Journal of the Operations Research
Society of Japan
Vol. 38, No. 3, September 1995

ON THE UNBIASEDNESS OF THE PARAMETRIC DIVISOR METHOD FOR THE APPORTIONMENT PROBLEM

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(Received January 18, 1993; Revised November 7, 1994)

Abstract Apportionment problem has a very long history of more than 200 years and has been challenged actively by many applied mathematicians and operations researchers. In the last 200 years many apportionment methods have been proposed and various types of properties, which they should desirably satisfy, have been proposed and investigated although the apportionment problem itself has not been completely solved yet.

Unbiasedness is an important property for the apportionment method, and has been studied intensively by Balinski and Young and others. In this paper we investigate the unbiasedness of parametric divisor method, comparing with that of most traditional apportionment methods. First we characterize parametric divisor method by explicitly showing the corresponding local measure of bias and deriving several useful facts concerning the unbiasedness of apportionment methods. Then we show the numerical results using Japan's House of Representatives data and exhibit parametric divisor method is preferable to other apportionment methods from the viewpoint of the unbiasedness by adjusting the parameter value appropriately in the certain range. Finally we discuss the possibility for the parametric divisor method to be accepted by our society in the future.

1 Introduction

In the present Japan's House of Representatives there are 511 seats available and there are 130 political constituencies in the whole country. Among these 130 political constituencies the number of seats assigned to each ranges from 1 to 6 and in most constituencies they are between 3 and 5. Total number of seats, *i.e.*, the house size, started from 466 in 1947, then it was increased several times reaching 511 in 1976, 512 in 1986 and currently 511 again after 1992. With the rapid and high economic growth in Japan in the 1960's, however, the distribution of our population drastically changed, *i.e.* our population has become heavily concentrated in big city areas while rural areas have been burdened by a serious "isolation" problem. In 1947 the largest number of seats per capita was 1.51 times the smallest. Since then, the largest-smallest ratio has begun to increase, reaching 3.21 in 1960, 4.83 in 1970, and 5.12 in 1985. Thus the "weight of one vote" in the populated areas became much less than that in the rural areas and the issue of weight gap between growing and declining political constituencies has become one of the most socially and politically controversial problems in our country. Our government revised the electoral system in order to reduce the weight gap among constituencies by increasing the house size by 19 in 1964 and by 20 with the total number of seats 511 in 1975. Then in 1986 our government increased the number of seat allocations in 8 political constituencies and decreased it in 7 constituencies making the total 512, and again in 1992 the number of seat allocations was increased in 9 political constituencies and decreased in 10 constituencies with the total number reaching 511 again.

In 1983 our Supreme Court gave a decision responding to the appeal that a weight gap of more than 3.0 may be unconstitutional. Since then, several similar decisions have been given in various judicial courts. After the Recruit scandal in 1988 political reform including the change to the new electoral voting system has been a serious concern for the nation's political community, our ruling Liberal Democratic Party in the Diet has also recognized the importance of this problem. After very harsh arguments and struggles among politicians a new political reform bill passed the Diet in January, 1994. Our new electoral voting system includes reducing the total number of seats from the present 511 to 500 as a combination of 300 single seat constituency and 200 proportional representation seats. The new election system is believed to promote a new realignment of several political parties.

Given the total number of seats and the distribution of each constituency's population the apportionment problem tries to allocate seats "fairly" among political constituencies. Let the set of N political constituencies be $S = \{1, 2, \ldots, N\}$, and let the population of political constituency $i \in S$ be p_i. Then the apportionment problem is to partition a given positive integer K into nonnegative integral parts $\{d_i \mid i \in S\}$ such that

$$(1.1) \qquad \sum_{i \in S} d_i = K$$

$$(1.2) \qquad d_i \geq 0, \ integer, \ i \in S$$

and such that these parts are "as near as possible" proportional to a set of nonnegative integers $\{p_1, p_2, \ldots, p_N\}$, respectively.

Denoting the total population by P, and the total number of seats, *i.e.*, the house size, by K, the "ideal" number of seats allocated to the constituency i, *i.e.*, the "exact quota" q_i, is given by

$$(1.3) \qquad q_i = \frac{p_i K}{P} \qquad i \in S$$

where P is the total population given by

$$(1.4) \qquad P = \sum_{i \in S} p_i$$

Hence we have

$$(1.5) \qquad \sum_{i \in S} q_i = K$$

Usually exact quotas $\{q_i \mid i \in S\}$ all have fractional parts. Therefore, the problem becomes how to round the fractions $\{q_i \mid i \in S\}$ to their "nearby" integral values keeping their sum equal to a given value K.

Balinski and Young have done quite extensive work on the above problem (see *e.g.*, [1,2,3,4,5,6,7,8,9]), and many apportionment methods have been proposed as mentioned in the next section (see *e.g.*, [8,10,11,12,13,14,15] for surveying the methods). Several properties are required for the apportionment method to satisfy. For example, we want the apportionment method to have the quota property that the number of seats given to each constituency is either rounded-up or rounded-down by an exact quota. We may want the house monotone property that a constituency should not be given less representation if the total number of seats increases and the distribution of population of each constituency remains the same. Bias is another important factor for the apportionment method. Namely the apportionment method cannot be accepted if it tends to be always biased in favor of the larger or the smaller constituencies. There are various "natural" requirements for acceptable apportionment methods. Some of these "requirements", however, are inconsistent. No matter which apportionment method is accepted, it will possess certain "defects". Namely, we may have to decide in advance which properties must be satisfied, and which "defects" are acceptable before we employ our own apportionment method.

In this paper we consider the bias of several apportionment methods, focused on the parametric divisor method proposed in [13]. We investigate the unbiasedness of several traditional apportionment methods and compare them with the parametric divisor method. We then propose a range of appropriate parameter values for the parametric divisor method in order to obtain higher unbiasedness as well as impartialness and fairness with respect to the population size of each constituency. In Section 2 we review several representative apportionment methods and their bias property. Then we show our results related to characterizing parametric divisor method from the viewpoint of the local measure of inequity. In Section 3 we give the results of our numerical experiments using Japan's House of Representatives data, and compare these results with the apportionment methods described therein. In the last Section, we conclude our paper by giving certain evaluations obtained from our results and numerical experiments.

2 Apportionment methods and the bias

One of the most common apportionment methods is the largest fraction method suggested by A. Hamilton at the United States Congress in 1791, and employed by the Congress from 1851 until 1910, which we shall denote by LFM. The LFM first assigns each constituency $i \in S$ its lower quota $\lfloor q_i \rfloor$, where $\lfloor q \rfloor$ denotes the largest integer less than or equal to q. Then we define the fraction of each constituency t_i as follows.

$$t_i = q_i - \lfloor q_i \rfloor \qquad i \in S \tag{2.1}$$

Sorting the set $\{t_i \mid i \in S\}$ from the largest, arbitrarily for the equal elements, we define the set of suffices of the first $K - \sum_{i \in S} \lfloor q_i \rfloor$ constituencies in the ordering by T. Then the LFM allocates an additional seat to the constituencies belonging to the set T; namely, the whole allocation $\{d_i \mid i \in S\}$ of the LFM is given as follows.

$$d_i = \begin{cases} \lfloor q_i \rfloor + 1 & i \in T \\ \lfloor q_i \rfloor & i \notin T \end{cases} \tag{2.2}$$

Let us define the general divisor method. First, we give a divisor λ in order to compute the quotient of each constituency $i \in S$ with the population p_i as $q_i(\lambda) = \frac{p_i}{\lambda}$. Then, we round the quotients according to values of the number of seats in each constituency. Let us denote the integer value obtained from the quotient $q_i(\lambda) = \frac{p_i}{\lambda}$ by $[q_i(\lambda)]_r = [\frac{p_i}{\lambda}]_r$. Then, in order that these quotients can be an apportionment, the following must hold.

$$\sum_{i \in S} [q_i(\lambda)]_r = \sum_{i \in S} [\frac{p_i}{\lambda}]_r = K \tag{2.3}$$

