12].

Remark 2. Equation (2.1) can be obtained by means of an integro-differential equation. Let $F(t,x) = P(U(t) \le x)$, $F_0(t,x) = P(U(t) \le x, I(t) = 0)$ and $B(x) = P(S \le x)$. A continuity argument as in Takács [13], [14] requiring differentiability of $F(t,x)$ and $F_0(t,x)$ gives

$$\frac{\partial}{\partial t} F(t,x) = \frac{\partial}{\partial x} F(t,x) - \frac{\partial}{\partial x} F_0(t,x) - \lambda \left[F(t,x) - \int_{[0,x)} B(x-y) d_y F(t,y) \right].$$

Laplace transformation with respect to x results in

$$\frac{\partial}{\partial t} E[e^{-\xi U(t)}] = \phi(\xi) E[e^{-\xi U(t)}] - \xi E[e^{-\xi U(t)} \mathbf{1}_{\{I(t)=0\}}],$$

so that (2.1) follows. This we find a more complicated way of finding (2.1). □

Remark 3. Let $C(t)$ be the cumulative time spent in the non-serving state up to time t, i.e. $C(t) = \int_0^t [1 - I(s)]ds$. Then the Laplace-Stieltjes transform for the joint distribution of $U(t)$ and $C(t)$ satisfies the following equation

$$E[\mathrm{e}^{-\xi U(t)-\omega C(t)}] = \mathrm{e}^{\phi(\xi)t}\left[E[\mathrm{e}^{-\xi U(0)}] - (\xi+\omega)\int_0^t \mathrm{e}^{-\phi(\xi)s} E[\mathrm{e}^{-\xi U(s)-\omega C(s)}\mathbf{1}_{\{I(s)=0\}}]ds\right].$$

This equation can be derived by means of the method of collective marks. In this case we need two different kinds of catastrophes, C_ξ and C_ω, which occur according to independent Poisson processes with parameters ξ and ω, respectively. These extra Poisson processes and the original queueing process are also independent. Without loss of generality we can assume that $U(0) = 0$. Let $S(t)$ be the sum of the service times of all customers who arrive in the time interval $(0, t)$. We introduce the following random events
A $\stackrel{\text{def}}{=}$ no C_ξ occurs during the service time of the customers who arrive in $(0, t)$,
B $\stackrel{\text{def}}{=}$ no C_ξ occurs during $[0, t) \cap \{s \geq 0: I(s) = 0\}$,
C $\stackrel{\text{def}}{=}$ no C_ω occurs during $[0, t) \cap \{s \geq 0: I(s) = 0\}$.
Clearly

$$P(\mathrm{A}) = P(\mathrm{A} \cap \mathrm{B} \cap \mathrm{C}) + P(\mathrm{A} \cap \overline{\mathrm{B} \cap \mathrm{C}}), \tag{2.12}$$

where $\overline{\mathrm{B} \cap \mathrm{C}}$ is the complement of the event $\mathrm{B} \cap \mathrm{C}$. We have

$$P(\mathrm{A}) = E[\mathrm{e}^{-\xi S(t)}] = \mathrm{e}^{-\lambda(1-\beta(\xi))t} \tag{2.13}$$

and

$$P(\mathrm{A} \cap \mathrm{B} \cap \mathrm{C}) = E[\mathrm{e}^{-\xi S(t)-(\xi+\omega)C(t)}] = E[\mathrm{e}^{-\xi(t+U(t))-\omega C(t)}], \tag{2.14}$$

where the last equality holds because $S(t) + C(t) = t + U(t)$. Since the event $\overline{\mathrm{B} \cap \mathrm{C}}$ occurs if and only if a catastrophe (C_ξ or C_ω) occurs during $[0, t) \cap \{s \geq 0: I(s) = 0\}$, we have

$$\begin{aligned} P(\mathrm{A} \cap \overline{\mathrm{B} \cap \mathrm{C}}) &= \int_0^t E[\mathrm{e}^{-\xi S(t)-(\xi+\omega)C(s)}\mathbf{1}_{\{I(s)=0\}}](\xi+\omega)ds \\ &= (\xi+\omega)\int_0^t E[\mathrm{e}^{-\xi(S(t)-S(s))-\xi(s+U(s))-\omega C(s)}\mathbf{1}_{\{I(s)=0\}}]ds \\ &= (\xi+\omega)\int_0^t E[\mathrm{e}^{-\xi U(s)-\omega C(s)}\mathbf{1}_{\{I(s)=0\}}]\mathrm{e}^{-\lambda(1-\beta(\xi))(t-s)}\mathrm{e}^{-\xi s}ds. \end{aligned} \tag{2.15}$$

The relations (2.12)-(2.15) and simple algebra give us the desired result. □

If $\rho \geq 1$, then $U(t)$ does not converge in distribution as $t \to \infty$. In that case, however, it is interesting to know the behaviour of the workload process $U(t)$ for large t. We have the following theorem.

Theorem 2. (a) If $\rho = 1$, $c = \lambda E[S^2] < \infty$ and for every $\omega > 0$ the limit

$$l(\omega) = \lim_{\xi \downarrow 0} \sqrt{\xi}\, U_0^*(\xi, \omega\sqrt{\xi}) \tag{2.16}$$

exists and is independent of ω, then

$$\frac{1}{\sqrt{ct}}U(t) \xrightarrow{\mathrm{d}} |Y| \quad \text{as } t \to \infty, \tag{2.17}$$

where Y has the standard normal distribution.
(b) If $\rho > 1$ and

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t P(I(s)=0)ds = 0, \tag{2.18}$$

then $U(t)/t \xrightarrow{\mathrm{d}} \rho - 1$.
Proof. We first show that if $l(\omega)$ is independent of ω, then $l(\omega) = \sqrt{c/2}$. Let $\delta(\xi)$ be the LSt of the length of a busy period in the standard $M/G/1$ queue. Note that $\delta(\xi)$ satisfies the well-known equation (see Takács [14], p. 58)

$$\delta(\xi) = \beta(\xi + \lambda - \lambda\delta(\xi)). \tag{2.19}$$

Substituting $\xi + \lambda - \lambda\delta(\xi)$ for ξ in (2.1), then multiplying both sides by $\mathrm{e}^{-\xi t}$ and finally letting $t \to \infty$ we have for $\xi > 0$

$$E[\mathrm{e}^{-(\xi+\lambda-\lambda\delta(\xi))U(0)}] = \{\xi + \lambda - \lambda\delta(\xi)\}\int_0^\infty \mathrm{e}^{-\xi s}E[\mathrm{e}^{-(\xi+\lambda-\lambda\delta(\xi))U(s)}\mathbf{1}_{\{I(s)=0\}}]ds.$$

