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STATIONARY INDEX AND ORIENTATION OF EQUALITY CONSTRAINED MULTIPARAMETRIC NONLINEAR PROGRAMS

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Abstract In this paper, we study a local property of the zero set of a differentiable map $F : R^{n+d} \rightarrow R^n$. We prove that, under a regular value condition, for each $\boldsymbol{x} \in F^{-1}(\boldsymbol{o})$, there exist a neighborhood U of $\boldsymbol{x}$ and a sign $c \in \{-1,1\}$ such that *sign* $det[\dot{\boldsymbol{x}}(\boldsymbol{p})^{\top}_{\sigma_d}] = c \cdot sgn\sigma \cdot sign$ det $D_xF(\boldsymbol{x}(\boldsymbol{p}))_{\sigma^n}$ for all permutation σ of degree $(n+d)$, where $\boldsymbol{p}$ is a d-dimensional parametrization parameter vector of the zero set $F^{-1}(\boldsymbol{o})$ in an open subset V of $\boldsymbol{R}^d$ and $[\dot{\boldsymbol{x}}(\boldsymbol{p})^{\top}_{\sigma_d}] := (\partial x_j/\partial p_l)^{\top}(j \in \sigma^{-1}(n+1,\cdots,n+d), l \in \{1,\cdots,d\}), D_xF(\boldsymbol{x}(\boldsymbol{p}))_{\sigma^n} = [\partial F_i(\boldsymbol{x}(\boldsymbol{p}))/\partial x_k](i \in \{1,\cdots,n\}, k \in \sigma^{-1}(1,\cdots,n))$. This results naturally leads to an index theory. We show a local property of the change of the Morse index and the orientation of critical point set w.r.t. the multiparametric function $f : \boldsymbol{R}^{n+d} \rightarrow \boldsymbol{R}$. Finally, we discuss the change of the stationary index of the equality constrained multiparametric nonlinear programs.

1 Introduction.

In this paper, we discuss a local property of the zero set of a differentiable map $F : \mathbb{R}^{n+d} \rightarrow \mathbb{R}^n$.

For the case $d = 1$, we can see many previous studies, so-called "homotopy methods". For solving nonlinear equations $\boldsymbol{f}(\boldsymbol{x}) = \mathbf{o}$ where $\boldsymbol{f} : \mathbb{R}^n \rightarrow \mathbb{R}^n$, we embed $\boldsymbol{f}$ into a one-parameter family of homotopy equations $H(\boldsymbol{x},t) = \mathbf{o}$, for example $H(\boldsymbol{x},t) = \boldsymbol{f}(\boldsymbol{x}) - t\boldsymbol{f}(\boldsymbol{x}_0)$ or $H(\boldsymbol{x},t) = t(\boldsymbol{x} - \boldsymbol{x}_0) + (1-t)\boldsymbol{f}(\boldsymbol{x})$. Some basic and essential results have been given by Garcia and Gould [3]. For method of complementary pivoting, it is in a limiting, we can see [2, 18, 19].

In this paper, we treat of the case that $d \geq 2$. We prove that, under a regular value condition, for each $\boldsymbol{x} \in F^{-1}(\mathbf{o})$, there exist a neighborhood U of $\boldsymbol{x}$ and a sign $c \in \{-1,1\}$ such that

$$sign\ det[\dot{\boldsymbol{x}}(\boldsymbol{p})^{\top}_{\sigma_d}] = c \cdot sgn\sigma \cdot sign\ det D_{\boldsymbol{x}}F(\boldsymbol{x}(\boldsymbol{p}))_{\sigma^n}$$

for all permutation σ of degree $(n+d)$, where $\boldsymbol{p}$ is a d-dimensional parametrization parameter vector of the zero set $F^{-1}(\mathbf{o})$ in an open subset V of $\mathbb{R}^d$, *i.e.*, there is a parametrization map $\boldsymbol{x}(\cdot)$ defined on V such that $\boldsymbol{x}(V) = U \cap F^{-1}(\mathbf{o})$) and

$$[\dot{\boldsymbol{x}}(\boldsymbol{p})^{\top}_{\sigma_d}] := (\partial x_j/\partial p_l)^{\top}\ (j \in \sigma^{-1}(n+1,\cdots,n+d) \text{ and } l \in \{1,\cdots,d\}),$$
$$D_{\boldsymbol{x}}F(\boldsymbol{x}(\boldsymbol{p}))_{\sigma^n} := [\partial F_i(\boldsymbol{x}(\boldsymbol{p}))/\partial x_k]\ (i \in \{1,\cdots,n\} \text{ and } k \in \sigma^{-1}(1,\cdots,n).$$

This result naturally leads to an index theory.

As an application of the result above, we deal with the equality constrained multiparametric nonlinear programs. Multiparametric nonlinear programs are concerned with the analysis of the behavior of stationary solutions under perturbation [7, 14] and also mathematical economics, *e.g.*, Pareto optimality [23] and [24, Section 6]. We show the behavior of stationary solutions and the change of the stationary index which characterizes a property of stationary solution, such as a local minimum, a saddle solution or a local maximum.

The organization of this paper is as follows. In Section 2, we prove the main theorem (Theorem 2.3) and make a trivial extension. In Section 3, we discuss a local characterization of the change of the Morse index w.r.t. multiparametric functions. In Section 4, we deal with the equality constrained multiparametric nonlinear programs and show the change of stationary index (Theorem 4.3) as an application of Sections 2 and 3.