Now we generalize the rounding process by defining the divisor function $v(d)$ as follows. Let $v(d)$ be a monotone increasing function defined for all integers $d \geq 0$ and also satisfying $d \leq v(d) \leq d+1$. Then, for any positive real number x, there corresponds a unique integer d such that $v(d-1) \leq x \leq v(d)$. Namely we assume that $v(d)$ can take either d or $d+1$ in case $x = v(d)$. We define the above rounding process by

$$[\frac{p_i}{\lambda}]_r = d_i \qquad i \in S \tag{2.4}$$

where

$$v(d_i - 1) \leq \frac{p_i}{\lambda} \leq v(d_i) \qquad i \in S \tag{2.5}$$

The apportionment method described above is called the divisor method based upon the divisor function $v(d)$. The divisor method can be defined equivalently as follows. From (2.4) and (2.5), the parameter λ has to satisfy the following relation for all $i \in S$

$$(2.6) \qquad \begin{cases} \dfrac{p_i}{v(d_i)} \le \lambda \le \dfrac{p_i}{v(d_i - 1)} & \text{for } d_i > 0 \\ \dfrac{p_i}{v(d_i)} \le \lambda & \text{for } d_i = 0 \end{cases}$$

This means that we have the following max-min inequality

$$(2.7) \qquad \max_{d_j \ge 0} \frac{p_j}{v(d_j)} \le \min_{d_i > 0} \frac{p_i}{v(d_i - 1)}$$

where we permit dividing by 0 and assume that $\frac{p_i}{0} > \frac{p_j}{0}$ if $p_i > p_j$. Defining the rank function $r(p_i, d_i)$ for $i \in S$ as

$$(2.8) \qquad r(p_i, d_i) = \frac{p_i}{v(d_i)} \qquad i \in S$$

we can write the above relation (2.8) as follows.

$$(2.9) \qquad \max_{d_i \ge 0} r(p_i, d_i) \le \min_{d_j > 0} r(p_j, d_j - 1)$$

We denote the above apportionment method based upon the rank function $r(p, d)$ obtained from the divisor function $v(d)$ by $A(v, \boldsymbol{p}, K)$, which expresses a function giving N integral components $d_1, \ldots, d_N$ as an image of a given population distribution vector $\boldsymbol{p} = (p_1, \ldots, p_N)$ and a total number of seats K. The function $A(v, \boldsymbol{p}, K)$ based upon the rank function $r(p, d)$ related with the divisor function $v(d)$ can be written as follows.

$$(2.10) \qquad A(v, \boldsymbol{p}, K) = \{\boldsymbol{d} \mid \sum_{i \in S} d_i = K, \max_{d_j \ge 0} r(p_j, d_j) \le \min_{d_i > 0} r(p_i, d_i - 1)\}$$

where $\boldsymbol{d}$ indicates an allocation vector given by $\boldsymbol{d} = (d_1, \ldots, d_N)$.

There exists an alternative way of expressing the general apportionment methods based upon the divisor function $v(d_i)$ recursively. Let d_i^k indicate the number of seats allocated to the political constituency $i \in S$ given the total number of seats $k \in \{0, 1, \ldots, K\}$. Then an iterative algorithm for the general divisor method can be written as follows.

Algorithm (general divisor method)

Step 1 $d_i^k = 0, \ k \in \{0, 1, \ldots, K\}, \ i \in S. \ k = 0.$

Step 2

$$(2.11) \qquad r(p_t, d_t) = \max_{i \in S} r(p_i, d_i)$$

$$(2.12) \qquad \begin{cases} d_t^{k+1} = d_t^k + 1 \\ d_i^{k+1} = d_i^k & i \ne t, \ i \in S \end{cases}$$

Step 3 $k = k + 1$. If $k = K$, then stop. Otherwise, go to step 2.

As shown in the above algorithm, for $k = 0$, the allocation must be zero for every constituency. Given that an allocation $\boldsymbol{d}^k = (d_1^k, \ldots, d_N^k)$ has been determined for a total number of seats k, an allocation for a size $k + 1$ is found by giving one more seat to the constituency i for which the rank function $r(p_i, d_i)$ is a maximum.

Based upon different divisor functions we can define an infinite number of different divisor methods (see *e.g.*, [1,2,3,4,7,8,10,13,15]). There are five traditional divisor methods as well as parametric divisor method (PDM) as shown in Table 1. The method of greatest divisors, which we denote by GDM, was also called the Jefferson method in Balinski and Young's papers. The method of major fractions, which we denote by MFM, was called the Webster method in their papers. Balinski and Young called the equal proportion method (EPM), the harmonic method (HMM), and the smallest divisor method (SDM) after the names of their advocates, *i.e.*, the Hill method, the Dean method, and the Adams method, respectively.

Table 1. Divisor method, divisor function and measure of inequity

Divisor method	Divisor function $v(d)$	Measure of inequity $E(p_i, d_i; p_j, d_j)$
GDM	$d+1$	$\frac{d_i p_j}{p_i} - d_j$
MFM	$d+0.5$	$\frac{d_i}{p_i} - \frac{d_j}{p_j}$
EPM	$\sqrt{d(d+1)}$	$\frac{\frac{d_i}{p_i} - \frac{d_j}{p_j}}{\min\{\frac{p_i}{d_i}, \frac{p_j}{d_j}\}}$
HMM	$\frac{d(d+1)}{d+0.5}$	$\frac{p_i}{d_i} - \frac{p_j}{d_j}$
SDM	d	$d_i - \frac{p_i d_j}{p_j}$
PDM	$d+t$	$d_i + t - (d_j + t)\frac{p_i}{p_j}$

Using a parameter t such that $0 \leq t \leq 1$, the divisor function of the parametric divisor method (refer to [13]), which we denote by PDM, can be written as follows.

$$v_{PD}(d, t) = d + t \tag{2.13}$$

Comparing the above function $v_{PD}(d, t)$ with those in Table 1, we find that $t = 0$, $1/2$, and 1 correspond to those functions of the SDM, MFM and GDM, respectively.

Now the apportionment method based upon PDM can be described as follows. Let the parameter for PDM be $\lambda = \lambda_{PD}$, then λ_{PD} can be determined as the maximum λ satisfying

$$\sum_{i \in S} \lfloor \frac{p_i}{\lambda} + 1 - t \rfloor \geq K \tag{2.14}$$

If (2.14) holds as an equality for $\lambda = \lambda_{PD}$, then the allocation $\{d_i \mid i \in S\}$ is given by

$$d_i = \lfloor \frac{p_i}{\lambda_{PD}} + 1 - t \rfloor \qquad i \in S \tag{2.15}$$

If (2.14) holds as a strict inequality, then let

$$E = \{i \mid i \in S, \frac{p_i}{\lambda_{PD}} : \text{integer}\} \tag{2.16}$$

Since there exist more than one i such that $\frac{p_i}{\lambda_{PD}}$ is integer valued, $E \neq \phi$. Suppose

$$\sum_{i \in S} \lfloor \frac{p_i}{\lambda_{PD}} \rfloor = K' > K \tag{2.17}$$

then we must decide that $K' - K$ constituencies lose a seat. Hence let D be a subset of E with $|D| = K' - K$ (we can apply an ad-hoc rule to determine this), then the apportionment can be given as

$$(2.18) \qquad d_i = \begin{cases} \dfrac{p_i}{\lambda_{PD}} & i \notin D,\ i \in E \\ \dfrac{p_i}{\lambda_{PD}} - 1 & i \in D \end{cases}$$

Also we know parameter λ_{PD} satisfies

$$(2.19) \qquad \frac{p_j}{d_j + t} \leq \lambda_{PD} \leq \frac{p_i}{d_i - 1 + t} \qquad i, j \in S$$

Hence the following relation holds

$$(2.20) \qquad \max_{j \in S} \frac{p_j}{d_j + t} \leq \min_{i \in S} \frac{p_i}{d_i - 1 + t}$$

There are several properties for each apportionment method to satisfy. If the allocation $\{d_i \mid i \in S\}$ given by the method M satisfies

$$(2.21) \qquad d_i \geq \lfloor q_i \rfloor \qquad i \in S$$

then we say that apportionment method M satisfies the lower quota. Suppose the allocation $\{d_i \mid i \in S\}$ satisfies

$$(2.22) \qquad d_i \leq \lfloor q_i \rfloor + 1 \qquad i \in S$$

then method M is said to satisfy the upper quota. If method M satisfies both the lower and the upper quota properties, we say that method M satisfies the quota property. No divisor method described above satisfies the quota property, while the LFM does it.

An apportionment method M is said to satisfy the house monotone property if no political constituency $i \in S$ decreases its allocation when the house size increases from k to $k+1$. The violation of this property is often referred to as the "Alabama paradox". The word "Alabama paradox" originates from the fact that when the U.S. Congress was using the LFM in 1881, the state of Alabama was allocated 8 representatives, while they received 7 when the total went to 300 from 299. Therefore, the LFM does not satisfy this property. All other divisor methods satisfy it.