It follows that

$$U_0^*(\xi, \xi+\lambda-\lambda\delta(\xi)) = \frac{E[\mathrm{e}^{-(\xi+\lambda-\lambda\delta(\xi))U(0)}]}{\xi+\lambda-\lambda\delta(\xi)}. \tag{2.20}$$

If $\rho = 1$ and $E[S^2] < \infty$, then using (2.19) it can be shown that for $0 < \epsilon < \omega_0 = \sqrt{2/c}$

$$(\omega_0 - \epsilon)\sqrt{\xi} \le \xi + \lambda - \lambda\delta(\xi) \le (\omega_0 + \epsilon)\sqrt{\xi} \tag{2.21}$$

for sufficiently small ξ. This together with (2.20) implies

$$l(\omega_0 + \epsilon) \le \frac{1}{\omega_0} \le l(\omega_0 - \epsilon).$$

Hence $l(\omega) = \sqrt{c/2}$ if $l(\omega)$ is independent of ω.
Substituting $\xi/\sqrt{ct}$ for ξ in (2.1) we obtain

$$E[\mathrm{e}^{-\xi U(t)/\sqrt{ct}}] = \mathrm{e}^{\phi(\xi/\sqrt{ct})t}\left[E[\mathrm{e}^{-\xi U(0)/\sqrt{ct}}] - \frac{\xi}{\sqrt{ct}}\int_0^t \mathrm{e}^{-\phi(\xi/\sqrt{ct})s}E[\mathrm{e}^{-\xi U(s)/\sqrt{ct}}\mathbf{1}_{\{I(s)=0\}}]ds\right].$$

Define

$$\phi_t(\xi) \stackrel{\text{def}}{=} \phi(\xi/\sqrt{ct})t$$

and

$$F_t^\omega(x) \stackrel{\text{def}}{=} \sqrt{\frac{t}{c}}\int_0^x E[\mathrm{e}^{-\omega U(ts)/\sqrt{ct}}\mathbf{1}_{\{I(ts)=0\}}]ds. \tag{2.22}$$

Then

$$E[e^{-\xi U(t)/\sqrt{ct}}] = e^{\phi_t(\xi)}\left[E[e^{-\xi U(0)/\sqrt{ct}}] - \xi\int_0^1 e^{-\phi_t(\xi)x}d_xF_t^{\xi}(x)\right]. \tag{2.23}$$

It can be shown that if the limit (2.16) exists for every $\omega > 0$, then

$$\lim_{t\to\infty}\int_0^\infty e^{-\xi x}d_xF_t^{\omega}(x) = \frac{1}{\sqrt{c\xi}}l(\omega/\sqrt{c\xi}) \quad \text{for } \xi > 0 \text{ and } \omega > 0. \tag{2.24}$$

In particular, if $l(\omega)$ is independent of ω, then $l(\omega) = \sqrt{c/2}$ and we have

$$\lim_{t\to\infty}\int_0^\infty e^{-\xi x}d_xF_t^{\omega}(x) = \frac{1}{\sqrt{2\xi}} = \int_0^\infty \frac{e^{-\xi x}}{\sqrt{2\pi x}}dx.$$

Applying an extended continuity theorem (see Feller [8], Theorem 2a, p. 410) we obtain

$$\lim_{t\to\infty} F_t^{\omega}(x) = F(x) \stackrel{\text{def}}{=} \sqrt{\frac{2x}{\pi}} \quad \text{for } x \geq 0 \text{ and } \omega > 0.$$

Note that

$$\lim_{t\to\infty}\phi_t(\xi) = \frac{1}{2}\xi^2. \tag{2.25}$$

Applying Helly's theorems we have for $\xi > 0$ and $0 < \epsilon < \xi^2/2$

$$\begin{aligned}\int_0^1 e^{-(\frac{1}{2}\xi^2+\epsilon)x}dF(x) &\leq \liminf_{t\to\infty}\int_0^1 e^{-\phi_t(\xi)x}d_xF_t^{\omega}(x)\\ &\leq \limsup_{t\to\infty}\int_0^1 e^{-\phi_t(\xi)x}d_xF_t^{\omega}(x) \leq \int_0^1 e^{-(\frac{1}{2}\xi^2-\epsilon)x}dF(x).\end{aligned}$$

Letting $\epsilon \downarrow 0$ results in

$$\lim_{t\to\infty}\int_0^1 e^{-\phi_t(\xi)x}d_xF_t^{\omega}(x) = \int_0^1 e^{-\frac{1}{2}\xi^2 x}dF(x). \tag{2.26}$$

We now have

$$\lim_{t\to\infty} E[e^{-\xi U(t)/\sqrt{ct}}] = e^{\xi^2/2}\left[1 - \xi\int_0^1 e^{-\xi^2 x/2}dF(x)\right] = \xi e^{\xi^2/2}\int_1^\infty \frac{e^{-\xi^2 x/2}}{\sqrt{2\pi x}}dx.$$

From the last formula we obtain by means of the substitution $x = (1 + u/\xi)^2$

$$\lim_{t\to\infty} E[e^{-\xi U(t)/\sqrt{ct}}] = \frac{2}{\sqrt{2\pi}}\int_0^\infty e^{-\xi u}e^{-u^2/2}du,$$

so that (a) is proved.

