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A GREEDY ALGORITHM FOR MINIMIZING A SEPARABLE CONVEX FUNCTION OVER A FINITE JUMP SYSTEM

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Abstract We present a greedy algorithm for minimizing a separable convex function over a finite jump system $(E, \mathcal{F})$, where E is a nonempty finite set and $\mathcal{F}$ is a nonempty finite set of integral points in $\mathbf{Z}^E$ satisfying a certain exchange axiom. The concept of jump system was introduced by A. Bouchet and W. H. Cunningham. A jump system is a generalization of an integral bisubmodular polyhedron, an integral polymatroid, a (poly-)pseudomatroid and a delta-matroid, and has combinatorially nice properties. The algorithm starts with an arbitrary feasible solution and a current feasible solution incrementally moves toward an optimal one in a greedy way. We also show that the greedy algorithm terminates after changing an initial feasible solution at most

$$\sum_{e \in E}\{u(e) - l(e)\}$$

times, where for each $e \in E$

$$u(e) = \max_{x \in \mathcal{F}} x(e), \qquad l(e) = \min_{x \in \mathcal{F}} x(e).$$

1. Introduction

A. Bouchet and W. H. Cunningham [3] have recently introduced the concept of *jump system*. A jump system is a pair $(E, \mathcal{F})$ of a nonempty finite set E and a nonempty set $\mathcal{F}$ of integral points in $\mathbf{Z}^E$ satisfying an exchange axiom (a precise definition will be given later). We call a jump system $(E, \mathcal{F})$ *finite* if $\mathcal{F}$ is finite. For a finite jump system $(E, \mathcal{F})$, it is known ([3]) that the convex hull $\mathrm{Co}(\mathcal{F})$ of $\mathcal{F}$ is a bounded bisubmodular polyhedron, i.e., for a given finite jump system $(E, \mathcal{F})$ there exists a *bisubmodular function* $f : 3^E \to \mathbf{Z}$ such that

$$\mathrm{Co}(\mathcal{F}) = \{x \mid x \in \mathbf{R}^E,\ \forall (X, Y) \in 3^E : x(X) - x(Y) \leq f(X, Y)\} \tag{1.1}$$

(the precise definition of bisubmodular function will also be given later). However, the set of integral points of $\mathrm{Co}(\mathcal{F})$ is not necessarily equal to $\mathcal{F}$. Some non-extreme integral points of $\mathrm{Co}(\mathcal{F})$ may be missing in $\mathcal{F}$. Therefore, a jump system is a proper generalization of an *integral bisubmodular polyhedron* [3, 4], which is the set of integral points of a bisubmodular polyhedron defined by an integral bisubmodular function. Hence, it generalizes an integral polymatroid [6], a pseudomatroid [4] and a delta-matroid [2]. An interesting example of a jump system arises from matchings in graphs (see [3, 5]).

Recently, we presented a greedy algorithm for minimizing a separable convex function over an integral bisubmodular polyhedron ([1]). We show in this paper that the algorithm given in [1] also works over a finite jump system $(E, \mathcal{F})$. Our algorithm starts with an arbitrary initial feasible point and repeats coordinate-wise augmentations and/or exchanges in a greedy way. In our previous paper [1] we did not give an estimation of the number of the required transformations of feasible solutions but by examining the behavior of the greedy algorithm we will show that the greedy algorithm for a finite jump system $(E, \mathcal{F})$ terminates after changing an initial feasible solution at most

$$\sum_{e \in E} \{u(e) - l(e)\} \tag{1.2}$$

times, where for each $e \in E$

$$u(e) = \max_{x \in \mathcal{F}} x(e), \qquad l(e) = \min_{x \in \mathcal{F}} x(e). \tag{1.3}$$

2. Definitions

Let E be a nonempty finite set. Denote by 3^E the set of all the ordered pairs (X, Y) of disjoint subsets X and Y of E. Let $f : 3^E \to \mathbf{Z}$ be a function from 3^E to the set $\mathbf{Z}$ of integers such that $f(\emptyset, \emptyset) = 0$ and for each $(X_i, Y_i) \in 3^E$ (i=1,2)

$$\begin{aligned} &f(X_1, Y_1) + f(X_2, Y_2) \\ &\quad \geq f((X_1 \cup X_2) - (Y_1 \cup Y_2), (Y_1 \cup Y_2) - (X_1 \cup X_2)) + f(X_1 \cap X_2, Y_1 \cap Y_2). \end{aligned} \tag{2.1}$$

We call such an f a *bisubmodular function*, which was first considered by Chandrasekaran and Kabadi [4]. Define a polyhedron

$$\mathrm{P}_*(f) = \{x \mid x \in \mathbf{Z}^E, \forall (X, Y) \in 3^E : x(X) - x(Y) \leq f(X, Y)\} \tag{2.2}$$

associated with f, where $x(X) = \sum_{e \in X} x(e)$ for any $X \subseteq E$ and $x \in \mathbf{Z}^E$. Note that $x(\emptyset) = 0$ for any $x \in \mathbf{Z}^E$. We call the polyhedron $\mathrm{P}_*(f)$ an *integral bisubmodular polyhedron*.