We consider the local measures of inequity between pairs of constituencies. Let the population in the constituency $i \in S$ be p_i and the number of seats assigned be d_i. We say that constituency i is favored over j when the number of seats per individual in the constituency i is greater than or equal to that in j; namely, $\frac{d_i}{p_i} \geq \frac{d_j}{p_j}$, (*i.e.*, $\frac{p_i}{d_i} \leq \frac{p_j}{d_j}$). Huntington considered making ratios such as $\frac{d_i}{p_i}$ or $\frac{p_i}{d_i}$ as equal as possible over all constituencies. That these ratios are nearly equal means that, ideally, the relative or the absolute differences concerning $\frac{d_i}{p_i}$ or $\frac{p_i}{d_i}$ become zero. Generally, we denote the measure of inequity between two constituencies i and j as $E(p_i, d_i; p_j, d_j)$. Then Huntington's rule says that we should transfer a seat from a more favored constituency i to a less favored constituency j when it brings a smaller measure of inequity. Namely, when $\frac{d_i}{p_i} \geq \frac{d_j}{p_j}$ and

$$(2.23) \qquad E(p_i, d_i; p_j, d_j) > E(p_i, d_i - 1; p_j, d_j + 1)$$

we should transfer a seat from i to j. The objective of Huntington's rule is to minimize the measure of inequity between pairs of constituencies. So the "desirable apportionment" is obtained when no switching of seats between constituencies can improve the measure of inequity between any such pair of constituencies. The attainment of this state is referred to as a stable allocation of seats.

Huntington's rule was applied to several forms of the measure of inequity $E(p_i, d_i; p_j, d_j)$ as shown in Table 1. The local measure of inequity in Table 1 assumes that the constituency i is favored over j. For each measure of inequity in Table 1 we can obtain a stable assignment of seats. Moreover, the resulting stable apportionment obtained from each function indicating the measure of inequity, corresponds to the solution for the apportionment methods GDM, MFM, EPM, HMM and SDM, respectively.

Now for the PDM we define that constituency i is favored over j when $\frac{d_i+t}{p_i} \geq \frac{d_j+t}{p_j}$ for the given parameter t. Obviously this condition is equivalent to the original one when t=0 for the PDM. Then we assume that we should transfer a seat from i to j when $\frac{d_i+t}{p_i} \geq \frac{d_j+t}{p_j}$ and the local measure of inequity $E_{PD}(p_i, d_i; p_j, d_j; t)$ satisfies

$$E_{PD}(p_i, d_i; p_j, d_j; t) > E_{PD}(p_i, d_i - 1; p_j, d_j + 1; t) \tag{2.24}$$

where

$$E_{PD}(p_i, d_i; p_j, d_j; t) = d_i + t - (d_j + t)\frac{p_i}{p_j} \tag{2.25}$$

for all pairs i and j with i favored over j.

Using the above definitions we obtain the following theorem.

Theorem 2.1 *For the pair of constituencies i and j with populations p_i and p_j, apportionments d_i and d_j, respectively, the following holds.*

$$E_{PD}(p_i, d_i; p_j, d_j; t) \leq E_{PD}(p_i, d_i - 1; p_j, d_j + 1; t) \qquad i, j \in S \tag{2.26}$$

if and only if

$$\frac{p_j}{d_j + t} \leq \frac{p_i}{d_i - 1 + t} \qquad i, j \in S \tag{2.27}$$

Proof We do not admit any transfer of a seat from constituency i to j if

$$d_i + t - (d_j + t)\frac{p_i}{p_j} \leq d_j + t + 1 - (d_i + t - 1)\frac{p_j}{p_i} \tag{2.28}$$

Namely for all $i \in S$ and $j \in S$ we obtain as follows.

$$d_j(\frac{p_i}{p_j} + 1) + t(\frac{p_i}{p_j} - \frac{p_j}{p_i}) \geq d_i(\frac{p_j}{p_i} + 1) - (1 + \frac{p_j}{p_i})$$

$$\frac{d_j}{p_j} + \frac{t(p_i - p_j)}{p_i p_j} \geq \frac{d_i - 1}{p_i}$$

$$\frac{d_j + t}{p_j} \geq \frac{d_i - 1 + t}{p_i}$$

$$\frac{p_j}{d_j + t} \leq \frac{p_i}{d_i - 1 + t}$$

Therefore comparing the above relation with (2.20) we know that such an apportionment $\boldsymbol{d} = (d_1, \ldots, d_N)$ must be a PDM. Conversely, every PDM apportionment satisfies the above inequality (2.27), and its solution admits no transfer of a seat. □

Suppose that relation (2.27) holds for all $i \in S$ and $j \in S$. Then the assignment corresponds to the optimal convergent apportionment. Hence, comparing (2.27) with (2.20), we can conclude that the above case in Theorem 2.1 is equivalent to the case that the divisor function is given as $v_{PD}(d) = d+t$. In other words, the pairwise transfering procedure given by the criterion in Theorem 2.1 gives the same apportionment solution as PDM.

Whether an apportionment method is generally in favor either of the larger constituencies or of the smaller ones is very important in order to evaluate the bias of the apportionment method. Namely no apportionment method can be accepted if it has a persistent bias toward the larger or the smaller constituencies. However measuring the bias of the apportionment method is difficult. There has not yet been established a completely acceptable definition or a model for determining whether or not an apportionment method is biased.

Given two apportionment methods M and M' Balinski and Young [8] defines that the method M' favors smaller constituencies relative to M if for allocations $\boldsymbol{d} = (d_i)$ and $\boldsymbol{d}' = (d'_i)$ by the methods M and M', respectively, the following holds

$$p_i < p_j \Rightarrow d'_i \geq d_i \text{ or } d'_j \leq d_j$$

Let the divisor functions for the apportionment methods M and M' be $v(d)$ and $v'(d)$, respectively, then we obtain the following theorem due to Balinski and Young (corresponding to Theorem 5.1 in [8]).

Theorem 2.2 *Let M and M' be two divisor methods with divisor functions $v(d)$ and $v'(d)$, respectively, satisfying $v(d) = v'(d) + c$ for $0 < c < 1$. Then the method M' favors small constituencies relative to M.*

Proof Let $\boldsymbol{d} \in A(v, \boldsymbol{p}, K)$ and $\boldsymbol{d}' \in A(v', \boldsymbol{p}, K)$ be the allocations obtained from methods M and M', respectively. Suppose for $\boldsymbol{d} = (d_i)$ and $\boldsymbol{d}' = (d'_i)$ the method M' does not favor small constituencies relative to M. Then for some $p_i < p_j$ we have $d'_i < d_i$ and $d'_j > d_j$. Combining with the fact $d_i \leq d_j$, we have $d'_i < d_i \leq d_j < d'_j$. Hence $d'_j - 1 > d'_i \geq 0$ leads to $v'(d'_j - 1) \geq 1$ since $d \leq v'(d) \leq d+1$. These allocations satisfy the following relations

$$\frac{p_j}{v'(d'_j - 1)} \geq \frac{p_i}{v'(d'_i)} \qquad \text{and} \qquad \frac{p_j}{v(d_j - 1)} \geq \frac{p_i}{v(d_i)}$$

Using the above relation we obtain the following

$$\frac{p_j}{p_i} \geq \frac{v'(d'_j - 1)}{v'(d'_i)} = \frac{v(d'_j - 1) - c}{v(d'_i) - c} > \frac{v(d'_j - 1)}{v(d'_i)} \geq \frac{v(d_j)}{v(d_i - 1)}$$

$$\frac{p_j}{v(d_j)} > \frac{p_i}{v(d_i - 1)}$$

The above contradicts the assumption. Thus the assumption $d'_i < d_i$ and $d'_j > d_j$ never occurs, which completes the proof. □

The above theorem leads to the following Corollary.

Corollary 2.3 *Let the PDM's M_1 and M_2 be based upon divisor functions $v(d) = d + t_1$ and $v(d) = d + t_2$ for parameters t_1 and t_2 such that $0 \leq t_1 < t_2 \leq 1$, respectively. Then the method M_1 favors small constituencies relative to M_2.*

Applying the above corollary to the apportionment methods SDM, PDM, MFM and GDM with divisor functions $v(d)$ as d, $d+t$ for $0 < t < \frac{1}{2}$, $d + \frac{1}{2}$ and $d+1$, respectively, we know that these methods tend to favor small constituencies in this order since we have $d < d+t < d+\frac{1}{2} < d+1$ for all d. The following theorems give the properties related to the parameter value λ_t satisfying (2.14) and the allocation of seats obtained from the PDM.

