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CONTINUOUS REVIEW CYCLIC INVENTORY MODELS WITH EMERGENCY ORDER

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Abstract Many kinds of reorder point - reorder quantity policies with an emergency order, in which we can decide when the expedited order point is reached and then we place an expedited replenishment order, have been presented in earlier contributions. Very few studies on cyclic inventory control with an emergency order, however, have been reported. The purpose of this paper is to establish some analytical results on the continuous review cyclic inventory policy with two kinds of lead times. We derive necessary and sufficient conditions for the existence of the optimal ordering time which minimizes the long-run average or the expected total discounted costs. Also, we give explicit formulae of the expected costs by specifying demand processes. Poisson and Brownian motion processes are assumed as the demand processes. Finally, we numerically calculate the optimal ordering time and the optimal order quantity, and refer to the sensitivity of model parameters for the optimal policy.

1. Introduction

Many kinds of reorder point - reorder quantity policies with an emergency order, in which we can decide when the expedited order point is reached and then we place an expedited replenishment order, have been presented in earlier contributions. By taking account of the emergency order, Barankin [3], Hadley and Whitin [7], Neuts [12] and Rosenshine and Obee [15] have considered somewhat different inventory models. Recently, Moinzadeh and Nahmias [11] have discussed an approximated (Q_1, Q_2, R_1, R_2) policy with an emergency order, which is a natural extension of the ordinary (Q, R) policy. The result simplifies the more general analysis by Whittmore and Saunders [17].

Very few studies on cyclic inventory control with an emergency order, however, have been reported so far as we know. The purpose of this paper is to establish some analytical results on the continuous review cyclic inventory policy with two kinds of lead times along the seminal contributions by Allen and D'Esopo [1, 2]. The inventory model considered here is the following: Order items are regularly delivered after a lead time if the stock is not still depleted until a prespecified time. On the other hand, if the stock level becomes 0 until the prespecified time, the expedited order is made at that time. The first problem is to determine the ordering time for the regular order. We derive necessary and sufficient conditions for the existence of the optimal ordering time which minimizes the long-run average or the expected total discounted costs.

The second problem is to seek the optimal ordering time and the optimal order quantity jointly minimizing the expected costs. In order to do it, the cumulative demand process must be specified. In this paper, Poisson and Brownian motion processes are assumed as the cumulative demand processes. The former is often used in the context of inventory theory. On the other hand, the latter corresponds to the assumption that the cumulative demand process obeys a normal distribution. In fact, the normal distribution is assumed in many static inventory models. Bather [4] and Puterman [14] have developed the (s, S) inventory models in the same assumption. Dohi *et al.* [6] have discussed a two-bin inventory system under the assumption that the cumulative demand follows a reflecting Brownian motion process. We give explicit formulae of the expected costs under the two kinds of demand processes above.

Finally, we numerically calculate the optimal ordering time and the optimal order quantity so as to minimize the expected costs. Also, we numerically refer to the sensitivity of model parameters for the optimal policy. The mathematical techniques used in this paper are similar to those often used in reliability theory. In fact, the cyclic inventory models considered here generalize some elementary order–replacement models for a one-unit system.

2. Model Description

2.1 Notation and assumption

Let us consider a continuous review single-product inventory system. The inventory level is decreased by the satisfaction of a demand. Without loss of generality, the inventory level is initially set to Q (> 0). Let $\{N(t), t \geq 0 \mid N(0) = 0\}$ be a cumulative demand at time t and be a stochastic process with an absorbing boundary at $N(t) = Q$. The inventory level process without jump (delivering items), $\{X(t), t \geq 0\}$, is defined as follows:

$$X(t) = Q - N(t), \quad (0 \leq t \leq \tau), \tag{1}$$

where

$$\tau = \inf\{t \geq 0; X(t) = 0 \mid X(0) = Q\}. \tag{2}$$

The inventory management begins operating at time 0, and the planning horizon is infinite. If the stock is depleted up to a prespecified time $t_0 \in (0, \infty]$, the emergency order is placed at the time when the stock is depleted and after a lead time L_1 (> 0), the amount Q of an item is delivered. We call t_0 *ordering time* in this paper. On the other hand, if the stock is not depleted before the time t_0, the regular order is immediately made at the time t_0, and the amount Q is delivered after a lead time $L_2 (> 0)$. In this situation, Q items are delivered at the time $t_0 + L_2$. The one cycle is defined as the time period from time 0 to the time when the inventory level becomes Q next, and repeats itself continually.