A $\{0, \pm 1\}$-vector with a unique nonzero component is called a *step*. For any $x, y \in \mathbf{Z}^E$ a *step s from x to y* is a step such that $\sum_{e \in E} |x(e) + s(e) - y(e)| = \sum_{e \in E} |x(e) - y(e)| - 1$. We denote by $\mathrm{St}(x, y)$ the set of all the steps from x to y. A *jump system* on E is a pair $(E, \mathcal{F})$ of a nonempty finite set E and a nonempty $\mathcal{F} \subseteq \mathbf{Z}^E$ which satisfies the 2-*step axiom*:

(2-SA) For any $x, y \in \mathcal{F}$ and $s \in \mathrm{St}(x, y)$ with $x + s \notin \mathcal{F}$ there exists $t \in \mathrm{St}(x + s, y)$ such that

$$x + s + t \in \mathcal{F}. \tag{2.3}$$

We see from a result in [3] that an integral bisubmodular polyhedron satisfies the axiom (2-SA). However, it should be noted that $\mathcal{F}$ does not always constitute of all the integral points of its convex hull (see [3]).

3. A Greedy Algorithm

Let $w : \mathbf{R}^E \to \mathbf{R}$ be a separable convex function given by

$$w(x) = \sum_{e \in E} w_e(x(e)), \tag{3.1}$$

where for each $e \in E$ w_e is a convex function on $\mathbf{R}$. Consider a discrete optimization problem described as

$$\begin{aligned} \mathbf{P}: \quad &\text{Minimize} \quad \sum_{e \in E} w_e(x(e)) \\ &\text{subject to} \quad x \in \mathcal{F}, \end{aligned} \tag{3.2}$$

where $(E, \mathcal{F})$ is a finite jump system, i.e., a jump system with a finite $\mathcal{F}$. We describe a greedy algorithm for solving the above problem $\mathbf{P}$. The validity is shown in the next section.

Denote by S the set of all the steps in $\mathbf{Z}^E$.

A greedy algorithm

Input: a finite jump system $(E, \mathcal{F})$, a separable convex function $w : E \to \mathbf{R}$ and a vector $x^0 \in \mathcal{F}$.

Output: an optimal solution x of Problem **P**.

Step 0: Put $x \leftarrow x^0$.

Step 1: If neither of the following two conditions is satisfied, then stop (x is an optimal solution).

(1) There exists a step $s \in S$ such that $x + s \in \mathcal{F}$ and $w(x+s) < w(x)$.

(2) There exist steps $s, t \in S$ such that $x + s \notin \mathcal{F}, x + s + t \in \mathcal{F}$ and $w(x+s+t) < w(x)$.

Step 2: Put

$$w_1 \leftarrow \min\{w(x+s) \,|\, \text{step } s \text{ satisfying Condition (1) in Step 1}\}, \tag{3.3}$$

$$w_2 \leftarrow \min\{w(x+s) \,|\, \text{steps } s, t \text{ satisfying Condition (2) in Step 1}\}, \tag{3.4}$$

where the minimum over the empty set is defined to be $+\infty$.

Put $\hat{w} \leftarrow \min\{w_1, w_2\}$.

If we have $\hat{w} = w_1$, let $\hat{s}$ be the step s that attains the minimum of (3.3), put $x \leftarrow x + \hat{s}$ and go to Step 1.

If $\hat{w} \neq w_1$, let $\hat{s}$ and $\hat{t}$ be the steps s and t that attain the minimum of (3.4), put $x \leftarrow x + \hat{s} + \hat{t}$ and go to Step 1.

(End)

It should be noted that in (3.4) not $w(x+s+t)$ but $w(x+s)$ is minimized and that each step s in the above algorithm is chosen in a greedy way.

Denote by x^k the current x obtained after the kth execution of Step 2 of the greedy algorithm. During the execution of the greedy algorithm, if the current x^k is not an optimal solution, then x^k is changed into either $x^{k+1} \leftarrow x^k + \hat{s}$ or $x^{k+1} \leftarrow x^k + \hat{s} + \hat{t}$ in Step 2. Denote such steps $\hat{s}$ and $\hat{t}$ by s^k and t^k. Then, we have

Remark 3.1: For any $s \in S$ such that $x^k + s \in \mathcal{F}$,

$$w(x^k) - w(x^k + s^k) \geq w(x^k) - w(x^k + s). \tag{3.5}$$

□

Remark 3.2: For any $s, t \in S$ such that $x^k + s \notin \mathcal{F}$, $x^k + s + t \in \mathcal{F}$ and $w(x^k) > w(x^k + s + t)$ we have

$$w(x^k) - w(x^k + s^k) \geq w(x^k) - w(x^k + s). \tag{3.6}$$

□

4. Validity of the Greedy Algorithm

In this section we prove the validity of the greedy algorithm. It should be noted that the algorithm terminates in finitely many steps since $\mathcal{F}$ is finite and the value of the objective function is reduced every time Step 2 is executed. For each step $s \in S$ let $e(s)$ be the element e of E such that $s(e) = 1$ or -1.

Theorem 4.1: *The greedy algorithm described in Section 3 finds an optimal solution of Problem* **P**.

Proof: Let x be the solution found by the greedy algorithm when it terminates.

Claim: Suppose that x is not an optimal solution of Problem **P**. Then, there exists an optimal solution $x^*(\neq x)$ that satisfies the following three conditions:
(i) If $s \in \mathrm{St}(x^*, x)$ and $x^* + s \in \mathcal{F}$, then $w(x^*) < w(x^* + s)$.
(ii) If $s \in \mathrm{St}(x^*, x)$, $t \in \mathrm{St}(x^* + s, x)$, $x^* + s \notin \mathcal{F}$ and $x^* + s + t \in \mathcal{F}$, then $w(x^*) < w(x^* + s + t)$.
(iii) There exists some $s \in \mathrm{St}(x^*, x)$ such that $w(x^*) < w(x^* + s)$.