Theorem 2.4 *Let the PDM based upon the divisor function* (2.13) *be given by finding a maximum parameter λ_t satisfying* (2.14). *Then the maximum parameter λ_t is strictly decreasing with respect to t.*

Proof Let the parameters t and u satisfy $0 \leq t < u \leq 1$. Then parameters λ_t and λ_u be the maximums satisfying

$$\sum_{i \in S} \lfloor \frac{p_i}{\lambda_t} + 1 - t \rfloor \geq K$$

and

$$\sum_{i \in S} \lfloor \frac{p_i}{\lambda_u} + 1 - u \rfloor \geq K$$

respectively. Suppose $\lambda_u > \lambda_t$. Then since $u > t$ we have for each $i \in S$

$$\frac{p_i}{\lambda_t} + 1 - t > \frac{p_i}{\lambda_u} + 1 - u$$

$$\lfloor \frac{p_i}{\lambda_t} + 1 - t \rfloor \geq \lfloor \frac{p_i}{\lambda_u} + 1 - u \rfloor$$

The above relation implies that

$$\sum_{i \in S} \lfloor \frac{p_i}{\lambda_t} + 1 - t \rfloor \geq \sum_{i \in S} \lfloor \frac{p_i}{\lambda_u} + 1 - u \rfloor \geq K$$

which contradicts that λ_t is the maximum. □

Theorem 2.5 *Let the PDM's M_t and M_u be based upon the divisor functions $v_t(d) = d+t$ and $v_u(d) = d + u$ for parameters t and u such that $0 \leq t < u \leq 1$, respectively. Let the allocation of seats to the constituency $i \in S$ with the population p_i by the methods M_t and M_u be denoted by d_i^t and d_i^u, respectively. Suppose $p_i < p_j$ and $d_j^t - d_j^u \geq 1$, then the following holds.*

$$d_i^t - d_i^u \leq d_j^t - d_j^u \tag{2.29}$$

Proof Suppose (2.29) does not hold, *i.e.*, $d_i^t - d_i^u > 1$ when $d_j^t - d_j^u \geq 1$. Let the maximum parameters for the methods M_t and M_u be λ_t and λ_u, respectively. Then the allocation of seats for the constituency i can be given as

$$d_i^t = \lfloor \frac{p_i}{\lambda_t} + 1 - t \rfloor$$

and

$$d_i^u = \lfloor \frac{p_i}{\lambda_u} + 1 - u \rfloor$$

respectively. Since $d_j^t - d_j^u \geq 1$, we have

$$\frac{p_j}{\lambda_t} + 1 - t > \frac{p_j}{\lambda_u} + 1 - u$$

which implies

$$u - t > p_j(\frac{1}{\lambda_u} - \frac{1}{\lambda_t})$$

On the other hand from the assumption $d_i^t - d_i^u > 1$ we have to have

$$\frac{p_i}{\lambda_t} + 1 - t - (\frac{p_i}{\lambda_u} + 1 - u) \geq 1$$

which implies

$$u - t > p_i(\frac{1}{\lambda_u} - \frac{1}{\lambda_t}) + 1$$

The above contradicts $0 < u - t < 1$ since $\lambda_t \geq \lambda_u$ from Theorem 2.4. This completes the proof. □

The above relation (2.29) can be written as

$$d_i^t - d_j^t \leq d_i^u - d_j^u$$

which implies that $d_i^t - d_j^t$ for $p_i < p_j$ increases with respect to t when d_j^t changes.

3 Numerical experiments

There are 130 political constituencies in Japan's House of Representative (HOR), each (CNST.) of which has a population (PPL.), quota(QTA.) and a current allocation (CRT) of representatives as shown in Table 2. Applying six apportionment methods (GDM, MFM, EPM, HMM, SDM and LFM) to Japan's HOR data based upon the 1990 Census, we obtain the results given in Table 2. First we recognize that in Japan's current allocation of HOR seats to smaller constituencies, which are mostly in rural areas, are favored over larger constituencies, which are mainly in urban areas. The results in Table 2 show that the apportionment methods GDM, MFM, EPM, HMM and SDM are, in this order, relatively more favorable to those constituencies with larger population, and Japan's current allocation of HOR seats is rather close to that of the SDM. The apportionment method LFM always satisfies the quota property since the allocation by the LFM is either rounded up or rounded down from the exact quota, *i.e.*, stays within the quota. We believe that the LFM is the most unbiased method since it satisfies the quota property although, unfortunately, it violates the house property. The result in Table 2 also shows that the method LFM gives similar apportionment to MFM or EPM. In the 1910's and 1920's in the United States there was a very intense controversy over whether the MFM or the EPM was more unbiased (see, *e.g.*, [8,10]). From our numerical results and historical arguments done so far, we can say that "impartial (unbiased to both larger or smaller constituencies) and appropriate" apportionment methods should be either MFM or EPM, or between or around these methods.

Applying the PDM to our HOR data we obtain the apportionment results as given in Table 3 using the parameter value t for $0 \leq t \leq 1$. The results in Table 3 indicate that the PDM with a smaller parameter value t is more favorable to smaller constituencies while that with a larger parameter value t is more favorable to larger constituencies as obtained from Theorem 2.2.

Table 3. Final apportionments by parametric divisor method

CNST.	PARAMETER										
	1.0	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1	0.0
HKID-1	10	10	10	9	9	9	9	9	9	9	8
FKOK-1	9	9	9	8	8	8	8	8	8	8	8
TKYO-11	8	8	8	8	8	8	8	8	8	7	7
KNGW-2	8	8	8	8	8	8	8	8	7	7	7
CHBA-1	8	8	8	8	8	8	8	7	7	7	7
CHBA-4	8	8	8	8	8	7	7	7	7	7	7
KNGW-4	8	8	8	7	7	7	7	7	7	7	7
HYOG-2	8	8	7	7	7	7	7	7	7	7	7

CNST.	PARAMETER										
	1.0	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1	0.0
KYOT-2	8	8	7	7	7	7	7	7	7	7	7
OSAK-3	8	7	7	7	7	7	7	7	7	7	7
KNGW-3	7	7	7	7	7	7	7	7	7	7	7
SITM-2	7	7	7	7	7	7	7	7	7	6	6
MIYG-1	7	7	7	7	7	7	7	6	6	6	6
OSAK-5	7	7	7	7	7	7	7	6	6	6	6
TKYO-7	7	7	7	7	7	7	7	6	6	6	6
TKYO-10	7	7	7	7	7	7	6	6	6	6	6
OSAK-4	7	7	7	6	6	6	6	6	6	6	6
SITM-4	6	6	6	6	6	6	6	6	6	6	6
HYOG-1	6	6	6	6	6	6	6	6	6	6	6
AITI-2	6	6	6	6	6	6	6	6	6	6	6
SZOK-1	6	6	6	6	6	6	6	6	6	6	6
KNGW-1	6	6	6	6	6	6	6	5	5	5	5
NARA-1	6	6	6	6	5	5	5	5	5	5	5
AITI-4	6	5	5	5	5	5	5	5	5	5	5
SITM-5	5	5	5	5	5	5	5	5	5	5	5
GIFU-1	5	5	5	5	5	5	5	5	5	5	5
SITM-1	5	5	5	5	5	5	5	5	5	5	5
HRSM-1	5	5	5	5	5	5	5	5	5	5	5
MIEE-1	5	5	5	5	5	5	5	5	5	5	5
SIGA-1	5	5	5	5	5	5	5	5	5	5	5
OKNW-1	5	5	5	5	5	5	5	5	5	5	5
SZOK-2	5	5	5	5	5	5	5	5	5	5	5
OSAK-7	5	5	5	5	5	5	5	5	5	5	5
OSAK-2	5	5	5	5	5	5	5	5	5	5	5
IBRK-1	5	5	5	5	5	5	5	5	5	5	5
KNGW-5	5	5	5	5	5	5	5	5	5	5	5
KMMT-1	5	5	5	5	5	5	5	5	5	5	5
AITI-6	5	5	5	5	5	5	5	5	5	5	5
HKID-5	5	5	5	5	5	5	5	5	5	4	4
TCHG-1	5	5	5	5	5	5	5	4	4	4	4
SZOK-3	5	5	5	5	5	5	4	4	4	4	4
TKYO-4	5	5	5	5	5	5	4	4	4	4	4
AITI-1	5	4	4	5	4	4	4	4	4	4	4
TKYO-3	4	4	4	4	4	4	4	4	4	4	4
AITI-3	4	4	4	4	4	4	4	4	4	4	4
TKYO-2	4	4	4	4	4	4	4	4	4	4	4
IBRK-3	4	4	4	4	4	4	4	4	4	4	4
FKOK-2	4	4	4	4	4	4	4	4	4	4	4
OKYM-2	4	4	4	4	4	4	4	4	4	4	4
AOMR-1	4	4	4	4	4	4	4	4	4	4	4
HKID-4	4	4	4	4	4	4	4	4	4	4	4
HYOG-3	4	4	4	4	4	4	4	4	4	4	4