To prove (b) we substitute ξ/t for ξ in (2.1). Then

$$E[e^{-\xi U(t)/t}] = e^{\phi(\xi/t)t}\left[E[e^{-\xi U(0)/t}] - \frac{\xi}{t}\int_0^t e^{-\phi(\xi/t)s}E[e^{-\xi U(s)/t}1_{\{I(s)=0\}}]ds\right]. \tag{2.27}$$

If $\rho > 1$, then

$$\lim_{t\to\infty} \phi(\xi/t)t = (1-\rho)\xi < 0 \quad \text{for } \xi > 0.$$

We have

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{1}{t}\int_0^t e^{\phi(\xi/t)(t-s)} E[e^{-\xi U(s)/t}\mathbf{1}_{\{I(s)=0\}}]ds \\
&\leq \limsup_{t\to\infty} \frac{1}{t}\int_0^t e^{\phi(\xi/t)(t-s)} P(I(s)=0)ds \\
&\leq \limsup_{t\to\infty} \frac{1}{t}\int_0^t P(I(s)=0)ds = 0.
\end{aligned}$$

We now obtain from (2.27) by letting $t \to \infty$

$$\lim_{t\to\infty} E[e^{-\xi U(t)/t}] = e^{-(\rho-1)\xi}. \qquad \square$$

3. Application to selected vacation models

In this section we demonstrate the generality and convenience of formula (2.9) for the workload process. We show how this new result can be applied to rederive and unify some existing results for selected vacation models. We also verify whether the conditions of Theorem 2 are satisfied in these special cases.

We shall analyse five $M/G/1$ vacation models. In the first three models the service is assumed to be exhaustive, while in the last two models the service is non-exhaustive. For more information about $M/G/1$ queues with vacations we refer the reader to Takagi [16], Chapter 2.

Throughout this section, X is an exponential random variable with parameter ξ and X is independent of the queueing process.

Example 1. Multiple vacations and exhaustive service

In this queueing model, the server takes a vacation of a random length of time when he finishes serving a customer and finds the system empty. At the end of a vacation, the server returns to the system and starts serving those, if any, who have arrived during the vacation. If there are no waiting customers at the end of a vacation, the server takes a new vacation. Vacations are taken repeatedly until the server finds at least one waiting customer at the end of a vacation. The lengths of vacations are i.i.d. non-negative random variables. The sequence of arrival times, the sequence of service times and the sequence of vacation times are independent stochastic processes. Let $V^*(\xi) = E[\exp(-\xi V)]$, where V is the random vacation time. It is assumed that $P(V=0) < 1$.

Note that in this queueing model $I(t) = 0$ if and only if the server is on vacation at time t. Because of the independence of X and the workload process $U(t)$ we have

$$E[\mathbf{1}_{\{I(X)=0\}}e^{-\omega U(X)}] = \int_0^\infty E[\mathbf{1}_{\{I(t)=0\}}e^{-\omega U(t)}]\, d[1-e^{-\xi t}] = \xi U_0^*(\xi,\omega). \qquad (3.1)$$

Let T_n be the moment at which the nth vacation starts and V_n the length of the nth vacation. From (3.1) we have

$$\begin{aligned}
\xi U_0^*(\xi,\omega) &= \sum_{n=1}^{\infty} E[\mathbf{1}_{\{T_n \le X < T_n + V_n\}} e^{-\omega U(X)}] \\
&= \sum_{n=1}^{\infty} E[\mathbf{1}_{\{T_n \le X\}} e^{-\omega U(T_n)}] \int_{[0,\infty)} \int_0^y e^{-\lambda(1-\beta(\omega))x} \xi e^{-\xi x} dx\, d_y P(V_n \le y) \\
&= \sum_{n=1}^{\infty} P(T_n \le X) \frac{\xi\{1 - V^*(\xi + \lambda - \lambda\beta(\omega))\}}{\xi + \lambda - \lambda\beta(\omega)} \\
&= \frac{E[e^{-(\xi+\lambda-\lambda\delta(\xi))U(0)}]}{1 - V^*(\xi + \lambda - \lambda\delta(\xi))} \cdot \frac{\xi\{1 - V^*(\xi + \lambda - \lambda\beta(\omega))\}}{\xi + \lambda - \lambda\beta(\omega)}.
\end{aligned} \tag{3.2}$$

Substituting $U_0^*(\xi,\omega)$ into (2.9) we get

$$\begin{aligned}
U^*(\xi,\omega) =& \frac{E[e^{-\omega U(0)}]}{\xi - \omega + \lambda - \lambda\beta(\omega)} - \frac{\omega}{\xi - \omega + \lambda - \lambda\beta(\omega)} \\
&\times \frac{E[e^{-(\xi+\lambda-\lambda\delta(\xi))U(0)}]}{1 - V^*(\xi + \lambda - \lambda\delta(\xi))} \cdot \frac{1 - V^*(\xi + \lambda - \lambda\beta(\omega))}{\xi + \lambda - \lambda\beta(\omega)}.
\end{aligned} \tag{3.3}$$

Two special cases of (3.3) with $U(0) = 0$ and $E[e^{-\xi U(0)}] = \beta(\xi)^i$ were obtained earlier by Keilson and Ramaswamy [10] and Takagi [17], respectively.

One can show that if $\rho < 1$, $E[V] < \infty$ and the distributions of service and vacation times are non-arithmetic, then the length of the busy cycle (T_n, T_{n+1}) also has a non-arithmetic distribution with finite mean value. In that case, using a renewal argument as in Cohen [4], it can be shown that $U(t)$ converges in distribution as $t \to \infty$. The LSt for the limiting distribution is given by

$$E[e^{-\omega U}] = \lim_{\xi \downarrow 0} \xi U^*(\xi,\omega) = \frac{(1-\rho)\omega}{\omega - \lambda + \lambda\beta(\omega)} \cdot \frac{1 - V^*(\lambda - \lambda\beta(\omega))}{\lambda(1-\beta(\omega))E[V]}. \tag{3.4}$$

From (3.4) we have

$$E[U] = \frac{\lambda E[S^2]}{2(1-\rho)} + \frac{\rho E[V^2]}{2E[V]}. \tag{3.5}$$

If $\rho = 1$, $c = \lambda E[S^2] < \infty$ and $E[V] < \infty$, then from (3.2) we get

$$l(\omega) = \lim_{\xi \downarrow 0} \sqrt{\xi} U_0^*(\xi, \omega\sqrt{\xi}) = \sqrt{\lambda E[S^2]/2},$$

which is independent of ω. Hence, by part (a) of Theorem 2, $U(t)/\sqrt{ct} \xrightarrow{d} |Y|$.

If $\rho > 1$ and $E[V] < \infty$, then

$$\int_0^\infty P(I(t) = 0)dt = \lim_{\xi \downarrow 0} U_0^*(\xi, 0) = E[V] \frac{E[e^{-\lambda(1-\delta(0))U(0)}]}{1 - V^*(\lambda - \lambda\delta(0))},$$

which is finite because $\delta(0) = \lim_{\xi \downarrow 0} \delta(\xi) < 1$. Hence $U(t)/t \xrightarrow{\mathrm{d}} \rho - 1$.