(Proof of Claim) We can easily find an optimal solution x^* that satisfies (i) and (ii) (see the similar argument in the proof of Theorem 4.1 in [1]).

We will show that this x^* also satisfies (iii). On the contrary, suppose that x^* does not satisfy (iii), i.e., for any $s \in \mathrm{St}(x^*, x)$

$$w(x^* + s) \leq w(x^*). \tag{4.1}$$

We will prove that this leads to a contradiction.

Since $\mathcal{F}$ satisfies (2-SA) and x^* satisfies (i), for any $s \in \mathrm{St}(x^*, x)$ we have $x^* + s \notin \mathcal{F}$ and there exists $s' \in \mathrm{St}(x^* + s, x)$ such that $x^* + s + s' \in \mathcal{F}$. Let s be an element of $\mathrm{St}(x^*, x)$ that satisfies

$$w(x^*) - w(x^* + s) = \max_{t \in \mathrm{St}(x^*, x)} \{w(x^*) - w(x^* + t)\}, \tag{4.2}$$

and choose $s' \in \mathrm{St}(x^* + s, x)$ such that $x^* + s + s' \in \mathcal{F}$.

Let us consider the following [Case a] and [Case b].

[Case a]: $s \neq s'$.
As in the proof of the claim in Theorem 4.1 in [1], we can show that

$$w(x^* + s + s') \leq w(x^*). \tag{4.3}$$

This contradicts the fact that x^* satisfies (ii).

[Case b]: $s = s'$.
From (ii) we must have

$$w(x^*) < w(x^* + 2s). \tag{4.4}$$

Since

$$-s \in \mathrm{St}(x, x^* + s) = \mathrm{St}(x, x^*), \tag{4.5}$$

by the separable convexity of w we have

$$w(x^* + 2s) - w(x^* + s) \leq w(x) - w(x - s). \tag{4.6}$$

It follows from (4.1), (4.4) and (4.6) that

$$0 \leq w(x^*) - w(x^* + s) < w(x^* + 2s) - w(x^* + s) \leq w(x) - w(x - s). \tag{4.7}$$

Since x is the solution found by the greedy algorithm when it terminates, we have $x - s \notin \mathcal{F}$. So, from (4.5) and (2-SA) there exists $-t \in \mathrm{St}(x - s, x^*) = \mathrm{St}(x, x^*)$ such that

$$x - s - t \in \mathcal{F}. \tag{4.8}$$

We have from the separable convexity of w

$$w(x - t) - w(x) \leq w(x^*) - w(x^* + t) \tag{4.9}$$

and from (4.2)

$$w(x^*) - w(x^* + t) \le w(x^*) - w(x^* + s). \tag{4.10}$$

If $s \neq t$, then we have from (4.7), (4.9) and (4.10)

$$\begin{aligned} w(x) - w(x - s - t) &= \{w(x) - w(x - s)\} + \{w(x) - w(x - t)\} \\ &> \{w(x^*) - w(x^* + s)\} - \{w(x^*) - w(x^* + t)\} \\ &\ge 0, \end{aligned} \tag{4.11}$$

where the equality is due to the separability of w. If $s = t$, then from the separable convexity of w, (4.5) and (4.4) we have

$$w(x) - w(x - 2s) \ge w(x^* + 2s) - w(x^*) > 0. \tag{4.12}$$

Both (4.11) and (4.12) contradict the assumption that x is the solution found by the greedy algorithm when it terminates.
(The end of the proof of Claim)

Now, suppose that x is not an optimal solution of Problem **P**. Then, we can choose an optimal solution $x^*(\neq x)$ that satisfies the conditions (i)~(iii) of the above claim.

Let s be an element of $\mathrm{St}(x^*, x)$ that satisfies

$$w(x^* + s) - w(x^*) = \max_{t \in \mathrm{St}(x^*, x)} \{w(x^* + t) - w(x^*)\}. \tag{4.13}$$

As in the proof of Theorem 4.1 in [1], we have

$$x - s \notin \mathcal{F}, \tag{4.14}$$

$$w(x^*) < w(x^* + s), \tag{4.15}$$

$$w(x) - w(x - s) > 0, \tag{4.16}$$

$$w(x - s - t) \ge w(x), \tag{4.17}$$

where $-t \in \mathrm{St}(x - s, x^*)$ such that $x - s - t \in \mathcal{F}$.

Let us consider the following [Case 1] and [Case 2].

[Case 1]: $s = t$.
From (4.16) and (4.17) we have

$$w(x - 2s) > w(x - s). \tag{4.18}$$

Since $-s \in \mathrm{St}(x - s, x^*)$, it follows from (4.18) and the separable convexity of w that

$$w(x^*) - w(x^* + s) \ge w(x - 2s) - w(x - s) > 0, \tag{4.19}$$

which contradicts (4.15).

[Case 2]: $s \neq t$.
As in the proof of Theorem 4.1 in [1], we can show

$$w(x^*) - w(x^* + t) \ge w(x - t) - w(x) > 0, \tag{4.20}$$

and there exists $t' \in \mathrm{St}(x^* + t, x)$ such that $x^* + t + t' \in \mathcal{F}$.

[Case 2] is divided into [Case 2-1] and [Case 2-2].