CNST.	PARAMETER										
	1.0	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1	0.0
OKYM-1	4	4	4	4	4	4	4	4	4	4	4
NGSK-1	4	4	4	4	4	4	4	4	4	4	4
CHBA-3	4	4	4	4	4	4	4	4	4	4	4
HRSM-3	4	4	4	4	4	4	4	4	4	4	4
CHBA-2	4	4	4	4	4	4	4	4	4	4	4
TKYO-5	4	4	4	4	4	4	4	4	4	4	4
SAGA-1	4	4	4	4	4	4	4	4	4	4	4
FKOK-4	4	4	4	4	4	4	4	4	4	4	4
FKOK-3	4	4	4	4	4	4	4	4	4	4	4
TKYO-9	4	4	4	4	4	4	4	4	4	4	4
TCHG-2	4	4	4	4	4	4	4	4	4	4	4
YMNS-1	4	4	4	4	4	4	4	4	4	4	4
HYOG-4	4	4	4	4	4	4	4	4	4	4	4
KGSM-1	4	4	4	4	4	4	4	4	4	4	4
KYOT-1	4	4	4	4	4	4	4	4	4	4	4
YMGC-2	3	4	4	4	4	4	4	4	4	4	4
IWTE-1	3	3	3	4	4	4	4	4	4	4	4
TKSM-1	3	3	3	4	4	4	4	4	4	4	4
KOTI-1	3	3	3	4	4	4	4	4	4	4	4
FUKI-1	3	3	3	3	3	3	3	3	4	3	3
OITA-1	3	3	3	3	3	3	3	3	3	3	3
ISKW-1	3	3	3	3	3	3	3	3	3	3	3
TKYO-6	3	3	3	3	3	3	3	3	3	3	3
FKSM-1	3	3	3	3	3	3	3	3	3	3	3
SIMN-1	3	3	3	3	3	3	3	3	3	3	3
GIFU-2	3	3	3	3	3	3	3	3	3	3	3
NIGT-3	3	3	3	3	3	3	3	3	3	3	3
NIGT-1	3	3	3	3	3	3	3	3	3	3	3
FKSM-2	3	3	3	3	3	3	3	3	3	3	3
MYZK-1	3	3	3	3	3	3	3	3	3	3	3
GNMA-3	3	3	3	3	3	3	3	3	3	3	3
YMGC-1	3	3	3	3	3	3	3	3	3	3	3
AKTA-1	3	3	3	3	3	3	3	3	3	3	3
HKID-2	3	3	3	3	3	3	3	3	3	3	3
AITI-5	3	3	3	3	3	3	3	3	3	3	3
OSAK-1	3	3	3	3	3	3	3	3	3	3	3
YMGT-1	3	3	3	3	3	3	3	3	3	3	3
KMMT-2	3	3	3	3	3	3	3	3	3	3	3
OSAK-6	3	3	3	3	3	3	3	3	3	3	3
HRSM-2	3	3	3	3	3	3	3	3	3	3	3
GNMA-1	3	3	3	3	3	3	3	3	3	3	3
IBRK-2	3	3	3	3	3	3	3	3	3	3	3
WKYM-1	3	3	3	3	3	3	3	3	3	3	3
TOYM-1	3	3	3	3	3	3	3	3	3	3	3

CNST.	PARAMETER										
	1.0	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1	0.0
SITM-3	2	3	3	3	3	3	3	3	3	3	3
TOTR-1	2	3	3	3	3	3	3	3	3	3	3
NGSK-2	2	2	3	3	3	3	3	3	3	3	3
NGNO-1	2	2	2	2	3	3	3	3	3	3	3
IWTE-2	2	2	2	2	2	2	3	3	3	3	3
NGNO-3	2	2	2	2	2	2	3	3	3	3	3
FKSM-3	2	2	2	2	2	2	2	3	3	3	3
MIEE-2	2	2	2	2	2	2	2	3	3	3	3
MIYG-2	2	2	2	2	2	2	2	3	3	3	3
NIGT-2	2	2	2	2	2	2	2	3	3	3	3
KAGW-1	2	2	2	2	2	2	2	3	3	3	3
GNMA-2	2	2	2	2	2	2	2	2	2	3	3
HKID-3	2	2	2	2	2	2	2	2	2	3	3
EHIM-2	2	2	2	2	2	2	2	2	2	3	3
YMGT-2	2	2	2	2	2	2	2	2	2	3	3
EHIM-1	2	2	2	2	2	2	2	2	2	2	3
AOMR-2	2	2	2	2	2	2	2	2	2	2	2
NGNO-4	2	2	2	2	2	2	2	2	2	2	2
AKTA-2	2	2	2	2	2	2	2	2	2	2	2
TKYO-1	2	2	2	2	2	2	2	2	2	2	2
TOYM-2	2	2	2	2	2	2	2	2	2	2	2
NGNO-2	2	2	2	2	2	2	2	2	2	2	2
KGSM-2	2	2	2	2	2	2	2	2	2	2	2
KAGW-2	2	2	2	2	2	2	2	2	2	2	2
EHIM-3	2	2	2	2	2	2	2	2	2	2	2
WKYM-2	2	2	2	2	2	2	2	2	2	2	2
OITA-2	2	2	2	2	2	2	2	2	2	2	2
MYZK-2	2	2	2	2	2	2	2	2	2	2	2
TKYO-8	2	2	2	2	2	2	2	2	2	2	2
NIGT-4	2	2	2	2	2	2	2	2	2	2	2
ISKW-2	1	1	2	2	2	2	2	2	2	2	2
KGSM-3	1	1	1	1	1	2	2	2	2	2	2
HYOG-5	1	1	1	1	1	1	2	2	2	2	2
KGSM-4	1	1	1	1	1	1	1	2	2	2	2
Total	511	511	511	511	511	511	511	511	511	511	511

Comparing the results of Table 3 with the allocation by the *LFM* in Table 2, we can easily recognize that if the parameter value t satisfies $t > 0.5$, larger constituencies get more seats and smaller ones have less, while if $t < 0.4$, smaller constituencies obtain more seats and larger ones less. Thus we can conclude that the *PDM* should be taken into account for the parameter t such that $0.4 \leq t \leq 0.5$ since a parameter t larger than 0.5 makes the *PDM* too favorable to larger constituencies and t less than 0.4 makes the method too favorable to smaller constituencies (see Table 4).

Table 4. Final apportionments by parametric divisor method

CNST.	PARAMETER										
	0.50	0.49	0.48	0.47	0.46	0.45	0.44	0.43	0.42	0.41	0.40
HKID-1	10	10	10	10	10	10	10	10	9	9	9
FKOK-1	9	9	9	9	9	9	9	9	9	9	9
TKYO-11	8	8	8	8	8	8	8	8	8	8	8
KNGW-2	8	8	8	8	8	8	8	8	8	8	8
CHBA-1	8	8	8	8	8	8	8	8	8	8	8
CHBA-4	8	8	8	8	8	8	8	8	8	8	8
KNGW-4	8	8	8	8	8	8	8	8	8	8	8
HYOG-2	7	7	7	7	7	7	7	7	7	7	7
KYOT-2	7	7	7	7	7	7	7	7	7	7	7
OSAK-3	7	7	7	7	7	7	7	7	7	7	7
KNGW-3	7	7	7	7	7	7	7	7	7	7	7
SITM-2	7	7	7	7	7	7	7	7	7	7	7
MIYG-1	7	7	7	7	7	7	7	7	7	7	7
OSAK-5	7	7	7	7	7	7	7	7	7	7	7
TKYO-7	7	7	7	7	7	7	7	7	7	7	7
TKYO-10	7	7	7	7	7	7	7	7	7	7	7
OSAK-4	6	6	6	6	6	6	6	6	6	6	6
SITM-4	6	6	6	6	6	6	6	6	6	6	6
HYOG-1	6	6	6	6	6	6	6	6	6	6	6
AITI-2	6	6	6	6	6	6	6	6	6	6	6
SZOK-1	6	6	6	6	6	6	6	6	6	6	6
KNGW-1	6	6	6	6	6	6	6	6	6	6	6
NARA-1	6	6	6	6	6	6	6	6	6	6	6
AITI-4	5	5	5	5	5	5	5	5	5	5	5
SITM-5	5	5	5	5	5	5	5	5	5	5	5
GIFU-1	5	5	5	5	5	5	5	5	5	5	5
SITM-1	5	5	5	5	5	5	5	5	5	5	5
HRSM-1	5	5	5	5	5	5	5	5	5	5	5
MIEE-1	5	5	5	5	5	5	5	5	5	5	5
SIGA-1	5	5	5	5	5	5	5	5	5	5	5
OKNW-1	5	5	5	5	5	5	5	5	5	5	5
SZOK-2	5	5	5	5	5	5	5	5	5	5	5
OSAK-7	5	5	5	5	5	5	5	5	5	5	5
OSAK-2	5	5	5	5	5	5	5	5	5	5	5
IBRK-1	5	5	5	5	5	5	5	5	5	5	5
KNGW-5	5	5	5	5	5	5	5	5	5	5	5
KMMT-1	5	5	5	5	5	5	5	5	5	5	5
AITI-6	5	5	5	5	5	5	5	5	5	5	5
HKID-5	5	5	5	5	5	5	5	5	5	5	5
TCHG-1	5	5	5	4	4	4	4	4	4	4	4
SZOK-3	4	4	4	4	4	4	4	4	4	4	4