Define the indicator function $I_{\{A\}}$ for the event $\{A\}$. Fig. 1 shows the schematic illustration of the inventory model under consideration, where the events $\{A_1\}$, $\{A_2\}$ and $\{A_3\}$ indicate $\{\min_{0 \leq t \leq t_0} X(t) = 0\}$, $\{\min_{t_0 < t \leq t_0 + L_2} X(t) = 0$ and $\min_{0 \leq t \leq t_0} X(t) > 0\}$ and $\{\min_{0 \leq t \leq t_0 + L_2} X(t) > 0\}$, respectively. By taking account of the delivery scheduling mentioned above, the net inventory level for one cycle, $\{X_N(t), t \geq 0\}$, is the following: (i) In the event $\{A_1\}$, $X_N(t) = X(t)$ and 0 when $0 \leq t < \tau$ and $\tau \leq t < \tau + L_1$. (ii) In the event $\{A_2\}$, $X_N(t) = X(t)$ and 0 when $0 \leq t < \tau$ and $\tau \leq t < t_0 + L_2$. (iii) In the event $\{A_3\}$, $X_N(t) = X(t)$ and $X(t) + Q$ when $0 \leq t < t_0 + L_2$ and $t_0 + L_2 \leq t < \tau$.

Suppose that the time when $\{X(t), t \geq 0\}$ becomes 0 for the first time, *i.e.* the stock runs out, obeys a probability distribution function $\Pr\{\tau \leq t\} = F(t)$ with density function $f(t)$. Of course, specifying the cumulative demand process $N(t)$ determines $F(t)$ and $f(t)$. The costs considered here are the following; a cost k per unit time is incurred for the shortage period; a cost h is an inventory holding cost per unit product and unit time; and costs c_1 and c_2 per unit product are fixed ordering costs for expedited and regular orders, respectively. These assumptions on costs are rather standard in the context of inventory theory. Under the assumptions, we formulate two expected cost criteria.

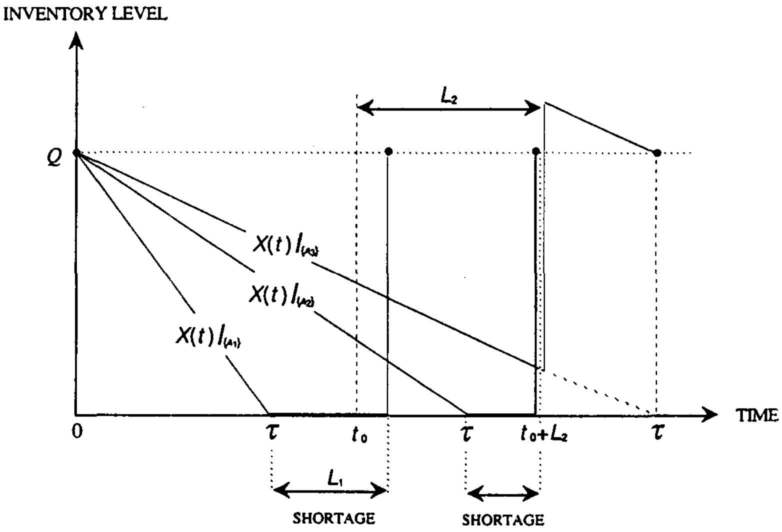

Fig. 1. State diagram for one cycle.

2.2 Long-run average cost

The long-run average cost (the expected cost per unit time in the steady-state) is, from the well-known result (*e.g.* [16]),

$$\begin{aligned} C(t_0, Q) &= \lim_{t\to\infty} \frac{\mathrm{E}[\text{the total cost on } (0,t]]}{t} \\ &= \phi(t_0, Q)/T(t_0, Q), \end{aligned} \tag{3}$$

where E is the mathematical expectation operator. The function $\phi(t_0, Q)$ is the expected cost for one cycle as

$$\begin{aligned} \phi(t_0, Q) &= h\Big\{\mathrm{E}[\int_0^{\tau} X(t)dt] + Q\int_{t_0+L_2}^{\infty} [t-(t_0+L_2)]dF(t)\Big\} \\ &\quad +k\Big\{\int_0^{t_0} L_1 dF(t) + \int_{t_0}^{t_0+L_2} (t_0+L_2-t)dF(t)\Big\} \\ &\quad +c_1 QF(t_0) + c_2 Q\bar{F}(t_0), \end{aligned} \tag{4}$$

where, in general,

$$\bar{\omega}(\cdot) = 1 - \omega(\cdot). \tag{5}$$

The function $T(t_0, Q)$ is the mean time of one cycle as

$$T(t_0, Q) = \int_0^{t_0} (t+L_1)dF(t) + \int_{t_0}^{t_0+L_2} (t_0+L_2)dF(t) + \int_{t_0+L_2}^{\infty} t dF(t). \tag{6}$$