[Case 2-1]: $t \neq t'$.
As in the proof of Theorem 4.1 in [1], we have $t' \in \mathrm{St}(x^*, x)$ and

$$w(x^* + t') - w(x^*) > w(x^* + s) - w(x^*). \tag{4.21}$$

This contradicts (4.13).
[Case 2-2]: $t = t'$.
From (ii) of the above claim and (4.20) we have

$$w(x^* + 2t) > w(x^* + t). \tag{4.22}$$

Since $t = t' \in \mathrm{St}(x^* + t, x)$, it follows from the separable convexity of w that

$$w(x) - w(x - t) \geq w(x^* + 2t) - w(x^* + t) > 0, \tag{4.23}$$

which contradicts (4.20). This completes [Case 2].
From the arguments for [Case 1] and [Case 2] x must be an optimal solution of Problem **P**. □

Now, we give a characterization of optimal solutions of Problem **P** in terms of a local optimality. A vector $x \in \mathcal{F}$ is called a *local optimal solution* of Problem **P** if the following two hold:
(L1) For any $s \in S$ such that $x + s \in \mathcal{F}$, we have $w(x) \leq w(x + s)$.
(L2) For any $s, t \in S$ such that $x + s \notin \mathcal{F}$ and $x + s + t \in \mathcal{F}$, we have $w(x) \leq w(x + s + t)$.

From Theorem 4.1 we have the following.

Corollary 4.2: *Every local optimal solution of Problem* **P** *is also an optimal solution of Problem* **P**. □

5. Properties of the Greedy Algorithm

During the execution of the greedy algorithm the current x^k, if not optimal, is changed into either $x^{k+1} \leftarrow x^k + \hat{s}$ or $x^{k+1} \leftarrow x^k + \hat{s} + \hat{t}$ ($x^k + \hat{s} \notin \mathcal{F}$) in Step 2. Recall that such steps $\hat{s}$ and $\hat{t}$ are denoted by s^k and t^k. The main purpose of this section is to show the following theorem, which is crucial for the estimation of the number of steps required by our algorithm.

Theorem 5.1: *If x^{k-1} is changed into x^k and further x^k into x^{k+1} successively by the greedy algorithm, then*

$$w(x^{k-1}) - w(x^{k-1} + s^{k-1}) \geq w(x^k) - w(x^k + s^k). \tag{5.1}$$

□

If $\mathcal{F}$ is the set of integral points of an integral polymatroid and the starting point x^0 is the origin $\mathbf{0}$, the greedy algorithm carries out only augmentations and hence (5.1) becomes

$$w(x^{k-1}) - w(x^k) \geq w(x^k) - w(x^{k+1}). \tag{5.2}$$

This is a fundamental property of the incremental greedy algorithm of Federgruen and Groenevelt [7].

First, we show basic properties of our greedy algorithm as Lemmas 5.2~5.6.

Lemma 5.2: *If $x^{k+1} = x^k + s^k + t^k$, then*

$$w(x^k + s^k) \leq w(x^k + t^k). \tag{5.3}$$

Proof: It suffices to consider the case when $\mathrm{e}(s^k) \neq \mathrm{e}(t^k)$. If $x^k + t^k \notin \mathcal{F}$, then from Remark 3.2 we have (5.3). If $x^k + t^k \in \mathcal{F}$, (5.3) is due to the definition of Step 2. □

Lemma 5.3:

$$w(x^k) > w(x^k + s^k). \tag{5.4}$$

Proof: If $x^{k+1} = x^k + s^k$, then (5.4) is trivial by the definition of Step 2. Suppose $x^{k+1} = x^k + s^k + t^k$.

Let us consider the following two cases (note that $t^k \neq -s^k$).

[Case 1]: $t^k \neq s^k$.
Since $w(x^{k+1}) < w(x^k)$, we have

$$\begin{aligned} 0 &> w(x^k + s^k + t^k) - w(x^k) && (5.5)\\ &= \{w(x^k + s^k) - w(x^k)\} + \{w(x^k + t^k) - w(x^k)\} && (5.6)\\ &\geq 2\{w(x^k + s^k) - w(x^k)\}, && (5.7) \end{aligned}$$

where (5.6) is due to the separable convexity of w and (5.7) to Lemma 5.2.

[Case 2]: $t^k = s^k$.
From $w(x^{k+1}) < w(x^k)$, we have

$$w(x^k + 2s^k) < w(x^k). \tag{5.8}$$

From (5.8) and the separable convexity of w we have (5.4). □

Lemma 5.4: *If*

$$\mathrm{e}(s^{k-1}) \neq \mathrm{e}(s^k), \tag{5.9}$$

$$x^{k-1} + s^k \notin \mathcal{F}, \tag{5.10}$$

$$x^{k-1} + s^k + s^{k-1} \in \mathcal{F} \tag{5.11}$$

and

$$w(x^k) - w(x^k + s^k) = w(x^{k-1}) - w(x^{k-1} + s^k), \tag{5.12}$$

then we have (5.1).
Proof: We have

$$\begin{aligned} & w(x^{k-1} + s^k + s^{k-1}) - w(x^{k-1}) \\ &= \{w(x^{k-1} + s^k) - w(x^{k-1})\} + \{w(x^{k-1} + s^{k-1}) - w(x^{k-1})\} && (5.13)\\ &= \{w(x^k + s^k) - w(x^k)\} + \{w(x^{k-1} + s^{k-1}) - w(x^{k-1})\} && (5.14)\\ &< 0, && (5.15) \end{aligned}$$

where (5.13), (5.14) and (5.15) are, respectively, due to (5.9), (5.12) and Lemma 5.3. Then, from Remark 3.2 we have

$$w(x^{k-1}) - w(x^{k-1} + s^k) \leq w(x^{k-1}) - w(x^{k-1} + s^{k-1}). \tag{5.16}$$