2 Zero Set of System of Differentiable Equations.

Let us consider a continuously differentiable (C^1-)map $F : \mathbb{R}^{n+d} \to \mathbb{R}^n$. Given $\boldsymbol{c} \in \mathbb{R}^n$, let

$$F^{-1}(\boldsymbol{c}) := \{\boldsymbol{x} \in \mathbb{R}^{n+d} : F(\boldsymbol{x}) = \boldsymbol{c}\},$$
$$\Gamma := \{\boldsymbol{x} \in \mathbb{R}^{n+d} : rank D_{\boldsymbol{x}} F(\boldsymbol{x}) < n\},$$

where $D_{\boldsymbol{x}}F$ is the $n \times (n+d)$ Jacobian matrix $[\partial F_i/\partial x_j]$ of F w.r.t. $\boldsymbol{x} \in \mathbb{R}^{n+d}$. The set Γ is said to be the set of *critical points* of F, and $F(\Gamma)$ the set of *critical values* of F. $\mathbb{R}^n \backslash F(\Gamma)$ is the set of *regular values* of F.

Lemma 2.1 (see [8, Theorem 3.2], also [10, page 100]): *Let $F : \mathbb{R}^{n+d} \to \mathbb{R}^n$ be a C^1-map and $\mathbf{o} \in \mathbb{R}^n$ a regular value of F. Then $F^{-1}(\mathbf{o})$ is a d-dimensional C^1-orientable manifold.* ∎

Thus, if F is C^1-map and $\mathbf{o} \in \mathbb{R}^n$ is a regular value of F, for $\boldsymbol{x} \in F^{-1}(\mathbf{o})$ there exists an open neighborhood U of $\boldsymbol{x}$ (w.r.t. $\mathbb{R}^{n+d}$) such that $U \cap F^{-1}(\mathbf{o})$ is C^1-diffeomorphic to some open subset V of $\mathbb{R}^d$, *i.e.*, $U \cap F^{-1}(\mathbf{o})$ is described by $\boldsymbol{x}(\boldsymbol{p})$ such that $\boldsymbol{x} : V \to U$ and whose derivative, denoted by $[\dot{\boldsymbol{x}}(\boldsymbol{p})] := (\partial x_j/\partial p_l)$, is of full rank ($= d$). Furthermore, for any $\boldsymbol{p} \in V$ such that $\boldsymbol{x}(\boldsymbol{p}) \in U \cap F^{-1}(\mathbf{o})$, we have $rank D_{\boldsymbol{x}} F(\boldsymbol{x}(\boldsymbol{p})) = n$ and $D_{\boldsymbol{x}} F(\boldsymbol{x}(\boldsymbol{p})) \cdot [\dot{\boldsymbol{x}}(\boldsymbol{p})] = \mathbf{o}$.

Remark 2.2. *By the Morse-Sard Theorem [8, Theorem 1.3], if $F : \mathbb{R}^{n+d} \to \mathbb{R}^n$ is a $(d+1)$-times continuously differentiable (C^{d+1})-map, then $F(\Gamma)$ has measure zero and $\mathbb{R}^n \backslash F(\Gamma)$ is residual. Therefore, for a sufficiently smooth map F, we may assume that $\mathbf{o}$ is a regular value of F.*

The following theorem is related to Theorem 2 of Garcia and Gould [3] for one parametric case.

Theorem 2.3. *Let $F : \mathbb{R}^{n+d} \to \mathbb{R}^n$ be a C^1-map and $\mathbf{o} \in \mathbb{R}^n$ be a regular value of F. Let U of $\boldsymbol{x}$ and V of $\boldsymbol{p}$ are as above. For each $\boldsymbol{p} \in V$,*

$$corank[\dot{\boldsymbol{x}}(\boldsymbol{p})^{\top}_{\sigma_d}] = corank D_{\boldsymbol{x}} F(\boldsymbol{x}(\boldsymbol{p}))_{\sigma^n},$$

where $corank \boldsymbol{A} := \min\{n, m\} - rank \boldsymbol{A}$ for an $(n \times m)$-matrix $\boldsymbol{A}$.

Moreover, there exists a sign $c \in \{-1, 1\}$ such that

$$sign\ det[\dot{\boldsymbol{x}}(\boldsymbol{p})^{\top}_{\sigma_d}] = c \cdot sgn\sigma \cdot sign\ det D_{\boldsymbol{x}} F(\boldsymbol{x}(\boldsymbol{p}))_{\sigma^n}$$

for any $\boldsymbol{p} \in V$ and any permutation σ of degree $(n+d)$.

Proof. It is clear by Lemmas (A), (B) of Appendix and the fact that, since the matrix $\begin{bmatrix} D_{\boldsymbol{x}} F(\boldsymbol{x}(\boldsymbol{p})) \\ \dot{\boldsymbol{x}}(\boldsymbol{p})^{\top} \end{bmatrix}$ is nonsingular, the sign of its determinant is constant on V. ∎

Corollary 2.4. *Let $F : \mathbb{R}^{n+d} \to \mathbb{R}^n$ be a C^1-map and $\mathbf{o} \in \mathbb{R}^n$ a regular value of F. Suppose for some subset $\{j_1, \cdots, j_d\}$ of $\{1, 2, \cdots, n+d\}$, the submatrix of $D_{\boldsymbol{x}}F(\boldsymbol{x})$ defined by*

$$[\partial F_i(\boldsymbol{x})/\partial x_j] \ (i \in \{1, \cdots, n\}, j \in \{1, \cdots, n+d\} \backslash \{j_1, \cdots, j_d\})$$

is nonsingular for all $\boldsymbol{x}$ in a particular open subset U. Then, $U \cap F^{-1}(\mathbf{o})$ can be parameterized by $\{x_{j_1}, \cdots, x_{j_d}\}$, in the sense of diffeomorphy.