CNST.	PARAMETER										
	0.50	0.49	0.48	0.47	0.46	0.45	0.44	0.43	0.42	0.41	0.40
TKYO-4	4	4	4	4	4	4	4	4	4	4	4
AITI-1	4	4	4	4	4	4	4	4	4	4	4
TKYO-3	4	4	4	4	4	4	4	4	4	4	4
AITI-3	4	4	4	4	4	4	4	4	4	4	4
TKYO-2	4	4	4	4	4	4	4	4	4	4	4
IBRK-3	4	4	4	4	4	4	4	4	4	4	4
FKOK-2	4	4	4	4	4	4	4	4	4	4	4
OKYM-2	4	4	4	4	4	4	4	4	4	4	4
AOMR-1	4	4	4	4	4	4	4	4	4	4	4
HKID-4	4	4	4	4	4	4	4	4	4	4	4
HYOG-3	4	4	4	4	4	4	4	4	4	4	4
OKYM-1	4	4	4	4	4	4	4	4	4	4	4
NGSK-1	4	4	4	4	4	4	4	4	4	4	4
CHBA-3	4	4	4	4	4	4	4	4	4	4	4
HRSM-3	4	4	4	4	4	4	4	4	4	4	4
CHBA-2	4	4	4	4	4	4	4	4	4	4	4
TKYO-5	4	4	4	4	4	4	4	4	4	4	4
SAGA-1	4	4	4	4	4	4	4	4	4	4	4
FKOK-4	4	4	4	4	4	4	4	4	4	4	4
FKOK-3	4	4	4	4	4	4	4	4	4	4	4
TKYO-9	4	4	4	4	4	4	4	4	4	4	4
TCHG-2	4	4	4	4	4	4	4	4	4	4	4
YMNS-1	4	4	4	4	4	4	4	4	4	4	4
HYOG-4	4	4	4	4	4	4	4	4	4	4	4
KGSM-1	4	4	4	4	4	4	4	4	4	4	4
KYOT-1	4	4	4	4	4	4	4	4	4	4	4
YMGC-2	3	3	3	4	4	4	3	3	4	4	4
IWTE-1	3	3	3	3	3	3	3	3	3	3	3
TKSM-1	3	3	3	3	3	3	3	3	3	3	3
KOTI-1	3	3	3	3	3	3	3	3	3	3	3
FUKI-1	3	3	3	3	3	3	3	3	3	3	3
OITA-1	3	3	3	3	3	3	3	3	3	3	3
ISKW-1	3	3	3	3	3	3	3	3	3	3	3
TKYO-6	3	3	3	3	3	3	3	3	3	3	3
FKSM-1	3	3	3	3	3	3	3	3	3	3	3
SIMN-1	3	3	3	3	3	3	3	3	3	3	3
GIFU-2	3	3	3	3	3	3	3	3	3	3	3
NIGT-3	3	3	3	3	3	3	3	3	3	3	3
NIGT-1	3	3	3	3	3	3	3	3	3	3	3
FKSM-2	3	3	3	3	3	3	3	3	3	3	3
MYZK-1	3	3	3	3	3	3	3	3	3	3	3
GNMA-3	3	3	3	3	3	3	3	3	3	3	3
YMGC-1	3	3	3	3	3	3	3	3	3	3	3
AKTA-1	3	3	3	3	3	3	3	3	3	3	3

CNST.	PARAMETER										
	0.50	0.49	0.48	0.47	0.46	0.45	0.44	0.43	0.42	0.41	0.40
HKID-2	3	3	3	3	3	3	3	3	3	3	3
AITI-5	3	3	3	3	3	3	3	3	3	3	3
OSAK-1	3	3	3	3	3	3	3	3	3	3	3
YMGT-1	3	3	3	3	3	3	3	3	3	3	3
KMMT-2	3	3	3	3	3	3	3	3	3	3	3
OSAK-6	3	3	3	3	3	3	3	3	3	3	3
HRSM-2	3	3	3	3	3	3	3	3	3	3	3
GNMA-1	3	3	3	3	3	3	3	3	3	3	3
IBRK-2	3	3	3	3	3	3	3	3	3	3	3
WKYM-1	3	3	3	3	3	3	3	3	3	3	3
TOYM-1	3	3	3	3	3	3	3	3	3	3	3
SITM-3	3	3	3	3	3	3	3	3	3	3	3
TOTR-1	3	3	3	3	3	3	3	3	3	3	3
NGSK-2	3	3	3	3	3	3	3	3	3	3	3
NGNO-1	2	2	2	2	2	2	2	2	2	2	2
IWTE-2	2	2	2	2	2	2	2	2	2	2	2
NGNO-3	2	2	2	2	2	2	2	2	2	2	2
FKSM-3	2	2	2	2	2	2	2	2	2	2	2
MIEE-2	2	2	2	2	2	2	2	2	2	2	2
MIYG-2	2	2	2	2	2	2	2	2	2	2	2
NIGT-2	2	2	2	2	2	2	2	2	2	2	2
KAGW-1	2	2	2	2	2	2	2	2	2	2	2
GNMA-2	2	2	2	2	2	2	2	2	2	2	2
HKID-3	2	2	2	2	2	2	2	2	2	2	2
EHIM-2	2	2	2	2	2	2	2	2	2	2	2
YMGT-2	2	2	2	2	2	2	2	2	2	2	2
EHIM-1	2	2	2	2	2	2	2	2	2	2	2
AOMR-2	2	2	2	2	2	2	2	2	2	2	2
NGNO-4	2	2	2	2	2	2	2	2	2	2	2
AKTA-2	2	2	2	2	2	2	2	2	2	2	2
TKYO-1	2	2	2	2	2	2	2	2	2	2	2
TOYM-2	2	2	2	2	2	2	2	2	2	2	2
NGNO-2	2	2	2	2	2	2	2	2	2	2	2
KGSM-2	2	2	2	2	2	2	2	2	2	2	2
KAGW-2	2	2	2	2	2	2	2	2	2	2	2
EHIM-3	2	2	2	2	2	2	2	2	2	2	2
WKYM-2	2	2	2	2	2	2	2	2	2	2	2
OITA-2	2	2	2	2	2	2	2	2	2	2	2
MYZK-2	2	2	2	2	2	2	2	2	2	2	2
TKYO-8	2	2	2	2	2	2	2	2	2	2	2
NIGT-4	2	2	2	2	2	2	2	2	2	2	2
ISKW-2	1	1	1	1	1	1	2	2	2	2	2
KGSM-3	1	1	1	1	1	1	1	1	1	1	1
HYOG-5	1	1	1	1	1	1	1	1	1	1	1

CNST.	PARAMETER										
	0.50	0.49	0.48	0.47	0.46	0.45	0.44	0.43	0.42	0.41	0.40
KGSM-4	1	1	1	1	1	1	1	1	1	1	1
Total	511	511	511	511	511	511	511	511	511	511	511

Balinski and Young [8] proposed a model of bias to conclude that Webster's method (MFM) is the most preferred method. In this section we apply a similar approach to theirs to Japan's House of Representative (HOR) data, then we investigate the unbiasedness of the apportionment method by comparing our PDM with other methods. Given the population distribution $\boldsymbol{p} = (p_1, ..., p_N)$ and its corresponding apportionment $\boldsymbol{d} = (d_1, ..., d_N)$ let S_L and S_S be two disjoint subsets of $S = \{1, ..., N\}$ indicating the larger constituencies and the smaller constituencies, respectively. Then the apportionment $\boldsymbol{d} = (d_1, ..., d_N)$ favors the smaller constituencies if $\frac{\sum_{S_S} d_j}{\sum_{S_S} p_j} > \frac{\sum_{S_L} d_i}{\sum_{S_L} p_i}$ and favors the larger constituencies $\frac{\sum_{S_S} d_j}{\sum_{S_S} p_j} < \frac{\sum_{S_L} d_i}{\sum_{S_L} p_i}$ where $\sum_{S_S} d_j$ indicates $\sum_{j \in S_S} d_j$ and so on. Among Japan's 130 (=N) HOR constituencies we define S_L as the largest one-third, *i.e.*, $\mid S_L \mid$=43, and S_S as the smallest one-third, *i.e.*, $\mid S_S \mid$=43.