Inequality (5.1) follows from (5.12) and (5.16). □

Lemma 5.5: *Suppose* $x^k = x^{k-1} + s^{k-1} + t^{k-1}$. *If*

$$\mathrm{e}(s^{k-1}) \neq \mathrm{e}(s^k),\ \ \mathrm{e}(t^{k-1}) \neq \mathrm{e}(s^k), \tag{5.17}$$

$$\mathrm{e}(s^{k-1}) \neq \mathrm{e}(t^{k-1}), \tag{5.18}$$

$$x^{k-1} + s^k \notin \mathcal{F} \tag{5.19}$$

and

$$x^{k-1} + s^k + t^{k-1} \in \mathcal{F}, \tag{5.20}$$

then we have (5.1).
Proof: It follows from (5.17) that

$$w(x^k) - w(x^k + s^k) = w(x^{k-1}) - w(x^{k-1} + s^k). \tag{5.21}$$

If $w(x^{k-1} + s^k + t^{k-1}) < w(x^{k-1})$, then (5.1) follows from Remark 3.2. Therefore, suppose $w(x^{k-1} + s^k + t^{k-1}) \geq w(x^{k-1})$. Then, we have from $\mathrm{e}(s^k) \neq \mathrm{e}(t^{k-1})$ and the separability of w

$$w(x^{k-1}) - w(x^{k-1} + s^k) \leq w(x^{k-1} + t^{k-1}) - w(x^{k-1}). \tag{5.22}$$

By the definition of the greedy algorithm we have $w(x^{k-1} + s^{k-1} + t^{k-1}) = w(x^k) < w(x^{k-1})$, and hence, from (5.18) and the separability of w we have

$$w(x^{k-1} + t^{k-1}) - w(x^{k-1}) < w(x^{k-1}) - w(x^{k-1} + s^{k-1}). \tag{5.23}$$

From (5.21), (5.22) and (5.23) we have (5.1). □

Lemma 5.6: *Suppose* $x^{k+1} = x^k + s^k + t^k$. *If*

$$x^{k-1} + s^k \notin \mathcal{F}, \tag{5.24}$$

$$x^{k-1} + s^k + t^k \in \mathcal{F}, \tag{5.25}$$

$$w(x^k) - w(x^k + s^k) = w(x^{k-1}) - w(x^{k-1} + s^k) \tag{5.26}$$

and

$$w(x^{k-1} + s^k + t^k) - w(x^{k-1}) = w(x^k + s^k + t^k) - w(x^k), \tag{5.27}$$

then we have (5.1).
Proof: We have

$$w(x^{k-1} + s^k + t^k) - w(x^{k-1}) < 0 \tag{5.28}$$

from (5.27) and the definition of the greedy algorithm. The same argument as in the proof of Lemma 5.4 applies to the rest of the proof, and hence, we have (5.1). □

To show Theorem 5.1 we consider the following five cases, each of which is treated as a lemma as indicated below.

(1) $\mathrm{e}(s^{k-1}) = \mathrm{e}(s^k)$ (Lemma 5.7).
(2) $\mathrm{e}(s^{k-1}) \neq \mathrm{e}(s^k)$.
  (2-1) $x^k = x^{k-1} + s^{k-1}$ and $x^{k+1} = x^k + s^k$ (Lemma 5.8).
  (2-2) $x^k = x^{k-1} + s^{k-1}$ and $x^{k+1} = x^k + s^k + t^k$ (Lemma 5.9).
  (2-3) $x^k = x^{k-1} + s^{k-1} + t^{k-1}$ and $x^{k+1} = x^k + s^k$ (Lemma 5.10).
  (2-4) $x^k = x^{k-1} + s^{k-1} + t^{k-1}$ and $x^{k+1} = x^k + s^k + t^k$ (Lemma 5.11).

Lemma 5.7: *Suppose that x^{k-1} is changed into x^k and further x^k into x^{k+1} by the greedy algorithm. If* $\mathrm{e}(s^{k-1}) = \mathrm{e}(s^k)$ *holds, then we have* (5.1), *i.e.*,

$$w(x^{k-1}) - w(x^{k-1} + s^{k-1}) \geq w(x^k) - w(x^k + s^k). \tag{5.29}$$

Proof: Suppose $s^k = -s^{k-1}$ and let $e = \mathrm{e}(s^{k-1})$. Then, from Lemma 5.3 we have

$$w_e((x^{k-1} + s^{k-1})(e)) < w_e(x^{k-1}(e)), \tag{5.30}$$

$$w_e((x^k - s^{k-1})(e)) < w_e(x^k(e)). \tag{5.31}$$

Furthermore, from the greedy algorithm we have

$$|x^k(e) - x^{k-1}(e)| \leq 2. \tag{5.32}$$

From (5.30) $\sim$ (5.32) and the convexity of w_e we obtain

$$x^k = x^{k-1} + 2s^{k-1}. \tag{5.33}$$

Therefore, we have

$$\begin{aligned} w(x^k) - w(x^k + s^k) &= w(x^k) - w(x^k - s^{k-1}) \\ &= w(x^k) - w(x^{k-1} + s^{k-1}) \\ &< w(x^{k-1}) - w(x^{k-1} + s^{k-1}) \end{aligned} \tag{5.34}$$

as required.

Also, if $s^k = s^{k-1}$, we have (5.1) from $s^{k-1} \in \mathrm{St}(x^{k-1}, x^k)$ and the separable convexity of w. □

Lemma 5.8: *Suppose that x^{k-1} is changed into $x^k = x^{k-1} + s^{k-1}$ and further x^k into $x^{k+1} = x^k + s^k$ by the greedy algorithm. Then,* (5.9) *implies* (5.1).