Proof. By Theorem 2.3, $(\partial x_{j_k}/\partial p_l)^\top$ $(k \in \{1, \cdots, d\})$ is nonsingular on U. Hence, the assertion is clear. ∎

3 Morse Index and Orientation.

By Theorem 2.3, we can locally characterize a property of the change of $sign\ det D_{\boldsymbol{x}}F(\boldsymbol{x})_{\sigma^n}$ by the change of $sign\ det[\dot{\boldsymbol{x}}(\boldsymbol{p})^\top_{\sigma_d}]$.

Let $f : \mathbb{R}^{n+d} \to \mathbb{R} : (\boldsymbol{x}, \boldsymbol{t}) \mapsto z$ be a C^2-multiparametric function, where $\boldsymbol{x} \in \mathbb{R}^n$ is a variable vector, $\boldsymbol{t} \in \mathbb{R}^d$ a parameter vector. Let

$$\begin{aligned} \Gamma_{\boldsymbol{x}}(f) &:= \{(\boldsymbol{x}, \boldsymbol{t}) \in \mathbb{R}^{n+d} : D_{\boldsymbol{x}}f(\boldsymbol{x}, \boldsymbol{t}) = \mathbf{o}\} && : \text{a } \textit{critical point set} \text{ of } f, \\ \Gamma_{\boldsymbol{x}}(D_{\boldsymbol{x}}f) &:= \{(\boldsymbol{x}, \boldsymbol{t}) \in \Gamma_{\boldsymbol{x}}(f) : rank D^2_{\boldsymbol{x}}f(\boldsymbol{x}, \boldsymbol{t}) < n\} && : \text{a } \textit{degenerate critical point set} \text{ of } f. \end{aligned}$$

$\Gamma_{\boldsymbol{x}}(f) \backslash \Gamma_{\boldsymbol{x}}(D_{\boldsymbol{x}}f)$ is said to be a *nondegenerate critical point set* of f. For each critical point $(\boldsymbol{x}^*, \boldsymbol{t}^*) \in \Gamma_{\boldsymbol{x}}(f)$, we define the *Morse index* to be the number of negative eigenvalues of $D^2_{\boldsymbol{x}}f(\boldsymbol{x}^*, \boldsymbol{t}^*)$ and denote it by $index(\boldsymbol{x}^* | f(\cdot, \boldsymbol{t}^*))$, see [20].

By Lemma 2.1 and the fact that $D_{\boldsymbol{x}}f : \mathbb{R}^{n+d} \to \mathbb{R}^n$ is a C^1-map, if $\mathbf{o}$ is a regular value of $D_{\boldsymbol{x}}f$, *i.e.*, $rank D_{(\boldsymbol{x},\boldsymbol{t})}D_{\boldsymbol{x}}f(\boldsymbol{x}, \boldsymbol{t})$ is of full rank for all $(\boldsymbol{x}, \boldsymbol{t})$ in the critical point set $\Gamma_{\boldsymbol{x}}(f)$, then $\Gamma_{\boldsymbol{x}}(f)$ is a d-dimensional differentiable manifold. Thus, as the previous section, $\Gamma_{\boldsymbol{x}}(f)$ can be locally parameterized by $\boldsymbol{p} \in \mathbb{R}^d$ such that $[\dot{\boldsymbol{x}}(\boldsymbol{p})^\top | \dot{\boldsymbol{t}}(\boldsymbol{p})^\top]$ is of full rank, *i.e.*, $rank[\dot{\boldsymbol{x}}(\boldsymbol{p})^\top | \dot{\boldsymbol{t}}(\boldsymbol{p})^\top] = d$.

Theorem 3.1. *Let $f : \mathbb{R}^{n+d} \to \mathbb{R}$ be a C^2-map. Suppose that $\mathbf{o} \in \mathbb{R}^n$ is a regular value of $D_{\boldsymbol{x}}f$. If $(\boldsymbol{x}^*, \boldsymbol{t}^*) \in \Gamma_{\boldsymbol{x}}(f)$ is a degenerate critical point of f ($(\boldsymbol{x}^*, \boldsymbol{t}^*) \in \Gamma_{\boldsymbol{x}}(D_{\boldsymbol{x}}f)$), then the Morse index can locally change at most $corank[\dot{\boldsymbol{t}}(\boldsymbol{p}^*)] (\le d)$. To be concrete, there are open sets V of $\mathbb{R}^d$, U of $\mathbb{R}^{n+d}$ and a C^1-parametrization $(\boldsymbol{x}(\cdot), \boldsymbol{t}(\cdot)) : V \to U$ such that $U \cap \Gamma_{\boldsymbol{x}}(f) = \{(\boldsymbol{x}(\boldsymbol{p}), \boldsymbol{t}(\boldsymbol{p})) | \boldsymbol{p} \in V\}$ and $(\boldsymbol{x}(\boldsymbol{p}^*), \boldsymbol{t}(\boldsymbol{p}^*)) = (\boldsymbol{x}^*, \boldsymbol{t}^*)$ for some $\boldsymbol{p}^* \in V$ and*

$$m \le index(\boldsymbol{x}(\boldsymbol{p}) | f(\cdot, \boldsymbol{t}(\boldsymbol{p}))) \le m + corank[\dot{\boldsymbol{t}}(\boldsymbol{p}^*)]$$

for any $\boldsymbol{p} \in V$ where m is the number of the negative eigenvalues of $D^2_{\boldsymbol{x}}f(\boldsymbol{x}(\boldsymbol{p}^*), \boldsymbol{t}(\boldsymbol{p}^*))$.