We originally define the measure of bias related to the apportionment method as follows.

$$m_S = \mid \sum_{S_S} d_j - \sum_{S_S} q_j \mid \qquad m_L = \mid \sum_{S_L} d_j - \sum_{S_L} q_j \mid \tag{3.1}$$

$$\Delta = \max\{m_S,\ m_L\} \tag{3.2}$$

Table 5 shows the above measure of bias for six traditional apportionment methods. In Table 4 we give the numerical results of the PDM for the parameter values $0.4 \leq t \leq 0.5$. Using the numerical results of the PDM in Table 3, we can calculate the measure of bias Δ_t for parameter values $0 \leq t \leq 1$ as shown in the Fig.1. From Table 4 we obtain Fig.2 showing the details of the measure of bias Δ_t for $0.35 \leq t \leq 0.5$. We can easily recognize that Δ_t takes the minimum at around $0.43 \leq t \leq 0.44$.

Table 5. Measure of biases Δ for the apportionment methods

Method	*GDM*	*MFM*	*EPM*	*HMM*	*SDM*	*LFM*
m_L	11.956	0.956	0.044	0.044	13.044	0.044
m_S	9.754	0.754	0.246	1.246	10.256	0.754
Δ	11.956	0.956	0.246	1.246	13.044	0.754

Now we define another measure of bias as follows.

$$k_S = \frac{\sum_{S_S} d_j}{\sum_{S_S} q_j} \qquad k_L = \frac{\sum_{S_L} d_j}{\sum_{S_L} q_j} \tag{3.3}$$

$$\Theta = (\frac{k_S}{k_L} - 1) \times 100 \tag{3.4}$$

The above definition is equivalent to Balinski and Young's ([8], p.126). Numerical results for the measure of bias Θ_t are given in Table 6 for six traditional apportionment methods. Fig.3 shows the measure of bias Θ_t for parameter values $0 \leq t \leq 1$. Fig.4 shows the details of Θ_t for $0.35 \leq t \leq 0.5$. We can conclude that the absolute value of the bias Θ, *i.e.*, $\mid \Theta_t \mid$ takes the minimum at around $0.43 \leq t \leq 0.44$.

Fig. 1 Measure of Bias Δt

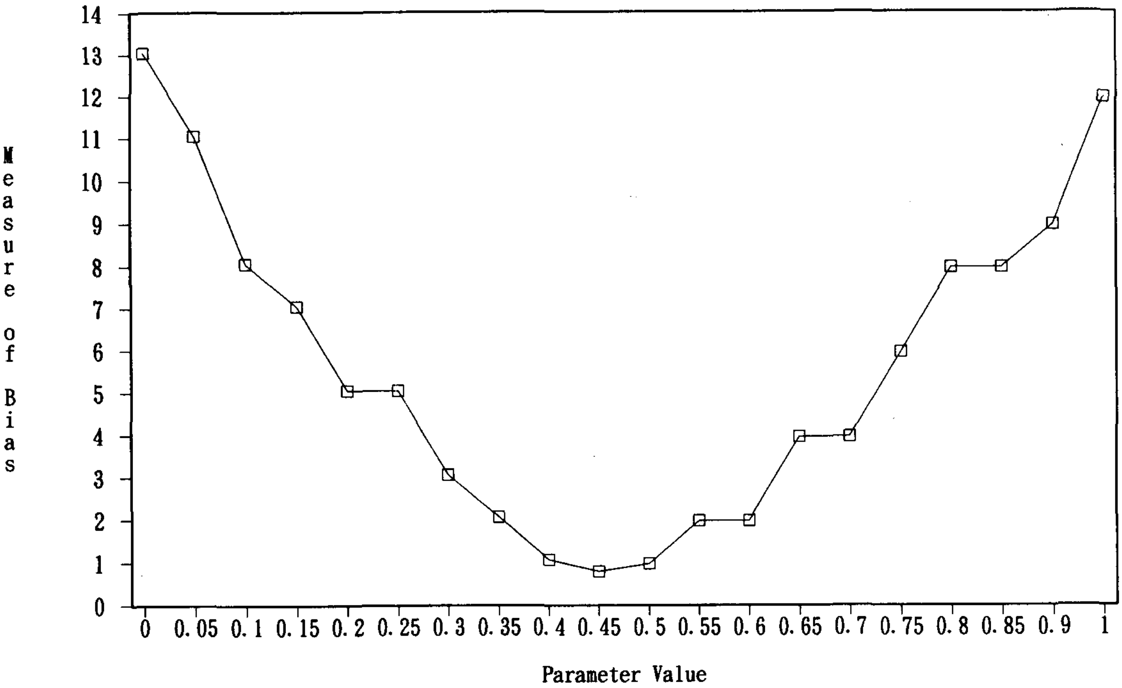


Table 6. Measure of biases Θ for the apportionment methods

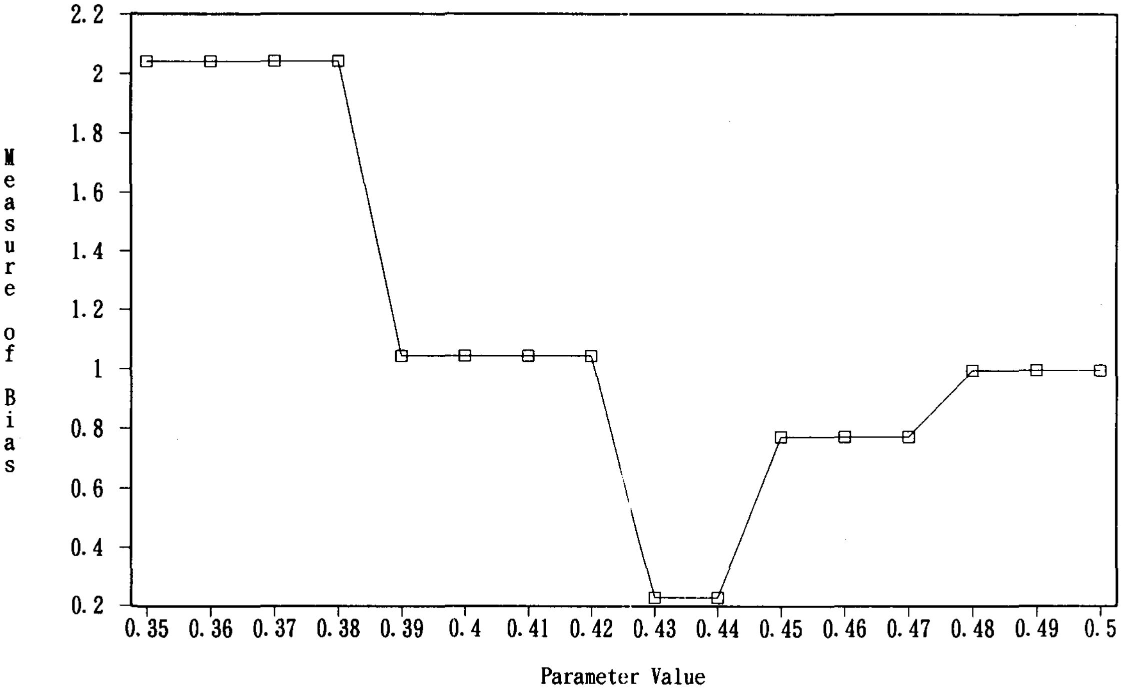
$Method$	GDM	MFM	EPM	HMM	SDM	LFM
k_L	1.046	1.004	1.000	1.000	0.950	1.000
k_S	0.897	0.992	1.003	1.013	1.108	0.992
Θ	-14.26	-1.19	0.26	1.31	16.65	-0.79

From the above results we believe we can say that the most unbiased apportionment method could be the PDM for the parameter values, approximately $0.43 \leq t \leq 0.44$. From the numerical results in Table 2 we know that the EPM gives exactly the same allocation of seats to our 130 constituencies as the LFM. Comparing the allocation of the MFM with that of the EPM, we know that the latter method gives one less seat to TCHG-1 (larger constituency) and one more seat to YMGC-2 (medium size constituency). From Table 4 we know that the PDM for parameter values $0.48 \leq t \leq 0.50$ gives exactly the same allocation as the MFM, while the PDM for parameter values $0.45 \leq t \leq 0.47$ gives the same allocation as the EPM, *i.e.*, the LFM. The PDM for parameter values $0.43 \leq t \leq 0.44$ gives one less seat to the medium sized constituency YMGC-2 and one more seat to the smaller constituency ISKW-2 compared with the EPM, *i.e.*, the LFM. Furthermore The PDM for parameter values $0.40 \leq t \leq 0.42$ gives one less seat to the largest constituency HKID-1 and one more seat to the medium sized constituency YMGC-2 compared with the EPM for parameter values $0.43 \leq t \leq 0.44$.