Proof: From (5.9) and the separability of w we have (5.12). If $x^{k-1} + s^k \in \mathcal{F}$, then from Remark 3.1 we have (5.1). Otherwise, since $x^{k+1} = x^{k-1} + s^{k-1} + s^k \in \mathcal{F}$, we have (5.1) from Lemma 5.4 . □

Lemma 5.9: *Suppose that x^{k-1} is changed into $x^k = x^{k-1} + s^{k-1}$ and further x^k into $x^{k+1} = x^k + s^k + t^k$ ($x^k + s^k \notin \mathcal{F}$) by the greedy algorithm. Then,* (5.9) *implies* (5.1).

Proof: From (5.9) we have (5.12).

First, suppose $t^k = s^{k-1}$. Then, we have $x^{k+1} = x^{k-1} + s^k + 2s^{k-1}$. Hence, $s^k \in \mathrm{St}(x^{k-1}, x^{k+1})$ and $s^{k-1} \in \mathrm{St}(x^{k-1} + s^k, x^{k+1}) = \{s^{k-1}\}$. So, if $x^{k-1} + s^k \notin \mathcal{F}$, then from (2-SA) we have $x^{k-1} + s^k + s^{k-1} \in \mathcal{F}$, which contradicts $x^k + s^k \notin \mathcal{F}$. Therefore, we have $x^{k-1} + s^k \in \mathcal{F}$ and (5.1) follows from Remark 3.1 and (5.12).

Next, suppose $t^k = -s^{k-1}$. Then, $x^{k+1} = x^k + s^k - s^{k-1} = x^{k-1} + s^k \in \mathcal{F}$ and (5.1) follows from Remark 3.1 and (5.12).

Therefore, we can suppose $\mathrm{e}(s^{k-1}) \neq \mathrm{e}(t^k)$. Let us consider the following two cases (i) and (ii).

(i): $x^{k-1} + s^k \in \mathcal{F}$.

In this case, Remark 3.1 and (5.12) give (5.1).

(ii): $x^{k-1} + s^k \notin \mathcal{F}$.

Since $s^k \in \mathrm{St}(x^{k-1}, x^{k+1})$, $\mathrm{St}(x^{k-1} + s^k, x^{k+1}) = \{s^{k-1}, t^k\}$ and $x^{k-1} + s^k + s^{k-1} \notin \mathcal{F}$, (2-SA) implies $x^{k-1} + s^k + t^k \in \mathcal{F}$. Also, since $\mathrm{e}(s^{k-1}) \neq \mathrm{e}(s^k)$ and $\mathrm{e}(s^{k-1}) \neq \mathrm{e}(t^k)$, we have (5.27). Hence, by Lemma 5.6 we have (5.1). □

Lemma 5.10: *Suppose that x^{k-1} is changed into $x^k = x^{k-1} + s^{k-1} + t^{k-1}$ ($x^{k-1} + s^{k-1} \notin \mathcal{F}$) and further x^k into $x^{k+1} = x^k + s^k$ by the greedy algorithm. Then,* (5.9) *implies* (5.1).

Proof: Suppose $\mathrm{e}(s^k) = \mathrm{e}(t^{k-1})$. We have $s^k \neq -t^{k-1}$ since otherwise we have a contradiction: $\mathcal{F} \not\ni x^{k-1} + s^{k-1} = x^{k+1} \in \mathcal{F}$. Hence, we have $s^k = t^{k-1}$. Since $\mathrm{e}(s^{k-1}) \neq \mathrm{e}(s^k)$ (i.e., $\mathrm{e}(s^{k-1}) \neq \mathrm{e}(t^{k-1})$), from Lemma 5.2 and the separable convexity of w we have

$$\begin{aligned}
(0 \ <) \ & w(x^k) - w(x^k + s^k) \\
\leq \ & w(x^k - s^k) - w(x^k) \\
= \ & w(x^k - t^{k-1}) - w(x^k) \\
= \ & w(x^{k-1} + s^{k-1}) - w(x^{k-1} + s^{k-1} + t^{k-1}) \\
= \ & w(x^{k-1}) - w(x^{k-1} + t^{k-1}) \\
\leq \ & w(x^{k-1}) - w(x^{k-1} + s^{k-1})
\end{aligned} \tag{5.35}$$

as desired.

Therefore, we can also suppose $\mathrm{e}(t^{k-1}) \neq \mathrm{e}(s^k)$. Then, we have (5.12). Let us consider the following two cases (i) and (ii).

(i): $x^{k-1} + s^k \in \mathcal{F}$.

In this case, from Remark 3.1 we have (5.1).

(ii): $x^{k-1} + s^k \notin \mathcal{F}$.

Since $s^k \in \mathrm{St}(x^{k-1}, x^{k+1})$ and $\mathrm{St}(x^{k-1} + s^k, x^{k+1}) = \{s^{k-1}, t^{k-1}\}$, from (2-SA) we have the following two subcases.

(ii-1): $x^{k-1} + s^k + s^{k-1} \in \mathcal{F}$.

In this case, (5.12) and Lemma 5.4 yield (5.1).

(ii-2): $x^{k-1} + s^k + t^{k-1} \in \mathcal{F}$.

If $t^{k-1} = s^{k-1}$, Case (ii-1) applies. Therefore, suppose $\mathrm{e}(t^{k-1}) \neq \mathrm{e}(s^{k-1})$. Then, we have (5.1) from Lemma 5.5. □

Lemma 5.11: *Suppose that x^{k-1} is changed into $x^k = x^{k-1} + s^{k-1} + t^{k-1}$ ($x^{k-1} + s^{k-1} \notin \mathcal{F}$) and further x^k into $x^{k+1} = x^k + s^k + t^k$ ($x^k + s^k \notin \mathcal{F}$) by the greedy algorithm. Then,* (5.9) *implies* (5.1).