Moreover, there exists a sign $c \in \{-1, 1\}$ such that

$$(-1)^{index(\boldsymbol{x}(\boldsymbol{p})|f(\cdot,\boldsymbol{t}(\boldsymbol{p})))} = c \cdot sign\ det[\dot{\boldsymbol{t}}(\boldsymbol{p})]$$

for any $(\boldsymbol{x}(\boldsymbol{p}), \boldsymbol{t}(\boldsymbol{p})) \in U \cap [\Gamma_{\boldsymbol{x}}(f) \backslash \Gamma_{\boldsymbol{x}}(D_{\boldsymbol{x}}f)]$.

Proof. The first assertion is obvious since $corank[\dot{\boldsymbol{t}}(\boldsymbol{p}^*)] = corank D^2_{\boldsymbol{x}}f(\boldsymbol{x}(\boldsymbol{p}^*), \boldsymbol{t}(\boldsymbol{p}^*))$ and the continuity of the elements of the matrix. Remark that

$$(-1)^{index(\boldsymbol{x}(\boldsymbol{p})|f(\cdot,\boldsymbol{t}(\boldsymbol{p})))} = sign\ det D^2_{\boldsymbol{x}}f(\boldsymbol{x}, \boldsymbol{t}).$$

By Theorem 2.3, $sign\ det[\dot{\boldsymbol{t}}(\boldsymbol{p})] = c \cdot sign\ det D^2_{\boldsymbol{x}}f(\boldsymbol{x}, \boldsymbol{t})$. Then the assertion is clear. ∎

Of course, the above formulation

$$m \le index(\boldsymbol{x}(\boldsymbol{p}) | f(\cdot, \boldsymbol{t}(\boldsymbol{p}))) \le m + corank[\dot{\boldsymbol{t}}(\boldsymbol{p}^*)]$$

is also true in the case that $(\boldsymbol{x}, \boldsymbol{t})$ is a nondegenerate critical point. In this case,

$$index(\boldsymbol{x}(\boldsymbol{p}) | f(\cdot, \boldsymbol{t}(\boldsymbol{p}))) = m,$$

i.e., the Morse index is locally constant around a nondegenerate critical point.

4 Application to Equality Constrained Multiparametric Nonlinear Programs.

In this section, we deal with the equality constrained multiparametric nonlinear programs as an application of Sections 2 and 3.

Multiparametric nonlinear programs are concerned with the analysis of the behavior of stationary solutions under data perturbations and also mathematical economics, *e.g.*, Pareto optimality [23] and [24, Section 6].

There are many papers treating of the one-parametric nonlinear programs, *e.g.*, [5, 9, 11, 12, 13, 15, 17, 21, 22, 27]. But for the multiparametric nonlinear programs, the situation becomes much more complicated and there are few papers, *e.g.*, [7, 14, 24].

The multiparametric nonlinear programs with inequality constraints (and equality constraints) are complicated, see [14, 17]. Hence, we treat of the multiparametric nonlinear programs with "equality constraints" only. To be concrete, we deal with the following:

$$\mathcal{NLP}(\boldsymbol{f}(\cdot,\boldsymbol{t})): \begin{array}{ll} \text{minimize} & f_0(\boldsymbol{x},\boldsymbol{t}) \\ \text{such that} & \boldsymbol{x} \in X(\boldsymbol{t}), \end{array}$$

where $\boldsymbol{x} \in \mathbb{R}^n$ is a variable vector, $\boldsymbol{t} \in \mathbb{R}^d$ a parameter vector, $X(\boldsymbol{t}) := \{\boldsymbol{x} \in \mathbb{R}^n : f_e(\boldsymbol{x},\boldsymbol{t}) = 0 \ (e \in E)\}$ a feasible set at a parameter $\boldsymbol{t} \in \mathbb{R}^d$, $|E| < \infty$ and f_0, f_e $(e \in E)$ C^2-functions from $\mathbb{R}^{n+d}$ to $\mathbb{R}$. We know that, under a certain kind of constraint qualifications, a local minimum becomes the so-called Karush-Kuhn-Tucker stationary solution (shortly, stationary solution).