2.3 Expected total discounted cost

Next let us consider the expected total discounted cost for an infinite time span. Let β (>0) be the discount factor. Then the expected total discounted cost for one cycle is

$$\pi(t_0, Q) = h\Big\{\mathrm{E}[\int_0^{\tau} \exp(-\beta t)X(t)dt] + Q\int_{t_0+L_2}^{\infty}\int_{t_0+L_2}^{t} \exp(-\beta s)ds dF(t)\Big\}$$

$$+k\Big\{\int_0^{t_0}\int_t^{t+L_1}\exp(-\beta s)ds dF(t)+\int_{t_0}^{t_0+L_2}\int_t^{t_0+L_2}\exp(-\beta s)dsdF(t)\Big\}$$
$$+c_1\int_0^{t_0}Q\exp(-\beta(t+L_1))dF(t)+c_2\int_{t_0}^{\infty}Q\exp(-\beta(t_0+L_2))dF(t). \quad (7)$$

Just after one cycle, a unit of cost is discounted as

$$\begin{aligned}\delta(t_0,Q) &= \int_0^{t_0}\exp(-\beta(t+L_1))dF(t)+\int_{t_0}^{t_0+L_2}\exp(-\beta(t_0+L_2))dF(t)\\ &\quad +\int_{t_0+L_2}^{\infty}\exp(-\beta t)dF(t). \end{aligned} \quad (8)$$

Thus, when the operation starts at time 0, the expected total discounted cost over an infinite time span is

$$\begin{aligned}V(t_0,Q) &= \sum_{n=0}^{\infty}\pi(t_0,Q)\delta(t_0,Q)^n\\ &= \pi(t_0,Q)/\bar{\delta}(t_0,Q).\end{aligned} \quad (9)$$

The well-known relationship between the long-run average and the expected total discounted costs is as follows:

$$C(t_0,Q)=\lim_{\beta\to 0}\beta V(t_0,Q). \quad (10)$$

3. Optimal Ordering Policies

For an arbitrary stochastic process $\{X(t),t\geq 0\}$, we obtain the optimal ordering policy which minimizes the expected costs formulated in the previous section.

First let us consider the case of long-run average cost. Define the numerator of the derivative of $C(t_0,Q)$ with respect to t_0, divided by $\bar{F}(t_0)$, as $q_c(t_0)$, *i.e.*

$$\begin{aligned}q_c(t_0) &= \Big\{hQ(R(t_0)-1)+[(c_1-c_2)Q+k(L_1-L_2)]r(t_0)+kR(t_0)\Big\}T(t_0,Q)\\ &\quad -[(L_1-L_2)r(t_0)+R(t_0)]\phi(t_0,Q),\end{aligned} \quad (11)$$

where

$$r(t_0)=f(t_0)/\bar{F}(t_0) \quad (12)$$

and

$$R(t_0)=\{F(t_0+L_2)-F(t_0)\}/\bar{F}(t_0) \quad (13)$$

are assumed to be differentiable, called *hazard rates* and have the same monotone properties (*e.g.* see [13]). Two special cases in the expected cost, $t_0=0$ and $t_0\to\infty$, are the following;

$$C(0,Q)=\phi(0,Q)/T(0,Q), \quad (14)$$

where

$$\phi(0,Q) = h\mathrm{E}[\int_0^{\tau}X(t)dt]+hQ\mathrm{E}[\tau]+hQ\int_0^{L_2}F(t)dt$$

$$+k\int_0^{L_2} F(t)dt + c_2Q - hQL_2, \tag{15}$$

$$T(0,Q) = \mathrm{E}[\tau] + \int_0^{L_2} F(t)dt; \tag{16}$$

and

$$C(\infty,Q) = \phi(\infty,Q)/T(\infty,Q), \tag{17}$$

where

$$\phi(\infty,Q) = h\mathrm{E}[\int_0^{\tau} X(t)dt] + c_1Q + kL_1, \tag{18}$$

$$T(\infty,Q) = \mathrm{E}[\tau] + L_1. \tag{19}$$

Moreover, we assume:

(A-1) $kL_2 + c_2Q < kL_1 + c_1Q$

(A-2) $0 < L_1 \leq L_2$

(A-3) $C(t_0,Q) < k$.