Example 2. Single vacation and exhaustive service

In this queueing model, the server takes only one vacation of random length of time after each busy period. That is, if upon returning from a vacation there are no waiting customers, the server stays idle until the first new customer arrives and then starts working. If customers arrive during a vacation, the server starts serving them as soon as that vacation terminates. Vacation times are i.i.d. non-negative random variables. The sequence of arrival times, the sequence of service times and the sequence of vacation times are assumed to be independent. Let $V^*(\xi) = E[\exp(-\xi V)]$, where V is the random vacation time.

Following the reasoning used in the previous example we have

$$\begin{aligned} U_0^*(\xi, \omega) = & \frac{E[\mathrm{e}^{-(\xi+\lambda-\lambda\delta(\xi))U(0)}]}{\xi + \lambda - \lambda\beta(\omega)} \\ & \times \frac{[1 - V^*(\xi + \lambda - \lambda\beta(\omega))](\xi + \lambda) + [\xi + \lambda - \lambda\beta(\omega)]V^*(\xi + \lambda)}{[1 - V^*(\xi + \lambda - \lambda\delta(\xi))](\xi + \lambda) + [\xi + \lambda - \lambda\delta(\xi)]V^*(\xi + \lambda)}. \end{aligned} \tag{3.6}$$

Substituting $U_0^*(\xi, \omega)$ into (2.9) we obtain

$$\begin{aligned} U^*(\xi, \omega) = & \frac{E[\mathrm{e}^{-\omega U(0)}]}{\xi - \omega + \lambda - \lambda\beta(\omega)} - \omega \frac{E[\mathrm{e}^{-(\xi+\lambda-\lambda\delta(\xi))U(0)}]}{\xi - \omega + \lambda - \lambda\beta(\omega)} \\ & \times \frac{[1 - V^*(\xi + \lambda - \lambda\beta(\omega))](\xi + \lambda) + [\xi + \lambda - \lambda\beta(\omega)]V^*(\xi + \lambda)}{[1 - V^*(\xi + \lambda - \lambda\delta(\xi))](\xi + \lambda) + [\xi + \lambda - \lambda\delta(\xi)]V^*(\xi + \lambda)}. \end{aligned} \tag{3.7}$$

The result for $U^*(\xi, \omega)$ in the special case $E[\mathrm{e}^{-\xi U(0)}] = \beta(\xi)^i$, where i is a non-negative integer, was obtained earlier by Takagi [17].

If $\rho < 1$ and $E[V] < \infty$, then $U(t)$ converges in distribution and the LSt for the limiting distribution is given by

$$E[\mathrm{e}^{-\omega U}] = \lim_{\xi \downarrow 0} \xi U^*(\xi, \omega) = \frac{(1-\rho)\omega}{\omega - \lambda + \lambda\beta(\omega)} \cdot \frac{1 - V^*(\lambda - \lambda\beta(\omega)) + V^*(\lambda)(1 - \beta(\omega))}{(1 - \beta(\omega))\{\lambda E[V] + V^*(\lambda)\}}.$$

From the last formula we obtain

$$E[U] = \frac{\lambda E[S^2]}{2(1-\rho)} + \rho \frac{\lambda E[V^2]}{2\{\lambda E[V] + V^*(\lambda)\}}.$$

Using (3.6) one can verify that

(a) if $\rho = 1$, $E[S^2] < \infty$ and $E[V] < \infty$, then

$$l(\omega) = \lim_{\xi \downarrow 0} \sqrt{\xi} U_0^*(\xi, \omega\sqrt{\xi}) = \sqrt{\lambda E[S^2]/2};$$

(b) if $\rho > 1$ and $E[V] < \infty$, then (2.18) holds.

Example 3. N-policy and set-up times

In this queueing model, the server remains idle after each busy period until the queue length builds up to a preassigned desired level N (this period is called a *build-up* period). Here N is a positive integer. Furthermore, a random set-up time T occurs before starting a busy period. The set-up times are i.i.d. non-negative random variables. The sequence of arrival times, the sequence of service times and the sequence of set-up times are independent. This queueing model was discussed previously, among others, by Heyman [9], Medhi and Templeton [11] and Takagi [17].

Note that $I(t)=0$ if and only if t is inside a build-up or set-up period. To compute $U^*(\xi,\omega)$ for this model we start with a simple situation: the system is empty at time 0. For this initial condition we have

$$\xi U_0^*(\xi,\omega)=E[\mathbf{1}_{\{I(X)=0\}}\mathrm{e}^{-\omega U(X)}]=\frac{E[\mathbf{1}_{\{I(X)=0,X\leq C_1\}}\mathrm{e}^{-\omega U(X)}]}{1-P(X>C_1)}, \tag{3.8}$$

where C_1 is the length of the first busy cycle, i.e. the time from 0 to the end of the first busy period.

It can be shown that

$$P(X>C_1)=(\lambda/\xi+\lambda)^N\delta(\xi)^N T^*(\xi+\lambda-\lambda\delta(\xi)) \tag{3.9}$$

and

$$\begin{aligned}E[\mathbf{1}_{\{I(X)=0,X\leq C_1\}}\mathrm{e}^{-\omega U(X)}]=&\sum_{k=0}^{N-1}(\lambda/\xi+\lambda)^k(\xi/\xi+\lambda)\beta(\omega)^k\\&+(\lambda/\xi+\lambda)^N\beta(\omega)^N\frac{\xi\{1-T^*(\xi+\lambda-\lambda\beta(\omega))\}}{\xi+\lambda-\lambda\beta(\omega)},\end{aligned} \tag{3.10}$$

where $T^*(\xi)=E[\exp(-\xi T)]$.