The aim of this section is to discuss the change of the stationary index on the stationary solution set Σ:

$$\begin{aligned} \Sigma &:= \{(\boldsymbol{x},\boldsymbol{t}) \in \mathbb{R}^{n+d} : \boldsymbol{x} \text{ be a Karush-Kuhn-Tucker solution to } \mathcal{NLP}(\boldsymbol{f}(\cdot,\boldsymbol{t}))\} \\ &= \{(\boldsymbol{x},\boldsymbol{t}) \in \mathbb{R}^{n+d} : \boldsymbol{x} \in X(\boldsymbol{t}),\ D_{\boldsymbol{x}} f_0(\boldsymbol{x},\boldsymbol{t}) = \sum_{e\in E} y_e D_{\boldsymbol{x}} f_e(\boldsymbol{x},\boldsymbol{t}) \text{ with } \exists \boldsymbol{y} \in \mathbb{R}^{|E|}\}. \end{aligned}$$

For the purpose, we consider another set Π, the set of all such triples $(\boldsymbol{x},\boldsymbol{y},\boldsymbol{t})$, *i.e.*,

$$\Pi \ := \ \{(\boldsymbol{x},\boldsymbol{y},\boldsymbol{t}) \in \mathbb{R}^{n+|E|+d} : D_{\boldsymbol{x}} f_0(\boldsymbol{x},\boldsymbol{t}) = \sum_{e\in E} y_e D_{\boldsymbol{x}} f_e(\boldsymbol{x},\boldsymbol{t})\}.$$

We call a triple $(\boldsymbol{x},\boldsymbol{y},\boldsymbol{t}) \in \Pi$ a stationary point and Π the stationary point set to $\mathcal{NLP}(\boldsymbol{f}(\cdot,\boldsymbol{t}))$. Using Kojima equation [16, Equation (1-2)], the set Π is equal to the zero set of the following C^1-map

$$F(\boldsymbol{x},\boldsymbol{y},\boldsymbol{t}) := \begin{pmatrix} D_{\boldsymbol{x}} f_0(\boldsymbol{x},\boldsymbol{t}) - \sum_{e\in E} y_e D_{\boldsymbol{x}} f_e(\boldsymbol{x},\boldsymbol{t}) \\ f_e(\boldsymbol{x},\boldsymbol{t}) \ (e \in E) \end{pmatrix},$$

i.e., $\Pi = F^{-1}(\mathbf{o})$. Note that $F(\boldsymbol{x},\boldsymbol{y},\boldsymbol{t})$ is the derivative of the Lagrange function w.r.t. $(\boldsymbol{x},\boldsymbol{y})$ where $\boldsymbol{y}$ denotes the associated Lagrange multiplier vector.

From now on, we assume the following condition;

Regular Value Condition

$\mathbf{o} \in \mathbb{R}^{n+|E|}$ is a regular value of the map F, *i.e.*, the Jacobian matrix

$$DF(\boldsymbol{x},\boldsymbol{y},\boldsymbol{t}) = \left(\begin{array}{c|c|c} D_{\boldsymbol{x}}^2 f_0(\boldsymbol{x},\boldsymbol{t}) - \sum_{e\in E} y_e D_{\boldsymbol{x}}^2 f_e(\boldsymbol{x},\boldsymbol{t}) & -D_{\boldsymbol{x}} f_E(\boldsymbol{x},\boldsymbol{t}) & * \\ \hline D_{\boldsymbol{x}} f_E(\boldsymbol{x},\boldsymbol{t})^\top & \mathbf{o} & * \end{array}\right)$$

is of full rank. ∎

Set $\boldsymbol{N}(\boldsymbol{x},\boldsymbol{t}) := D^2_{\boldsymbol{x}} f_0(\boldsymbol{x},\boldsymbol{t}) - \sum_{e\in E} y_e D^2_{\boldsymbol{x}} f_e(\boldsymbol{x},\boldsymbol{t})$ and

$$\boldsymbol{M}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{t}) := D_{(\boldsymbol{x},\boldsymbol{y})} F(\boldsymbol{x},\boldsymbol{y},\boldsymbol{t}) = \left(\begin{array}{c|c} \boldsymbol{N}(\boldsymbol{x},\boldsymbol{t}) & -D_{\boldsymbol{x}} f_E(\boldsymbol{x},\boldsymbol{t}) \\ \hline D_{\boldsymbol{x}} f_E(\boldsymbol{x},\boldsymbol{t})^\top & \mathbf{o} \end{array}\right).$$

By Lemma 2.1, the stationary point set $\Pi(= F^{-1}(\mathbf{o}))$ is a d-dimensional C^1-orientable manifold, if the Regular Value Condition holds.

Remark 4.1. *Under the Regular Value Condition, the following constraint qualification holds at all the feasible solution $\boldsymbol{x} \in X(\boldsymbol{t})$:*

Linear Independence Constraint Qualification (shortly LICQ)
We say that LICQ holds at $\boldsymbol{x}^* \in X(\boldsymbol{t}^*)$ if the set of vectors $\{D_{\boldsymbol{x}} f_e(\boldsymbol{x}^*,\boldsymbol{t}^*)\ (e \in E)\}$ is linearly independent. ∎

For the stability of the stationary solutions, the number of negative eigenvalues of the matrix $\boldsymbol{B}(\boldsymbol{x}^*,\boldsymbol{t}^*)^\top \boldsymbol{N}(\boldsymbol{x}^*,\boldsymbol{y}^*,\boldsymbol{t}^*)\boldsymbol{B}(\boldsymbol{x}^*,\boldsymbol{t}^*)$ plays a quite important role, see [1, 4, 6, 16, 26], where $(\boldsymbol{x}^*,\boldsymbol{t}^*)$ is a stationary solution of $\mathcal{NLP}(\boldsymbol{f}(\cdot,\boldsymbol{t}^*))$, $\boldsymbol{y}^*$ is the associated Lagrange multiplier vector and $\boldsymbol{B}(\boldsymbol{x}^*,\boldsymbol{t}^*)$ is a matrix whose column forms a basis of the tangent subspace

$$W(\boldsymbol{x}^*,\boldsymbol{t}^*) := \{\boldsymbol{v} \in I\!R^n : D_{\boldsymbol{x}} f_e(\boldsymbol{x}^*,\boldsymbol{t}^*)^\top \boldsymbol{v} = 0\ (e \in E)\} \text{ (see Kojima [16, Section 5]).}$$