The assumption (A-1) means that the cost for the expedited order should be larger than that for the regular order, which seems to be plausible. The assumption (A-2) is also needed since the expedited order must be made quickly. The final assumption implies that the expected cost per unit time in the steady-state is less than the shortage cost per unit time, and is often used in the context of reliability theory (see [9, 13]). If $C(t_0,Q) \geq k$, for sufficiently small ordering and inventory holding costs, the shortage always occurs in the steady-state. This fact tells us that the assumption (A-3) should be adopted to avoid a trivial case.

The sufficient condition for the existence of optimal ordering policy is presented as follows.

Theorem 3.1 For an arbitrary distribution $F(t)$, if $q_c(\infty) > 0$ or $q_c(0) < 0$, then there exists at least one optimal ordering time $t_0{}^*$ ($0 \leq t_0{}^* < \infty$ or $0 < t_0{}^* \leq \infty$) minimizing the expected cost $C(t_0,Q)$.

Proof is omitted for brevity. Thus, we have the following theorem on the optimal ordering time $t_0{}^*$ which minimizes the expected cost $C(t_0,Q)$ by using the monotone properties of the hazard rate $r(t)$.

Theorem 3.2 (1) Suppose that $F(t)$ has a strictly IHR (increasing hazard rate) property and that the assumptions from (A-1) to (A-3) are satisfied. Then:

(i) If $q_c(\infty) > 0$ and $q_c(0) < 0$, there exists a finite and unique optimal ordering time $t_0{}^*$ ($0 < t_0{}^* < \infty$) satisfying $q_c(t_0) = 0$ and the corresponding expected cost is

$$C(t_0{}^*,Q) = \frac{hQ(R(t_0{}^*) - 1) + kR(t_0{}^*) + [(c_1 - c_2)Q + k(L_1 - L_2)]r(t_0{}^*)}{R(t_0{}^*) + (L_1 - L_2)r(t_0{}^*)}. \tag{20}$$

(ii) If $q_c(0) \geq 0$, $t_0{}^* = 0$, *i.e.* the regular order is made at the same time instant as the beginning of the operation and the corresponding expected cost is given in Eq.(14).

(iii) If $q_c(\infty) \leq 0$, $t_0{}^* \to \infty$, *i.e.* the regular order is not made and the expedited one is only done at the same time instant as stock-exhaustion and the corresponding expected cost is given in Eq.(17).
(2) Suppose that $F(t)$ is DHR (decreasing hazard rate) and that the assumptions from (A-1) to (A-3) are satisfied. Then, if $\phi(0,Q)/T(0,Q) < \phi(\infty,Q)/T(\infty,Q)$, $t_0{}^* = 0$, otherwise, $t_0{}^* \to \infty$.

Proof. Differentiating $C(t_0,Q)$ with respect to t_0 and setting it equal to zero implies the equation $q_c(t_0) = 0$. Further, with respect to t_0, we have

$$\begin{aligned} q_c'(t_0) &= \Big\{hQR'(t_0) + r'(t_0)[(c_1 - c_2)Q + k(L_1 - L_2)]\Big\}T(t_0,Q) \\ &\quad + R'(t_0)[kT(t_0,Q) - \phi(t_0,Q)] + r'(t_0)(L_2 - L_1)\phi(t_0,Q), \end{aligned} \tag{21}$$

where the 'prime' denotes the symbol of the differentiation with respect to t_0. Since the hazard rate is strictly increasing, from (A-1) to (A-3), we have $q_c'(t_0) > 0$, *i.e.* $q_c(t_0)$ is strictly increasing. If $q_c(\infty) > 0$ and $q_c(0) < 0$, then there exists a finite and unique optimal ordering time $t_0{}^*$ ($0 < t_0{}^* < \infty$) satisfying $q_c(t_0) = 0$, since $q_c(t_0)$ is strictly increasing and continuous. Substituting the relation of $q_c(t_0{}^*) = 0$ into $C(t_0{}^*, Q)$ in Eq.(3) yields Eq.(20). If $q_c(0) \geq 0$, the expected cost $C(t_0,Q)$ is strictly increasing and the optimal ordering time is $t_0{}^* = 0$. If $q_c(\infty) \leq 0$, the expected cost $C(t_0,Q)$ is strictly decreasing and $t_0{}^* \to \infty$. The case of a decreasing hazard rate is also similar. Thus, the proof is completed. □