From (3.8)-(3.10) and using simple algebra we obtain

$$U_0^*(\xi,\omega)=\frac{1}{\xi+\lambda-\lambda\beta(\omega)}\cdot\frac{1-(\lambda\beta(\omega)/\xi+\lambda)^N T^*(\xi+\lambda-\lambda\beta(\omega))}{1-(\lambda\delta(\xi)/\xi+\lambda)^N T^*(\xi+\lambda-\lambda\delta(\xi))}. \tag{3.11}$$

If the server is serving at time 0, then it takes a time T_1 until the system becomes empty. In that case, $U_0^*(\xi,\omega)$ equals the right-hand side of (3.11) multiplied by $E[\mathrm{e}^{-\xi T_1}]$. In fact we have

$$U_0^*(\xi,\omega)=\frac{E[\mathrm{e}^{-(\xi+\lambda-\lambda\delta(\xi))U(0)}]}{\xi+\lambda-\lambda\beta(\omega)}\cdot\frac{1-(\lambda\beta(\omega)/\xi+\lambda)^N T^*(\xi+\lambda-\lambda\beta(\omega))}{1-(\lambda\delta(\xi)/\xi+\lambda)^N T^*(\xi+\lambda-\lambda\delta(\xi))}. \tag{3.12}$$

Substituting $U_0^*(\xi,\omega)$ into (2.9) we obtain $U^*(\xi,\omega)$. The result for $U^*(\xi,\omega)$ in the special case $E[\mathrm{e}^{-\xi U(0)}]=\beta(\xi)^i$, where i is a non-negative integer, was obtained earlier by Takagi [17].

It can be shown that if $\rho<1$ and $E[T]<\infty$, then $U(t)$ converges in distribution as $t\to\infty$. The LSt for the limiting distribution is given by

$$E[\mathrm{e}^{-\omega U}]=\lim_{\xi\downarrow 0}\xi U^*(\xi,\omega)=\frac{(1-\rho)\omega}{\omega+\lambda-\lambda\beta(\omega)}\cdot\frac{1-\beta(\omega)^N T^*(\lambda-\lambda\beta(\omega))}{\{N+\lambda E[T]\}(1-\beta(\omega))}.$$

From the last relation we obtain

$$E[U]=\frac{\lambda E[S^2]}{2(1-\rho)}+\frac{\rho N E[T]}{N+\lambda E[T]}+\frac{\rho(N(N-1)+\lambda^2E[T^2])}{2\lambda(N+\lambda E[T])}.$$

One can verify that

(a) if $\rho=1$, $E[S^2]<\infty$ and $E[T]<\infty$, then

$$l(\omega)=\lim_{\xi\downarrow 0}\sqrt{\xi}U_0^*(\xi,\omega\sqrt{\xi})=\sqrt{\lambda E[S^2]/2};$$

(b) if $\rho>1$ and $E[T]<\infty$, then

$$\int_0^\infty P(I(s)=0)ds=\lim_{\xi\downarrow 0}U_0^*(\xi,0)\leq\frac{E[S]}{1-\delta(0)^N S^*(\lambda-\lambda\delta(0))}<\infty.$$

Example 4. Multiple vacations and semi-exhaustive service

This is the same model as in Example 1 with the following change. If the server finds waiting customers at the end of a vacation, he starts working until the number of customers present is one less than the number of customers present upon his return from the last vacation. The server then leaves for a new vacation. This service discipline is called *semi-exhaustive*. It has been introduced by Takagi [15], who studies it in a cyclic polling system with switchover times. A service discipline which is slightly different from the one described above has been studied by Cohen [5]. He analyses a two-queue model with alternating semi-exhaustive service and obtains a number of interesting results.

We denote by T_n the moment at which the nth vacation starts and by N_n the number of customers present in the system at that time. To avoid complexity we assume that $T_1=0$ and $N_1=0$. As in (3.2) we have

$$U_0^*(\xi,\omega)=\sum_{n=1}^{\infty}E[\mathbf{1}_{\{T_n\leq X\}}\mathrm{e}^{-\omega U(T_n)}]\frac{1-V^*(\xi+\lambda-\lambda\beta(\omega))}{\xi+\lambda-\lambda\beta(\omega)}. \tag{3.13}$$

Note that

$$E[\mathbf{1}_{\{T_n\leq X\}}\mathrm{e}^{-\omega U(T_n)}]=E[\mathbf{1}_{\{T_n\leq X\}}\beta(\omega)^{N_n}]. \tag{3.14}$$

It can be shown that for $n\geq 1$

$$\begin{aligned}zE[\mathbf{1}_{\{T_{n+1}\leq X\}}z^{N_{n+1}}]=&[z-\delta(\xi)]V^*(\xi+\lambda)P(T_n\leq X,N_n=0)\\&+\delta(\xi)V^*(\xi+\lambda-\lambda z)E[\mathbf{1}_{\{T_n\leq X\}}z^{N_n}].\end{aligned} \tag{3.15}$$

Let

$$\sigma(\xi,z)\stackrel{\text{def}}{=}\sum_{n=1}^{\infty}E[\mathbf{1}_{\{T_n\leq X\}}z^{N_n}]. \tag{3.16}$$

From (3.14)-(3.16) and simple calculation we get

$$\sigma(\xi,z)=\frac{z-[\delta(\xi)-z]V^*(\xi+\lambda)\sigma(\xi,0)}{z-\delta(\xi)V^*(\xi+\lambda-\lambda z)}. \tag{3.17}$$

To find $\sigma(\xi,0)$ we consider the standard $M/G/1$ queue with arrival rate λ and the service time V. Let L_v and N_v be the length of a busy period and the number of customers served during that busy period, respectively. Write $\delta_v(\xi,z)$ for $E[\mathrm{e}^{-\xi L_v} z^{N_v}]$ and put

$$\gamma(\xi)=\delta_v(\xi,\delta(\xi)).$$

Because $z=\gamma(\xi)$ for $\xi\geq 0$ is a unique zero in [0,1] of the denominator in (3.17), it must be a zero of the numerator. Therefore

$$\sigma(\xi,0)=\frac{\gamma(\xi)}{(\delta(\xi)-\gamma(\xi))V^*(\xi+\lambda)}. \tag{3.18}$$

Substituting (3.18) into (3.17) results in

$$\sigma(\xi,z)=\frac{1}{1-V^*(\xi+\lambda-\lambda\gamma(\xi))}\cdot\frac{z-\gamma(\xi)}{z-\delta(\xi)V^*(\xi+\lambda-\lambda z)}. \tag{3.19}$$

We now have

$$\begin{aligned}U_0^*(\xi,\omega)=&\frac{1}{1-V^*(\xi+\lambda-\lambda\gamma(\xi))}\cdot\frac{\beta(\omega)-\gamma(\xi)}{\beta(\omega)-\delta(\xi)V^*(\xi+\lambda-\lambda\beta(\omega))}\\&\times\frac{1-V^*(\xi+\lambda-\lambda\beta(\omega))}{\xi+\lambda-\lambda\beta(\omega)}.\end{aligned} \tag{3.20}$$

Substituting $U_0^*(\xi,\omega)$ into (2.9) we obtain $U^*(\xi,\omega)$.