Definition 4.2 (Section 5 of [16]): *For the triple $(\boldsymbol{x}^*,\boldsymbol{y}^*,\boldsymbol{t}^*)$ such that $\boldsymbol{x}^*$ is a stationary solution to the problem $\mathcal{NLP}(\boldsymbol{f}(\cdot,\boldsymbol{t}^*))$ and $\boldsymbol{y}^*$ is an associated multiplier vector, we define the stationary index;*

$$s.index(\boldsymbol{x}^*|\mathcal{NLP}(\boldsymbol{f}(\cdot,\boldsymbol{t}^*))) := \begin{array}{r} \textit{the number of the negative eigenvalues of the matrix} \\ \boldsymbol{B}(\boldsymbol{x}^*,\boldsymbol{t}^*)^\top \boldsymbol{N}(\boldsymbol{x}^*,\boldsymbol{y}^*,\boldsymbol{t}^*)\boldsymbol{B}(\boldsymbol{x}^*,\boldsymbol{t}^*) \end{array}.$$

∎

Note that the definition of the stationary index does not depend on a choice of $\boldsymbol{B}(\boldsymbol{x}^*,\boldsymbol{t}^*)$. The stationary index is a natural generalization of the Morse index ([20]) and the quadratic index ([10]).

Set $\Sigma^s := \left\{(\boldsymbol{x},\boldsymbol{t}) \in \Sigma : \begin{array}{l} \boldsymbol{B}(\boldsymbol{x},\boldsymbol{t})^\top \boldsymbol{N}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{t})\boldsymbol{B}(\boldsymbol{x},\boldsymbol{t}) \text{ is nonsingular} \\ \text{with the unique associated Lagrange multiplier vector } \boldsymbol{y} \end{array}\right\}$. Then, the set Σ^s is the set of (strongly) stable stationary solutions ([16, Corollary 4.3]). The stationary index completely determines the type of stable stationary solution to $\mathcal{NLP}(\boldsymbol{f}(\cdot,\boldsymbol{t}^*))$, *i.e.*, if $\boldsymbol{x}^*$ is stable stationary solution to $\mathcal{NLP}(\boldsymbol{f}(\cdot,\boldsymbol{t}^*))$, then

(a) $\boldsymbol{x}^*$ is a local minimum if and only if $s.index(\boldsymbol{x}^*|\mathcal{NLP}(\boldsymbol{f}(\cdot,\boldsymbol{t}^*)) = 0$,
(b) $\boldsymbol{x}^*$ is a saddle solution if and only if $1 \le s.index(\boldsymbol{x}^*|\mathcal{NLP}(\boldsymbol{f}(\cdot,\boldsymbol{t}^*)) \le n - |E| - 1$,
(c) $\boldsymbol{x}^*$ is a local maximum if and only if $s.index(\boldsymbol{x}^*|\mathcal{NLP}(\boldsymbol{f}(\cdot,\boldsymbol{t}^*)) = n - |E|$,

see [17, Theorem 3.1]. Therefore, it is important to discuss the change of the stationary index for the analysis of the nonlinear programming under data perturbations.

Theorem 4.3. *Let $\boldsymbol{x}^*$ be a stationary solution to the multiparametric nonlinear program $\mathcal{NLP}(\boldsymbol{f}(\cdot,\boldsymbol{t}^*))$, i.e., $(\boldsymbol{x}^*,\boldsymbol{t}^*) \in \Sigma$. Suppose that Regular Value Condition holds for the problem. Then there are open sets V of $I\!R^d$, U of $I\!R^{n+d}$ and a C^1-parametrization $(\boldsymbol{x}(\cdot),\boldsymbol{t}(\cdot)) : V \to U$ such that $U \cap \Sigma = \{(\boldsymbol{x}(\boldsymbol{p}),\boldsymbol{t}(\boldsymbol{p}))|\boldsymbol{p} \in V\}$ and $(\boldsymbol{x}(\boldsymbol{p}^*),\boldsymbol{t}(\boldsymbol{p}^*)) = (\boldsymbol{x}^*,\boldsymbol{t}^*)$ for some $\boldsymbol{p}^* \in V$ and*

$$m \le s.index(\boldsymbol{x}(\boldsymbol{p})|\mathcal{NLP}(\boldsymbol{f}(\cdot,\boldsymbol{t}(\boldsymbol{p})))) \le m + corank[\dot{\boldsymbol{t}}(\boldsymbol{p}^*)]$$

for any $\boldsymbol{p} \in V$*, where* m *is the number of the negative eigenvalues of the matrix*

$$\boldsymbol{B}(\boldsymbol{x}^*, \boldsymbol{t}^*)^\top \boldsymbol{N}(\boldsymbol{x}^*, \boldsymbol{y}^*, \boldsymbol{t}^*)\boldsymbol{B}(\boldsymbol{x}^*, \boldsymbol{t}^*).$$

Moreover, there exists a sign $c \in \{-1, 1\}$ *such that*

$$(-1)^{s.index(\boldsymbol{x}(\boldsymbol{p})|\mathcal{NLP}(\boldsymbol{f}(\cdot,\boldsymbol{t}(\boldsymbol{p}))))} = c \cdot sign\ det[\dot{\boldsymbol{t}}(\boldsymbol{p})]$$

for any $(\boldsymbol{x}(\boldsymbol{p}), \boldsymbol{t}(\boldsymbol{p})) \in U \cap \Sigma^s$.