4 Summary and conclusion

In this paper we investigated the unbiasedness of the PDM based upon the parameter t given in (2.13). As mentioned in sections 2 and 3, PDM satisfies the house monotone

Fig. 2 Measure of Bias Δt



property for any t such that $0 \leq t \leq 1$ as it belongs to the divisor method. It does not guarantee the quota property as do other apportionment methods (with the exception of the LFM). In section 2 we characterized the PDM from the viewpoints of the local measure of inequity, which was originally proposed in our work (see [13]). Then we found the properties of the PDM related to the maximum parameter values and the allocation obtained from the PDM, which we expect would be useful to gain more insights into the allocation by the PDM. As shown in section 3, from the results of our numerical experiments as illustrated in Tables 5 and 6 we can conclude that the apportionment method LFM is located between MFM and EPM from the viewpoint of biasedness to the population size of the constituency. As history shows (see *e.g.* [8, 10]), there was a harsh controversy in the U.S. Congress in the 1950's over whether the MFM or the EPM should be accepted. Although Balinski and Young [8, 10] insist that the MFM is the only unbiased divisor method, we believe that generally the MFM is more favorable to larger constituencies as most numerical examples, although they are hypothetical and not real data, violate the upper quota property (see *e.g.* [10, 12]) and as our own numerical experiments also show.

In conclusion, we believe that the method LFM, which satisfies the quota property, gives "a most reasonable and impartial" assignment of seats to the constituency although it does not satisfy the house monotone property. Based upon our numerical experiments related to 1990 census data of Japan's House of Representatives we would like to strongly recommend the PDM with the parameter value $0.43 \leq t \leq 0.44$ since it gives almost the same assignment as the LFM as shown in section 3, and importantly, it satisfies the house monotone property. In this sense we evaluate the unbiasedness of the Condorcet method (see, *e.g.*, Balinski and Young [8]), which is equivalent to t=0.4 for the PDM and is close to our recommendation of the parameter value.

Presently we are investigating other properties of population monotonicity, constituency

Fig. 3 Measure of bias Θt

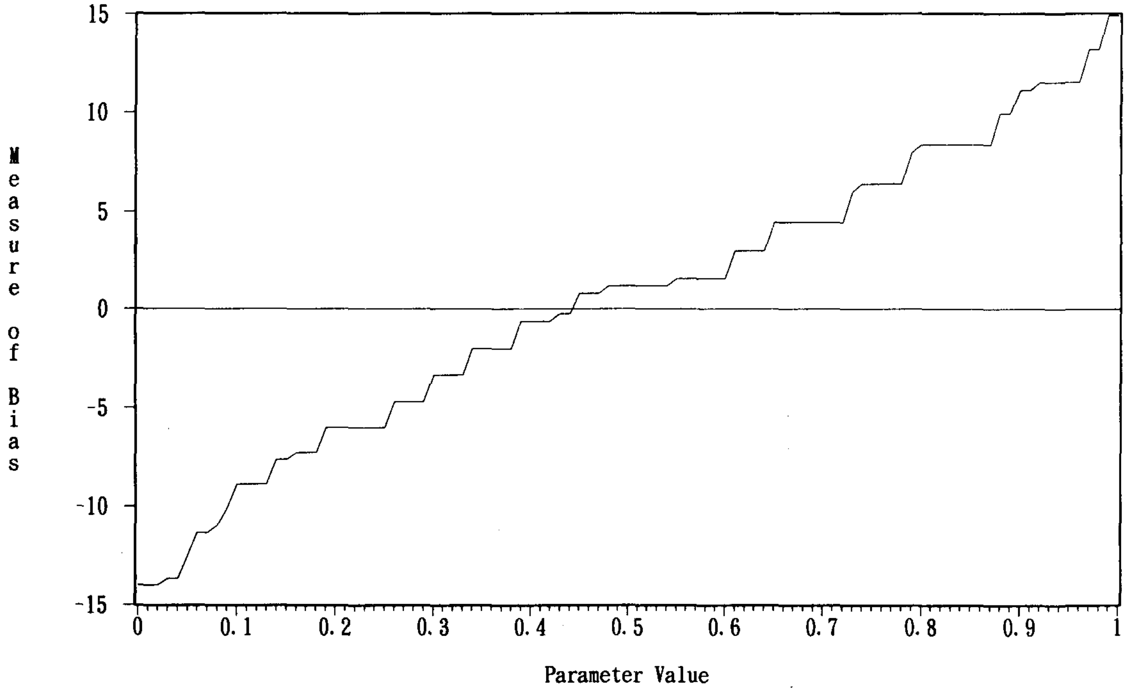
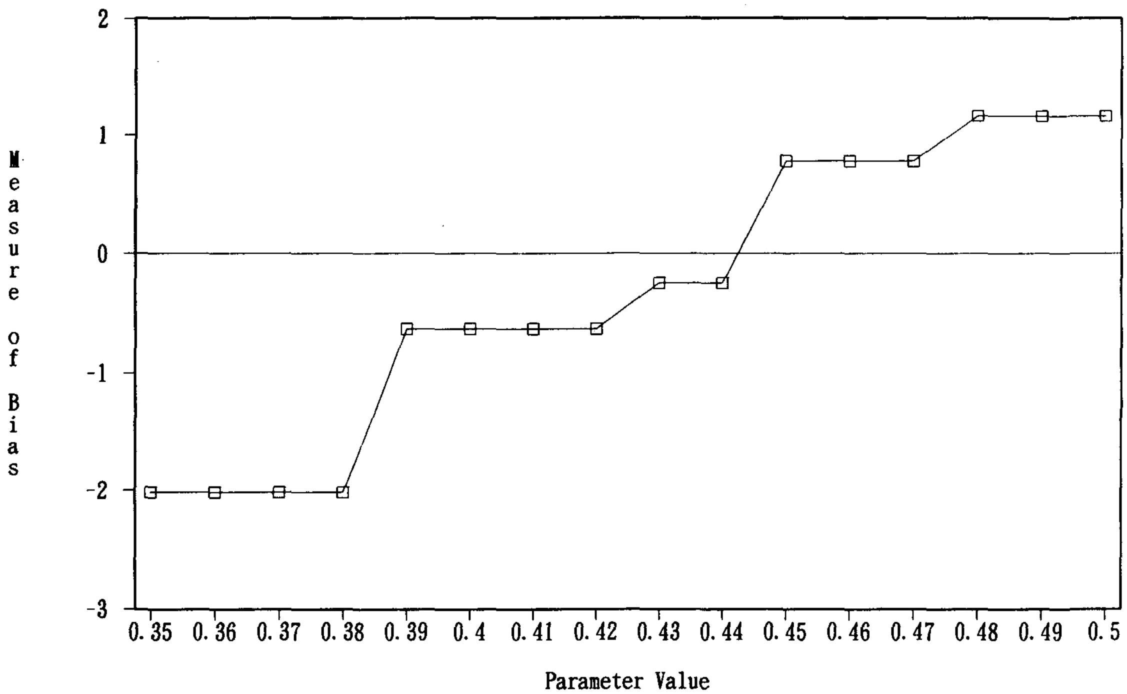


Fig. 4 Measure of Bias Θt



and so on for the *PDM* to see if this method can be made to more closely satisfy these properties.

Acknowledgements

The authors would like to express their sincere thanks to anonymous referees for their valuable and helpful comments.

This research was partly supported by Grant-in-Aid for Scientific Research by the Ministry of Education, Science and Culture, Grant No.04832011.

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