The LSt for the limiting distribution of $U(t)$ as $t\to\infty$ is given by

$$\begin{aligned}E[\mathrm{e}^{-\omega U}]=&\lim_{\xi\downarrow 0}\xi U^*(\xi,\omega)\\=&\frac{(1-\rho)\omega}{\omega-\lambda+\lambda\beta(\omega)}\cdot\frac{(1-\rho_v)(1-\beta(\omega))}{V^*(\lambda-\lambda\beta(\omega))-\beta(\omega)}\cdot\frac{1-V^*(\lambda-\lambda\beta(\omega))}{\lambda(1-\lambda\beta(\omega))E[V]},\end{aligned}$$

where $\rho<1$ and $\rho_v=\lambda E[V]<1$ are necessary for the existence of a limiting distribution.

From the last equation we obtain

$$E[U]=\frac{\lambda E[S^2]}{2(1-\rho)}+\rho\frac{\lambda E[V^2]}{2(1-\rho_v)}+\rho\frac{E[V^2]}{2E[V]}.$$

Let us now assume that $\rho=1$, $c=\lambda E[S^2]<\infty$ and $\rho_v=\lambda E[V]<\infty$. It is not difficult to verify that for $\omega>0$

$$l(\omega)=\lim_{\xi\downarrow 0}\sqrt{\xi}U_0^*(\xi,\omega\sqrt{\xi})=\begin{cases}\sqrt{c/2} & \text{if } \rho_v\leq 1,\\ \dfrac{\rho_v}{\sqrt{2/c}+(\rho_v-1)\omega} & \text{if } \rho_v>1.\end{cases} \tag{3.21}$$

If $\rho_v\leq 1$, then $l(\omega)$ is independent of ω and hence, by part (a) of Theorem 2, $U(t)/\sqrt{ct}\xrightarrow{\mathrm{d}}|Y|$, where Y has the standard normal distribution. If $\rho_v>1$, then $l(\omega)$

depends on ω. In this case, however, $U(t)/\sqrt{ct}$ still converges in distribution but the limit is not normal.

Theorem 3. Consider the multiple vacation model with semi-exhaustive service. Let $\rho = 1$, $E[V] < \infty$ and $c = \lambda E[S^2] < \infty$. Assume that $\rho_v = \lambda E[V] > 1$. Then

$$\frac{1}{\sqrt{ct}} U(t) \xrightarrow{\mathrm{d}} Y \quad \text{as } t \to \infty, \tag{3.22}$$

where Y has an absolutely continuous distribution with density

$$f_Y(x) = \frac{2}{(2-\rho_v)\sqrt{2\pi}} \exp(-\tfrac{1}{2}x^2) \left[1 - \exp(-\tfrac{\rho_v(2-\rho_v)}{2(\rho_v-1)^2} x^2)\right] \quad \text{if } \rho_v \neq 2 \tag{3.23}$$

and

$$f_Y(x) = \frac{2}{\sqrt{2\pi}} x^2 \exp(-\tfrac{1}{2}x^2) \quad \text{if } \rho_v = 2. \tag{3.24}$$

Proof. We first notice that the relations (2.23) and (2.24) are still valid when $\rho_v > 1$. From (2.24) and (3.21) we get

$$\lim_{t\to\infty} \int_0^\infty e^{-\xi x} d_x F_t^\omega(x) = \frac{\rho_v}{\sqrt{2\xi} + (\rho_v - 1)\omega}, \tag{3.25}$$

where $F_t^\omega(x)$ is as defined in (2.22). With the help of (29.3.37) in Abramowitz and Stegun [1] we find

$$\frac{1}{\sqrt{2\xi} + \omega} = \int_0^\infty e^{-\xi x} f(\omega, x) dx, \tag{3.26}$$

where

$$f(\omega, x) = \frac{1}{\sqrt{2\pi x}} - \frac{\omega}{\sqrt{\pi}} e^{\omega^2 x/2} \int_{\omega\sqrt{x/2}}^\infty e^{-y^2} dy. \tag{3.27}$$

We obtain from (3.25) and (3.26) by applying the extended continuity theorem

$$\lim_{t\to\infty} F_t^\omega(x) = F^\omega(x) = \rho_v \int_0^x f((\rho_v - 1)\omega, y) dy \quad \text{for } x \geq 0. \tag{3.28}$$

The same argument as in the proof of (2.26) gives

$$\lim_{t\to\infty} \int_0^1 e^{-\phi_t(\xi)x} d_x F_t^\omega(x) = \int_0^1 e^{-\xi^2 x/2} d_x F^\omega(x) \quad \text{for } \xi > 0 \text{ and } \omega > 0.$$

Letting $t \to \infty$ in (2.23) results in

$$\lim_{t\to\infty} E[e^{-\xi U(t)/\sqrt{ct}}] = e^{\xi^2/2}\left[1 - \xi\rho_v \int_0^1 e^{-\xi^2 x/2} f((\rho_v - 1)\xi, x) dx\right].$$

The last integral can be simplified by applying Fubini's theorem. After some routine simplification we obtain

$$\begin{aligned}\lim_{t\to\infty} E[e^{-\xi U(t)/\sqrt{ct}}] =& \frac{\exp(\xi^2/2)}{2-\rho_v} \operatorname{erfc}(\xi/\sqrt{2}) \\ &- \frac{(\rho_v - 1)\exp((\rho_v-1)^2\xi^2/2)}{2-\rho_v} \operatorname{erfc}((\rho_v - 1)\xi/\sqrt{2}),\end{aligned} \tag{3.29}$$

where $\mathrm{erfc}\,(x)$ is the complementary error function, i.e.

$$\mathrm{erfc}\,(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty \mathrm{e}^{-u^2} du.$$

Hence $U(t)/\sqrt{ct} \xrightarrow{\mathrm{d}} Y$ as $t \to \infty$. The probability density function of the random variable Y can be obtained from (3.29) by noting that

$$\frac{2}{\sqrt{2\pi}} \int_0^\infty \mathrm{e}^{-\xi x} \mathrm{e}^{-x^2/2} dx = \exp(\xi^2/2)\,\mathrm{erfc}\,(\xi/\sqrt{2}). \qquad \square$$

Example 5. Multiple vacations and gated service

This is the same model as in Example 1 with the following change. When returning from a vacation the server only gives service to those, if any, who were waiting when the server returned. After doing this the server leaves for another vacation. If upon returning from a vacation there are no waiting customers, the server leaves for a new vacation. We assume that at time 0 the system is empty and the server is about to take a vacation.