Proof. It is a direct consequence of Theorem 2.3 and the proof of Theorem 3.1. Note that the number of the negative (resp. zero) eigenvalues of $\boldsymbol{M}(\boldsymbol{x}^*, \boldsymbol{y}^*, \boldsymbol{t}^*)$ is equal to the number of the negative (resp. zero) eigenvalues of $\boldsymbol{B}(\boldsymbol{x}^*, \boldsymbol{t}^*)^\top \boldsymbol{N}(\boldsymbol{x}^*, \boldsymbol{y}^*, \boldsymbol{t}^*)\boldsymbol{B}(\boldsymbol{x}^*, \boldsymbol{t}^*)$. ∎

Remark 4.4. *We make a short remark for the general nonlinear programs;*

$$\mathcal{NLP}(\boldsymbol{f}(\cdot, \boldsymbol{t})): \quad \begin{array}{ll} minimize & f_0(\boldsymbol{x}, \boldsymbol{t}) \\ such\ that & f_e(\boldsymbol{x}, \boldsymbol{t}) = 0\ (e \in E), \\ & f_i(\boldsymbol{x}, \boldsymbol{t}) \leq 0\ (i \in I). \end{array}$$

In the case $I \neq \emptyset$*, the property we obtained by Theorem 4.3 are not obvious, since the Karush-Kuhn-Tucker stationary solution set to* $\mathcal{NLP}(\boldsymbol{f}(\cdot, \boldsymbol{t}))$ *is not defined by the zero set of* C^1*-map. If* $I \neq \emptyset$*, the stationary solution set is the zero set of a piecewise continuously differentiable (*PC^1*-)map (see Kojima [16]). But, for the general nonlinear programs (of course, the definition and the characterization theorem of the stability is more complicated, [16, Corollary 4.3 and Theorem 7.2]), the following generic property is obtained by [7, Corollary 3.8], see also [25];*

$$m \leq s.index(\boldsymbol{x}(\boldsymbol{p})|\mathcal{NLP}(\boldsymbol{f}(\cdot, \boldsymbol{t}(\boldsymbol{p})))) \leq m + d.$$

∎

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Appendix.

Lemma A: Let $[\boldsymbol{A}|\boldsymbol{B}]$ be an $n \times (n+m)$-matrix of full rank with an $n \times n$-matrix $\boldsymbol{A}$ and $[\boldsymbol{C}|\boldsymbol{D}]$ be an $m \times (n+m)$-matrix of full rank with an $m \times m$-matrix $\boldsymbol{D}$. Suppose that $\left[\begin{array}{c|c} \boldsymbol{A} & \boldsymbol{B} \\ \hline \boldsymbol{C} & \boldsymbol{D} \end{array}\right]$ is nonsingular and $[\boldsymbol{A}|\boldsymbol{B}] \cdot \left[\begin{array}{c} \boldsymbol{C}^\top \\ \hline \boldsymbol{D}^\top \end{array}\right] = \mathbf{o}$. Then $corank \boldsymbol{A} = corank \boldsymbol{D}$.

Proof. Without loss of generality, we may assume that $\boldsymbol{A} = \left[\begin{array}{c|c} \boldsymbol{E}_{n-r} & \mathbf{o} \\ \hline \mathbf{o} & \mathbf{o}_r \end{array}\right]$ and $\boldsymbol{D} = \left[\begin{array}{c|c} \boldsymbol{E}_{m-r'} & \mathbf{o} \\ \hline \mathbf{o} & \mathbf{o}_{r'} \end{array}\right]$ where $\boldsymbol{E}_p$ is a $p \times p$-identity matrix and $\mathbf{o}_q$ is a $q \times q$-zero matrix. We have only to show that $r = r'$. Remark that the square matrix $\left[\begin{array}{c|c} \boldsymbol{A} & \boldsymbol{B} \\ \hline \boldsymbol{C} & \boldsymbol{D} \end{array}\right]$ is of full rank, *i.e.*,

$$\left[\begin{array}{c|c} \begin{array}{c|c} \boldsymbol{E}_{n-r} & \mathbf{o} \\ \hline \mathbf{o} & \mathbf{o}_r \end{array} & \boldsymbol{B} \\ \hline \boldsymbol{C} & \begin{array}{c|c} \boldsymbol{E}_{m-r'} & \mathbf{o} \\ \hline \mathbf{o} & \mathbf{o}_{r'} \end{array} \end{array}\right]$$ is nonsingular.