Next we shall consider the case of expected total discounted cost. In the similar fashion to the long-run average cost, define the numerator of the derivative of $V(t_0,Q)$ with respect to t_0, divided by $\exp(-\beta t_0)\bar{F}(t_0)$, as $q_v(t_0)$, *i.e.*

$$\begin{aligned} q_v(t_0) &= \Big\{(k + hQ)\exp(-\beta L_2)R(t_0) - (h + \beta c_2)Q\exp(-\beta L_2) \\ &\quad + [(k/\beta - c_2 Q)\exp(-\beta L_2) - (k/\beta - c_1 Q)\exp(-\beta L_1)]r(t_0)\Big\}\bar{\delta}(t_0,Q) \\ &\quad - \Big\{\beta\exp(-\beta L_2)R(t_0) + (\exp(-\beta L_2) - \exp(-\beta L_1))r(t_0)\Big\}\pi(t_0,Q). \end{aligned} \tag{22}$$

Two special cases, $t_0 = 0$ and $t_0 \to \infty$, are the following:

$$V(0,Q) = \pi(0,Q)/\bar{\delta}(0,Q), \tag{23}$$

where

$$\begin{aligned} \pi(0,Q) &= h\mathrm{E}[\int_0^{\tau} \exp(-\beta t)X(t)dt] + hQ\exp(-\beta L_2)/\beta - hQ\int_{L_2}^{\infty}\exp(-\beta t)F(t)dt \\ &\quad + k\int_0^{L_2}\exp(-\beta t)F(t)dt + c_2 Q\exp(-\beta L_2), \end{aligned} \tag{24}$$

$$\bar{\delta}(0,Q) = 1 - \beta\int_{L_2}^{\infty}\exp(-\beta t)F(t)dt; \tag{25}$$

and

$$V(\infty,Q) = \pi(\infty,Q)/\bar{\delta}(\infty,Q), \tag{26}$$

where

$$\pi(\infty,Q) = h\mathrm{E}[\int_0^T \exp(-\beta t)X(t)dt] + k(1-\exp(-\beta L_1))\int_0^\infty \exp(-\beta t)F(t)dt + c_1 Q\int_0^\infty \exp(-\beta(t+L_1))dF(t), \tag{27}$$

$$\bar{\delta}(\infty,Q) = 1-\beta\exp(-\beta L_1)\int_0^\infty \exp(-\beta t)F(t)dt. \tag{28}$$

Instead of (A-1) – (A-3), we make the following assumptions:

(B-1) $k\int_0^{L_2} e^{-\beta t}dt + c_2 Q e^{-\beta L_2} < k\int_0^{L_1} e^{-\beta t}dt + c_1 Q e^{-\beta L_1}$

(B-2) $0 < L_1 \le L_2$

(B-3) $\beta V(t_0, Q) < k$.

The assumption (B-1) corresponds to (A-1) in the case of long-run average cost. The assumption (B-3) is a weak one comparing with (A-3), since it is common that the discount factor β is assumed to be less than one.

Similar to Theorems 3.1 and 3.2, we have the following results for the expected total discounted cost criterion, where the proofs are omitted.

Theorem 3.3 For an arbitrary distribution $F(t)$, if $q_v(\infty) > 0$ or $q_v(0) < 0$, then there exists at least one optimal ordering time $t_0{}^*$ ($0 \le t_0{}^* < \infty$ or $0 < t_0{}^* \le \infty$) minimizing the expected cost $V(t_0, Q)$.