Let T_n be the moment at which the nth vacation starts. We have

$$U_0^*(\xi, \omega) = \sum_{n=1}^\infty E[\mathbf{1}_{\{T_n \le X\}} \mathrm{e}^{-\omega U(T_n)}] \frac{1 - V^*(\xi + \lambda - \lambda\beta(\omega))}{\xi + \lambda - \lambda\beta(\omega)}. \tag{3.30}$$

Let N_n be the number of customers present in the system at the beginning of the nth vacation. Clearly

$$E[\mathbf{1}_{\{T_n \le X\}} \mathrm{e}^{-\omega U(T_n)}] = E[\mathbf{1}_{\{T_n \le X\}} \beta(\omega)^{N_n}]. \tag{3.31}$$

For $n \ge 1$ we have, omitting the details

$$\begin{aligned} E[\mathbf{1}_{\{T_{n+1} \le X\}} z^{N_{n+1}}] =& V^*(\xi + \lambda - \lambda\beta(\xi + \lambda - \lambda z)) \\ & \times E[\mathbf{1}_{\{T_n \le X\}} \beta(\xi + \lambda - \lambda z)^{N_n}]. \end{aligned} \tag{3.32}$$

Set

$$\begin{aligned} \psi_1(\xi, \omega) =& V^*(\xi + \lambda - \lambda\beta(\omega)), \\ \psi_k(\xi, \omega) =& \psi_{k-1}(\xi, \xi + \lambda - \lambda\beta(\omega)), \\ \psi(\xi, \omega) =& \sum_{n=1}^\infty \prod_{k=1}^n \psi_k(\xi, \omega). \end{aligned} \tag{3.33}$$

From (3.31)-(3.33) we obtain

$$\sum_{n=1}^\infty E[\mathbf{1}_{\{T_n \le X\}} \mathrm{e}^{-\omega U(T_n)}] = 1 + \psi(\xi, \xi + \lambda - \lambda\beta(\omega)), \tag{3.34}$$

and hence

$$U_0^*(\xi, \omega) = [1 + \psi(\xi, \xi + \lambda - \lambda\beta(\omega))] \frac{1 - V^*(\xi + \lambda - \lambda\beta(\omega))}{\xi + \lambda - \lambda\beta(\omega)}. \tag{3.35}$$

The time-dependent result for $U(t)$ is now given by

$$U^*(\xi,\omega) = \frac{1}{\xi-\omega+\lambda-\lambda\beta(\omega)} - \frac{\omega}{\xi-\omega+\lambda-\lambda\beta(\omega)} \times [1+\psi(\xi,\xi+\lambda-\lambda\beta(\omega))]\frac{1-V^*(\xi+\lambda-\lambda\beta(\omega))}{\xi+\lambda-\lambda\beta(\omega)}. \tag{3.36}$$

With some algebraic manipulations, one can verify that (3.36) is identical to (4.78) in Takine and Hasegawa [19].

If $\rho<1$ and $E[V]<\infty$, we have

$$\lim_{\xi\downarrow 0}\xi[1+\psi(\xi,\xi+\lambda-\lambda\beta(\omega))] = \frac{1-\rho}{E[V]}\prod_{n=2}^{\infty}\psi_n(0,\omega) \tag{3.37}$$

(see [17] or the appendix for a direct computation).

The LSt for the limiting distribution of the workload is given by

$$E[\mathrm{e}^{-\omega U}] = \lim_{\xi\downarrow 0}\xi U^*(\xi,\omega) = \frac{(1-\rho)\omega}{\omega-\lambda+\lambda\beta(\omega)}\cdot\prod_{n=2}^{\infty}\psi_n(0,\omega)\cdot\frac{1-V^*(\lambda-\lambda\beta(\omega))}{\lambda(1-\beta(\omega))E[V]}.$$

From the last formula we obtain

$$E[U] = \frac{\lambda E[S^2]}{2(1-\rho)} + \frac{\rho^2 E[V]}{1-\rho} + \rho\frac{E[V^2]}{2E[V]}.$$

Remark 4. In Example 5 the question concerning the convergence in distribution of $U(t)/\sqrt{ct}$ for $t\to\infty$ when $\rho=1$, $c=\lambda E[S^2]<\infty$ and $E[V]<\infty$ is still open. At present we are unable to answer this question.

Appendix: Proof of relation (3.37)

Because

$$\psi(\xi,\omega) = \psi_1(\xi,\omega)[1+\psi(\xi,\xi+\lambda-\lambda\beta(\omega))],$$

it suffices to show that

$$\lim_{\xi\downarrow 0}\xi\psi(\xi,\omega) = \frac{1-\rho}{E[V]}\prod_{n=1}^{\infty}\psi_n(0,\omega).$$

Let

$$\beta_1(\xi,\omega)\stackrel{\text{def}}{=}\beta(\omega),\quad \beta_n(\xi,\omega)\stackrel{\text{def}}{=}\beta(\xi+\lambda-\lambda\beta_{n-1}(\xi,\omega)), \tag{3.38}$$

where $\xi\geq 0$ and $\omega\geq 0$. Then

$$\psi_n(\xi,\omega) = V^*(\xi+\lambda-\lambda\beta_n(\xi,\omega)). \tag{3.39}$$

It can be shown by induction that if $\beta(\omega)\leq\delta(\xi)$, then

$$\psi_n(\xi,\omega) = V^*(\xi+\lambda-\lambda\beta_n(\xi,\omega)) \leq V^*(\xi+\lambda-\lambda\delta(\xi)).$$