Set the matrix

$$
\left[\begin{array}{cc|cc}
\boldsymbol{E}_{n-r} & \mathbf{o} & \multicolumn{2}{c}{\multirow{2}{*}{$\boldsymbol{B}$}} \\
\mathbf{o} & \mathbf{o}_r & & \\ \hline
\multicolumn{2}{c|}{\multirow{2}{*}{$\boldsymbol{C}$}} & \boldsymbol{E}_{m-r'} & \mathbf{o} \\
 & & \mathbf{o} & \mathbf{o}_{r'}
\end{array}\right]
=:
\begin{array}{c}
\begin{array}{cccc} n\text{-}r & r & m\text{-}r' & r' \end{array} \\
\left[\begin{array}{c|c|c|c}
\boldsymbol{E}_{n-r} & \mathbf{o} & \boldsymbol{B}(1) & \boldsymbol{B}(2) \\ \hline
\mathbf{o} & \mathbf{o}_r & \boldsymbol{B}(3) & \boldsymbol{B}(4) \\ \hline
\boldsymbol{C}(1) & \boldsymbol{C}(2) & \boldsymbol{E}_{m-r'} & \mathbf{o} \\ \hline
\boldsymbol{C}(3) & \boldsymbol{C}(4) & \mathbf{o} & \mathbf{o}_{r'}
\end{array}\right]
\end{array}
.\quad
\begin{array}{c} n\text{-}r \\ r \\ m\text{-}r' \\ r' \end{array}
$$

Assume that $r > r'$ (similar argument in case $r < r'$). From $[\boldsymbol{A}|\boldsymbol{B}] \cdot \left[\begin{array}{c} \boldsymbol{C}^\top \\ \hline \boldsymbol{D}^\top \end{array}\right] = \mathbf{o}$, we can easily see that

$$
\left[\begin{array}{c} \boldsymbol{C}(1)^\top | \boldsymbol{C}(3)^\top \\ \hline \mathbf{o} \end{array}\right] + \left[\begin{array}{c|c} \begin{array}{c} \boldsymbol{B}(1) \\ \hline \boldsymbol{B}(3) \end{array} & \mathbf{o} \end{array}\right] = \mathbf{o}.
$$

Hence, $\boldsymbol{B}(3) = \mathbf{o}$ (and $\boldsymbol{C}(3) = \mathbf{o}$). From the fact that $\left[\begin{array}{c|c} \boldsymbol{A} & \boldsymbol{B} \\ \hline \boldsymbol{C} & \boldsymbol{D} \end{array}\right]$ is nonsingular, it follows that an $\{r \times (n+m)\}$-submatrix of $\left[\begin{array}{c|c} \boldsymbol{A} & \boldsymbol{B} \\ \hline \boldsymbol{C} & \boldsymbol{D} \end{array}\right]$,

$$
\begin{array}{c}
\left[\begin{array}{cccc} \mathbf{o} & \mathbf{o}_r & \mathbf{o} & \boldsymbol{B}(4) \end{array}\right] \\
\begin{array}{cccc} n\text{-}r & r & m\text{-}r' & r' \end{array}
\end{array}
\;\; r
$$

is of full rank. It is not possible since $r > r'$ (it is a contradiction). Then $r = r'$, *i.e.*,

$$
corank\boldsymbol{A} = corank\boldsymbol{D}.
$$

∎

Lemma B: Let $[\boldsymbol{A}|\boldsymbol{B}]$ be an $n \times (n+m)$-matrix with an $n \times n$ nonsingular matrix $\boldsymbol{A}$ and $[\boldsymbol{C}|\boldsymbol{D}]$ be an $m \times (n+m)$-matrix with an $m \times m$ nonsingular matrix $\boldsymbol{D}$. Suppose that $\left[\begin{array}{c|c} \boldsymbol{A} & \boldsymbol{B} \\ \hline \boldsymbol{C} & \boldsymbol{D} \end{array}\right]$ is nonsingular and $[\boldsymbol{A}|\boldsymbol{B}] \cdot \left[\begin{array}{c} \boldsymbol{C}^\top \\ \hline \boldsymbol{D}^\top \end{array}\right] = \mathbf{o}$. Then

$$
sign\ det \left[\begin{array}{c|c} \boldsymbol{A} & \boldsymbol{B} \\ \hline \boldsymbol{C} & \boldsymbol{D} \end{array}\right] = sign\ det\boldsymbol{A} \times sign\ det\boldsymbol{D}.
$$

Proof. Set matrices X and Y as follows; $\boldsymbol{X} := \left[\begin{array}{c|c} \boldsymbol{A} & \boldsymbol{B} \\ \hline \boldsymbol{C} & \boldsymbol{D} \end{array}\right]$ and $\boldsymbol{Y} := \left[\begin{array}{c|c} \boldsymbol{A}^\top & \boldsymbol{C}^\top \\ \hline \mathbf{o} & \boldsymbol{D}^\top \end{array}\right]$. Then, we can see $\boldsymbol{X} \cdot \boldsymbol{Y} = \left[\begin{array}{c|c} \boldsymbol{A} \cdot \boldsymbol{A}^\top & \mathbf{o} \\ \hline * & \boldsymbol{C} \cdot \boldsymbol{C}^\top + \boldsymbol{D} \cdot \boldsymbol{D}^\top \end{array}\right]$. Hence,

$$
det\boldsymbol{X} \times det\boldsymbol{Y} = det\boldsymbol{X}\boldsymbol{Y} = det[\boldsymbol{A} \cdot \boldsymbol{A}^\top] \times det[\boldsymbol{C} \cdot \boldsymbol{C}^\top + \boldsymbol{D} \cdot \boldsymbol{D}^\top] > 0
$$

(since $\boldsymbol{A}$ and $\boldsymbol{D}$ are nonsingular matrices and then $\boldsymbol{A} \cdot \boldsymbol{A}^\top$ and $\boldsymbol{C} \cdot \boldsymbol{C}^\top + \boldsymbol{D} \cdot \boldsymbol{D}^\top$ are positive definite). Therefore,

$$
sign\ det\boldsymbol{X} = sign\ det\boldsymbol{Y} = sign\ det\boldsymbol{A} \times sign\ det\boldsymbol{D}.
$$

∎

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