Theorem 3.4 (1) Suppose that $F(t)$ has a strictly IHR property and that the assumptions from (B-1) to (B-3) are satisfied. Then:

(i) If $q_v(\infty) > 0$ and $q_v(0) < 0$, there exists a finite and unique optimal ordering time $t_0{}^*$ ($0 < t_0{}^* < \infty$) satisfying $q_v(t_0) = 0$ and the corresponding expected cost is

$$V(t_0{}^*, Q) = \frac{(k+hQ)e^{-\beta L_2}R(t_0{}^*) + \xi r(t_0{}^*) - (h+\beta c_2)Qe^{-\beta L_2}}{(e^{-\beta L_2}-e^{-\beta L_1})r(t_0{}^*) + \beta e^{-\beta L_2}R(t_0{}^*)}, \tag{29}$$

where

$$\xi = \left\{(k-\beta c_2 Q)e^{-\beta L_2} - (k-\beta c_1 Q)e^{-\beta L_1}\right\}/\beta. \tag{30}$$

(ii) If $q_v(0) \ge 0$, then $t_0{}^* = 0$ and the corresponding expected cost is given in Eq.(23).

(iii) If $q_v(\infty) \le 0$, then $t_0{}^* \to \infty$ and the corresponding expected cost is given in Eq.(26).

(2) Suppose that $F(t)$ is DHR and that the assumptions from (B-1) to (B-3) are satisfied. Then, if $\pi(0,Q)/\bar{\delta}(0,Q) < \pi(\infty,Q)/\bar{\delta}(\infty,Q)$, $t_0{}^* = 0$, otherwise, $t_0{}^* \to \infty$.

Remark 3.5 When $h = 0$, these inventory models for respective expected costs are essentially reduced to the order-replacement models discussed by Kaio and Osaki [8, 9]. In other words, the inventory models under consideration are generalizations to the replacement systems with delay, since the corresponding expected costs in Eqs.(3) and (9) include the state variable, *i.e.* $X(t)$.

4. Inventory Control

In this section, we discuss the procedure to obtain the optimal ordering time and the optimal order quantity which jointly minimizes the expected costs. Then we must concretely specify the cumulative demand process $\{N(t), t \geq 0\}$. Throughout this paper, we assume Poisson and Brownian motion processes as cumulative demand processes. In fact, we would expect that most real systems could be accurately described by either Poisson or Gaussian demand pattern, which are common assumptions in the inventory theory.

4.1 Poisson process model

The cumulative demand process $\{N(t), t \geq 0\}$ is a Poisson process with rate $\mu(> 0)$, where

$$P(x \mid t) = \Pr\{N(t) \leq x \mid N(0) = 0\} = \sum_{k=0}^{x} \frac{(\mu t)^k \exp(-\mu t)}{k!}. \tag{31}$$

Since τ is the time required for Q demands to occur, it follows that the distribution of stock-exhaustion time is the Erlang distribution with parameters μ and Q;

$$F(t) = \int_0^t \frac{\mu^Q s^{Q-1} \exp(-\mu s)}{\Gamma_1(Q)} ds, \tag{32}$$

where $\Gamma_1(\cdot)$ is the gamma function. It is well known that the Erlang distribution is strictly IHR for $Q > 1$. Without loss of generality, we assume $Q > 1$. Then, we have the following theorems.

Theorem 4.1 The long-run average cost as a function of the ordering time and the order quantity is given by Eq.(3) and the corresponding expected cost for one cycle and the mean time of one cycle under the assumption of Poisson cumulative demand are

$$\begin{aligned}
\phi(t_0, Q) \quad = \quad & \frac{hQ(Q+1)}{2\mu} + c_1 Q + kL_1 \\
& -(hQ+k)(t_0+L_2)\Gamma_2(Q, \mu(t_0+L_2))/\Gamma_1(Q) \\
& +\Big\{(c_1 - c_2)Q + k(t_0 + L_2 - L_1)\Big\}\Gamma_2(Q, \mu t_0)/\Gamma_1(Q) \\
& +(hQ+k)\Gamma_2(Q+1, \mu(t_0+L_2))/(\mu\Gamma_1(Q)) \\
& -k\Gamma_2(Q+1, \mu t_0)/(\mu\Gamma_1(Q)),
\end{aligned} \tag{33}$$

$$\begin{aligned}
T(t_0, Q) \quad = \quad & \Big\{\Gamma_1(Q+1) - \Gamma_2(Q+1, \mu t_0)\Big\}/(\mu\Gamma_1(Q)) \\
& +L_1\Big\{\Gamma_1(Q) - \Gamma_2(Q, \mu t_0)\Big\}/\Gamma_1(Q) + \Gamma_2(Q+1, \mu(t_0+L_2))/(\mu\Gamma_1(Q)) \\
& +(t_0+L_2)\Big\{\Gamma_2(Q, \mu t_0) - \Gamma_2(Q, \mu(t_0+L_2))\Big\}/\Gamma_1(Q),
\end{aligned} \tag{34}$$