Hence, we have for $\omega > 0$ and ξ sufficiently small

$$\begin{aligned}\psi(\xi,\omega) &= \sum_{n=1}^{m}\prod_{k=1}^{n}\psi_k(\xi,\omega) + \sum_{n=m+1}^{\infty}\prod_{k=1}^{n}\psi_k(\xi,\omega)\\ &\leq \sum_{n=1}^{m}\prod_{k=1}^{n}\psi_k(\xi,\omega) + \frac{V^*(\xi+\lambda-\lambda\delta(\xi))}{1-V^*(\xi+\lambda-\lambda\delta(\xi))}\prod_{k=1}^{m}\psi_k(\xi,\omega). \qquad (3.40)\end{aligned}$$

Multiplying both sides of (3.40) by ξ, then letting $\xi \downarrow 0$ and finally letting $m \to \infty$ we obtain

$$\limsup_{\xi\downarrow 0}\ \xi\psi(\xi,\omega) \leq \frac{1-\rho}{E[V]}\prod_{k=1}^{\infty}\psi_k(0,\omega).$$

To show an inequality for liminf we note that

$$|e^{-ax} - e^{-bx}| \leq |a-b|x \quad \text{for } a,b,x \geq 0. \qquad (3.41)$$

Using (3.38)-(3.39) and applying (3.41) successively we obtain

$$\begin{aligned}|\psi_n(\xi,\omega) - V^*(\xi+\lambda-\lambda\delta(\xi))| &\leq \lambda E[V]|\beta_n(\xi,\omega)-\delta(\xi)|\\ &\leq \lambda E[V]\lambda E[S]|\beta_{n-1}(\xi,\omega)-\delta(\xi)| \leq \lambda E[V]\rho^{n-1}.\end{aligned}$$

It follows that for m sufficiently large and $n > m$

$$\begin{aligned}\prod_{k=m+1}^{n}\psi_k(\xi,\omega) &\geq \prod_{k=m+1}^{n}\{V^*(\xi+\lambda-\lambda\delta(\xi)) - \lambda E[V]\rho^{k-1}\}\\ &\geq \left[1-\frac{\lambda E[V]}{V^*(\xi+\lambda-\lambda\delta(\xi))}[\rho^m+\cdots+\rho^{n-1}]\right]V^*(\xi+\lambda-\lambda\delta(\xi))^{n-m}\\ &\geq \left[1-\frac{\lambda E[V]}{V^*(\xi+\lambda-\lambda\delta(\xi))}\cdot\frac{\rho^m}{1-\rho}\right]V^*(\xi+\lambda-\lambda\delta(\xi))^{n-m}.\end{aligned}$$

Hence

$$\begin{aligned}\psi(\xi,\omega) &\geq \sum_{n=m+1}^{\infty}\prod_{k=1}^{n}\psi_k(\xi,\omega) \geq \frac{V^*(\xi+\lambda-\lambda\delta(\xi))}{1-V^*(\xi+\lambda-\lambda\delta(\xi))}\\ &\times\left[1-\frac{\rho^m}{1-\rho}\cdot\frac{\lambda E[V]}{V^*(\xi+\lambda-\lambda\delta(\xi))}\right]\prod_{k=1}^{m}\psi_k(\xi,\omega). \qquad (3.42)\end{aligned}$$

Multiplying both sides of (3.42) by ξ, then letting $\xi \downarrow 0$ and finally letting $m \to \infty$ we obtain

$$\liminf_{\xi\downarrow 0}\ \xi\psi(\xi,\omega) \geq \frac{1-\rho}{E[V]}\prod_{k=1}^{\infty}\psi_k(0,\omega). \qquad \square$$

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References

[1] Abramowitz, M. and Stegun, I.A., *Handbook of Mathematical Functions*, 9th printing, Dover Publications, 1970.

[2] Boxma, O.J., Workloads and waiting times in single-server systems with multiple customer classes, *Queueing Systems*, 5 (1989), 185-214.

[3] Boxma, O.J. and Groenendijk, W.P., Pseudo-conservation laws in cyclic-service systems, *J. Appl. Probab.*, 24 (1987), 949-964.

[4] Cohen, J.W., *On Regenerative Processes in Queueing Theory*, Lecture Notes in Economics and Mathematical Systems, Vol. 121, Springer-Verlag, 1976.

[5] Cohen, J.W., A two-queue model with semi-exhaustive alternating service, In *Performance '87*, eds. P.J. Courtois and G. Latouche, North-Holland, 1988.

[6] Cong, T.D., Application of the method of collective marks to some $M/G/1$ vacation models with exhaustive service, *Queueing systems*, 16 (1994), 67-81.

[7] Cong, T.D., *Collective marks and waiting-time problems*, PhD thesis, Amsterdam, 1995.

[8] Feller, W., *An Introduction to Probability Theory and Its Applications*, Vol. II, John Wiley & Sons, 1966.

[9] Heyman, D. P., Optimal operating policies for $M/G/1$ queueing systems, *Operations Research*, 16 (1968), 362-382.

[10] Keilson, J. and Ramaswamy, R., The backlog and depletion-time process for $M/G/1$ vacation models with exhaustive service discipline, *J. Appl. Probab.*, 25 (1988), 404-412.

[11] Medhi, J. and Templeton, J.G.C., A Poisson input queue under N-policy and with a general start up time, *Computers Ops. Res.*, 19 (1992), 35-41.

[12] Runnenburg, J.Th., Probabilistic interpretation of some formulae in queueing theory, *Bull. Inst. Internat. Statist.*, 37 (1958), 405-414.

[13] Takács, L., Investigation of waiting time problems by reduction to Markov processes, *Acta Math. Hung.*, 6 (1955), 101-129.

[14] Takács, L., *Introduction to the Theory of Queues*, Oxford University Press, 1962.

[15] Takagi, H., Mean message waiting time in a symmetric polling system, In *Performance '84*, ed. E. Gelenbe, North-Holland, 1985.

[16] Takagi, H., *Queueing analysis: A Foundation of Performance Evaluation*, Vol. 1, Part 1, North-Holland, 1991.

[17] Takagi, H., Time-dependent process of $M/G/1$ vacation models with exhaustive service, *J. Appl. Probab.*, 29 (1992), 418-429.

[18] Takagi, H., Takine, T. and Boxma, O.J., Distribution of the workload in multiclass queueing systems with server vacations, *Naval Research Logistics*, 39 (1992), 41-52.

[19] Takine, T. and Hasegawa, T., On the $M/G/1$ queue with multiple vacations and gated service discipline, *J. Ops. Res. Soc. Japan*, 35 (1992), 217-235.

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