Proof: If $s^k = t^{k-1}$, then by the same argument in Lemma 5.10 we have (5.1).

Moreover, if $s^k = -t^{k-1}$, it follows from the greedy algorithm that

$$\begin{aligned}
w(x^k) - w(x^k + s^k) \ &= \ w(x^k) - w(x^k - t^{k-1}) \\
&= \ w(x^{k-1} + s^{k-1} + t^{k-1}) - w(x^{k-1} + s^{k-1}) \\
&< \ w(x^{k-1}) - w(x^{k-1} + s^{k-1})
\end{aligned} \tag{5.36}$$

as desired.

Hence, we can also suppose $\mathrm{e}(t^{k-1}) \neq \mathrm{e}(s^k)$, which gives (5.12). Let us consider the following two cases (i) and (ii).

(i): $x^{k-1} + s^k \in \mathcal{F}$.

In this case, from Remark 3.1 we have (5.1).

(ii): $x^{k-1} + s^k \notin \mathcal{F}$.

Since $s^k \in \mathrm{St}(x^{k-1}, x^{k+1})$ and $\mathrm{St}(x^{k-1} + s^k, x^{k+1}) \subseteq \{s^{k-1}, t^{k-1}, t^k\}$, from (2-SA) we have the following two subcases.

(ii-1): $x^{k-1} + s^k + s^{k-1} \in \mathcal{F}$ or $x^{k-1} + s^k + t^{k-1} \in \mathcal{F}$.

We have (5.1) by Lemma 5.4 or Lemma 5.5 as in Case (ii) in the proof of Lemma 5.10.

(ii-2): $x^{k-1}+s^k+t^k \in \mathcal{F}$.

If $t^k \in \{s^{k-1}, t^{k-1}\}$, then Case (ii-1) applies. If $t^k = -s^{k-1}$, then $x^{k+1} = x^k + s^k - s^{k-1} = x^{k-1} + s^k + t^{k-1}$. If $t^k = -t^{k-1}$, then $x^{k+1} = x^k + s^k - t^{k-1} = x^{k-1} + s^k + s^{k-1}$. Hence, if $t^k \in \{-s^{k-1}, -t^{k-1}\}$, then again Case (ii-1) applies. Therefore, we can also suppose that $\mathrm{e}(t^k) \neq \mathrm{e}(s^{k-1})$ and $\mathrm{e}(t^k) \neq \mathrm{e}(t^{k-1})$. Then, we have (5.27) and hence, we have (5.1) by Lemma 5.6. □

From Lemmas 5.7 ∼ 5.11 we have Theorem 5.1.

6. The Main Result

For each $e \in E$ let

$$l(e) = \min_{x \in \mathcal{F}} x(e) \tag{6.1}$$

and

$$u(e) = \max_{x \in \mathcal{F}} x(e). \tag{6.2}$$

We will show the following theorem, which gives an upper bound of the number of the required transformations of feasible solutions.

Theorem 6.1: *The greedy algorithm executes Step 2 at most*

$$\sum_{e \in E} \left\{ u(e) - l(e) \right\} \tag{6.3}$$

times. □

This theorem can be shown by using the following two lemmas. Suppose that starting with an initial solution x^0, the algorithm terminates with x^n. Recall that x^k is changed into $x^{k+1} \leftarrow x^k + s^k$ or $x^{k+1} \leftarrow x^k + s^k + t^k$ for $0 \leq k < n$.

Let us consider the following sequence of ordered pairs

$$c^k = (x^k(\mathrm{e}(s^k)), s^k) \quad (0 \leq k < n). \tag{6.4}$$

For example, if $s^k = -\chi_e$, then $c^k = (x^k(e), -\chi_e)$, where χ_e is the unit vector with $\chi_e(e) = 1$ and $\chi_e(e') = 0$ $(e' \in E - \{e\})$. Denote by C the set of all the c^k $(0 \leq k < n)$, i.e.,

$$C = \{c^k \mid 0 \leq k < n\}. \tag{6.5}$$

Then we have

Lemma 6.2:

$$|C| \leq \sum_{e \in E} \left\{ u(e) - l(e) \right\}. \tag{6.6}$$

Proof: Suppose $(\alpha, s) \in C$. It follows from the greedy algorithm and the separable convexity of w that

- **(C1)** the pair of α and $\mathrm{e}(s)$ uniquely determines s, which is either $\chi_{\mathrm{e}(s)}$ or $-\chi_{\mathrm{e}(s)}$,
- **(C2)** $l(\mathrm{e}(s)) \leq \alpha \leq u(\mathrm{e}(s))$,
- **(C3)** α is not a minimizer of $w_{\mathrm{e}(s)}$ on the interval $[l(\mathrm{e}(s)), u(\mathrm{e}(s))]$.

Hence, we have (6.6). □

Lemma 6.3: *Let* $0 \leq j, h < n$. *If* $j \neq h$, *then* $c^j \neq c^h$.

Proof: Suppose that for some $c = (\alpha, s) \in C$ and some j, h with $0 \leq j < h < n$, we have

$$\alpha = x^j(\mathrm{e}(s)) = x^h(\mathrm{e}(s)), \tag{6.7}$$

$$s = s^j = s^h. \tag{6.8}$$

Put $e = \mathrm{e}(s)$ and suppose $s = \chi_e$ without loss of generality. We will show a contradiction. By the present assumption,

(1) when x^j is changed into x^{j+1}, $\alpha = x^j(e)$ is changed into $x^j(e) + 1$ (or $x^j(e) + 2$), and

(2) when x^h is changed into x^{h+1}, $\alpha = x^h(e)$ is changed into $x^h(e) + 1$ (or $x^h(e) + 2$).