where $\Gamma_2(\cdot, \cdot)$ is the incomplete gamma function defined by

$$\Gamma_2(a, b) = \int_b^\infty t^{a-1} \exp(-t) dt. \tag{35}$$

Theorem 4.2 The expected total discounted cost as a function of the ordering time and the order quantity is given by Eq.(9). Then, we have

$$\pi(t_0,Q) = H_\pi(t_0,Q) + O_\pi(t_0,Q) + S_\pi(t_0,Q), \tag{36}$$

where

$$\begin{aligned} H_\pi(t_0,Q) &= \frac{hQ}{\beta} + \frac{h(\mu+\beta)}{\beta^2}\Big\{\Big(\frac{\mu}{\mu+\beta}\Big)^Q - 1\Big\} \\ &\quad + \frac{hQe^{-\beta(t_0+L_2)}}{\beta}\Big\{1 - \frac{\mu^Q(t_0+L_2)^Q e^{-\mu(t_0+L_2)}}{\Gamma_1(Q+1)}\Big\} \\ &\quad - \frac{hQ\mu^Q}{\beta(\mu+\beta)^Q}\frac{\Gamma_2(Q,(\mu+\beta)(t_0+L_2))}{\Gamma_1(Q)} \\ &\quad - \frac{he^{-\beta(t_0+L_2)}}{\beta\Gamma_1(Q)}\Big\{\Gamma_1(Q+1) - \Gamma_2(Q+1,\mu(t_0+L_2))\Big\}, \end{aligned} \tag{37}$$

$$\begin{aligned} O_\pi(t_0,Q) &= \frac{c_1 Q\mu^Q e^{-\beta L_1}}{(\mu+\beta)^Q\Gamma_1(Q)}\Big\{\Gamma_1(Q) - \Gamma_2(Q,(\mu+\beta)t_0)\Big\} \\ &\quad + \frac{c_2 Q e^{-\beta(t_0+L_2)}\Gamma_2(Q,\mu t_0)}{\Gamma_1(Q)}, \end{aligned} \tag{38}$$

$$\begin{aligned} S_\pi(t_0,Q) &= \frac{k\mu^Q(1-e^{-\beta L_1})}{\beta(\mu+\beta)^Q\Gamma_1(Q)}\Big\{\Gamma_1(Q) - \Gamma_2(Q,(\mu+\beta)t_0)\Big\} \\ &\quad + \frac{k\mu^Q}{\beta(\mu+\beta)^Q\Gamma_1(Q)}\Big\{\Gamma_2(Q,(\mu+\beta)t_0) - \Gamma_2(Q,(\mu+\beta)(t_0+L_2))\Big\} \\ &\quad - \frac{ke^{-\beta(t_0+L_2)}}{\beta\Gamma_1(Q)}\Big\{\Gamma_2(Q,\mu t_0) - \Gamma_2(Q,\mu(t_0+L_2))\Big\}, \end{aligned} \tag{39}$$

and

$$\begin{aligned} \bar{\delta}(t_0,Q) &= 1 - \frac{\mu^Q e^{-\beta L_1}}{(\mu+\beta)^Q\Gamma_1(Q)}\Big\{\Gamma_1(Q) - \Gamma_2(Q,(\mu+\beta)t_0)\Big\} \\ &\quad - \frac{e^{-\beta(t_0+L_2)}}{\Gamma_1(Q)}\Big\{\Gamma_2(Q,\mu t_0) - \Gamma_2(Q,\mu(t_0+L_2))\Big\} \\ &\quad - \frac{\mu^Q}{(\mu+\beta)^Q\Gamma_1(Q)}\Gamma_2(Q,(\mu+\beta)(t_0+L_2)). \end{aligned} \tag{40}$$

Proofs of Theorems 4.1 and 4.2 are given in Appendix.