Hence, for some k with $j < k < h$, when x^k is changed into x^{k+1}, we have

(3) $x^k(e) = \alpha + 1$ is changed into $x^{k+1}(e) = \alpha$,

or

(3′) $x^k(e) = \alpha + 2$ is changed into $x^{k+1}(e) = \alpha$,

or

(3″) $x^k(e) = \alpha + 1$ is changed into $x^{k+1}(e) = \alpha - 1$.

From Lemma 5.3 and the separability of w we have

$$w_e(\alpha + 1) < w_e(\alpha). \tag{6.9}$$

Let us consider the following three cases.

[Case 1]: (3) holds.

In this case, (6.9) implies

$$x^{k+1} = x^k + s^k - \chi_e = x^k + s^k - s. \tag{6.10}$$

Also, from the greedy algorithm we have

$$w(x^k + s^k - \chi_e) < w(x^k), \tag{6.11}$$

i.e.,

$$w(x^k - \chi_e) - w(x^k) < w(x^k) - w(x^k + s^k), \tag{6.12}$$

where note that $e \neq \mathrm{e}(s^k)$. Furthermore, from (1) and (3), we have

$$\begin{aligned} w(x^j) - w(x^j + s^j) &= w_e(x^j(e)) - w_e(x^j(e) + 1) \\ &= w_e(\alpha) - w_e(\alpha + 1) \\ &= w(x^k - \chi_e) - w(x^k). \end{aligned} \tag{6.13}$$

Therefore, from (6.12) and (6.13) we have

$$w(x^j) - w(x^j + s^j) < w(x^k) - w(x^k + s^k). \tag{6.14}$$

This contradicts Theorem 5.1.

[Case 2]: (3′) holds.

(3′) implies

$$x^{k+1} = x^k - 2\chi_e \tag{6.15}$$

and

$$s^k = -\chi_e. \tag{6.16}$$

Also, we have

$$w_e(\alpha) < w_e(\alpha + 2). \tag{6.17}$$

It follows from (1), (3′), (6.16) and (6.17) that

$$\begin{aligned} w(x^j) - w(x^j + s^j) &= w_e(x^j(e)) - w_e(x^j(e) + 1) \\ &= w_e(\alpha) - w_e(\alpha + 1) \\ &< w_e(\alpha + 2) - w_e(\alpha + 1) \\ &= w(x^k) - w(x^k - \chi_e) \\ &= w(x^k) - w(x^k + s^k). \end{aligned} \tag{6.18}$$

This also contradicts Theorem 5.1.

[Case 3]: (3″) holds.

From (6.9) and the separable convexity of w we have

$$w_e(\alpha + 1) < w_e(\alpha - 1). \tag{6.19}$$

From (3″) we have

$$w(x^k) < w(x^{k+1}). \tag{6.20}$$

This contradicts the definition of the greedy algorithm.

From Cases 1, 2 and 3, for any integers j and h with $0 \leq j < h < n$ we have $c^j \neq c^h$. □

From Lemmas 6.2 and 6.3 we have Theorem 6.1.

Based on the results of Bouchet and Cunningham [3], we have the following theorem.

Theorem 6.4 ([3]): *For a finite jump system $(E, \mathcal{F})$ the convex hull of $\mathcal{F}$ coincides with a bisubmodular polyhedron in $\mathbf{R}^E$.* □

It follows from Theorem 6.4 that for a given finite jump system $(E, \mathcal{F})$ there exists a bisubmodular function f such that the convex hull of $\mathcal{F}$ is given by

$$\mathrm{Co}(\mathcal{F}) = \{x \mid x \in \mathbf{R}^E, \forall (X, Y) \in 3^E : x(X) - x(Y) \leq f(X, Y)\}. \tag{6.21}$$

By the use of such a bisubmodular function f we can also express the upper bound (6.3) given in Theorem 6.1 as

$$\sum_{e \in E} \big\{ f(\{e\}, \emptyset) + f(\emptyset, \{e\}) \big\}. \tag{6.22}$$

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References

[1] K. Ando, S. Fujishige and T. Naitoh: A greedy algorithm for minimizing a separable convex function over an integral bisubmodular polyhedron. *Journal of Operations Research Society of Japan* **37** (1994) 188-196.

[2] A. Bouchet: Greedy algorithm and symmetric matroids. *Mathematical Programming* **38** (1987) 147-159.

[3] A. Bouchet and W. H. Cunningham: Delta-matroids, jump systems and bisubmodular polyhedra. *SIAM Journal on Discrete Mathematics* **8** (1995) 17-32.

[4] R. Chandrasekaran and S. N. Kabadi: Pseudomatroids. *Discrete Mathematics* **71** (1988) 205-217.

[5] W. H. Cunningham and J. Green-Krótki: b-matching degree-sequence polyhedra. *Combinatorica* **11** (1991) 219-230.

[6] J. Edmonds: Submodular functions, matroids, and certain polyhedra. In: *Combinatorial Structures and Their Applications* (R. K. Guy et al., eds., Gordon and Breach, New York, 1970), pp. 69-87.

[7] A. Federgruen and H. Groenevelt: The greedy procedure for resource allocation problems — Necessary and sufficient conditions for optimality. *Operations Research* **34** (1986) 909-918.

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