Note that $H_\pi(t_0,Q)$, $O_\pi(t_0,Q)$ and $S_\pi(t_0,Q)$ are the holding, ordering and shortage costs, respectively. Since the hazard rate is strictly increasing, we can directly apply Theorems 3.2 and 3.4 to obtain the optimal ordering time. It is, however, difficult to obtain the optimal order quantity Q^* analytically. Therefore, we numerically examine the behavior of $C(t_0,Q)$ and $V(t_0,Q)$ for the order quantity. Figures 2 and 3 illustrate the convexity of the expected costs for the order quantity. Thus, if the expected costs are strictly convex in the order quantity, we can numerically obtain the *optimal inventory policy* $(t_0{}^*,Q^*)$ satisfying $(\partial^2 C(t_0,Q)/\partial t_0\partial Q)^2 - \partial^2 C(t_0,Q)/\partial t_0{}^2\cdot\partial^2 C(t_0,Q)/\partial Q^2 < 0$ and $\partial C(t_0,Q)/\partial Q = q_c(t_0) = 0$.

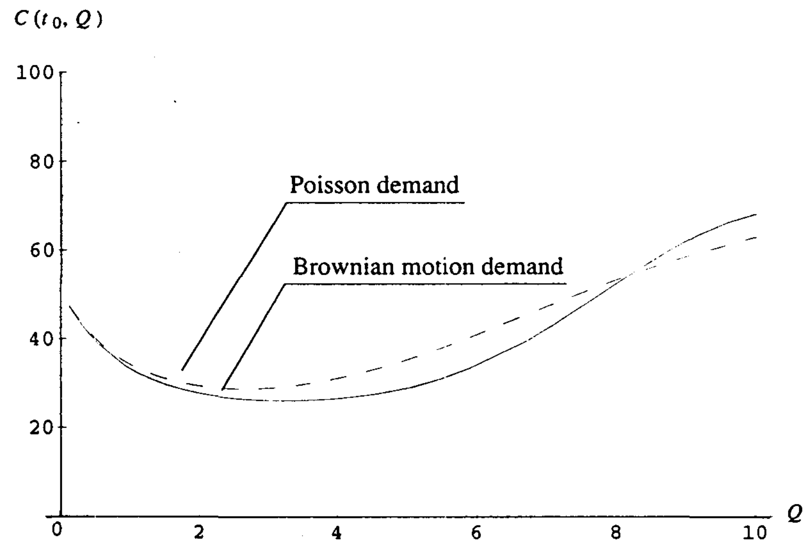

Fig. 2. Behavior of the long–run average cost for each demand pattern.

$$\begin{bmatrix} h = \$7, & c_1 = \$2, & c_2 = \$1, & L_1 = 2, & L_2 = 5 \\ k = \$50, & \mu = 0.8, & \sigma = 0.5, & t_0 = 10 & \end{bmatrix}$$

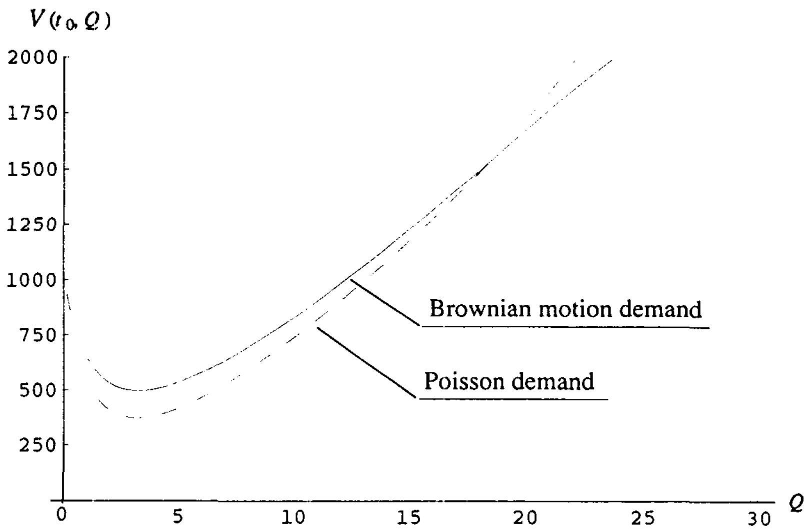

Fig. 3. Behavior of the expected total discounted cost for each demand pattern.

$$\begin{bmatrix} h = \$7, & c_1 = \$2, & c_2 = \$1, & L_1 = 2, & L_2 = 5 \\ k = \$50, & \mu = 0.8, & \sigma = 0.5, & \beta = 0.05, & t_0 = 30 \end{bmatrix}$$

4.2 Brownian motion model

If the cumulative demand obeys a Gaussian process, then

$$P(x \mid t) = \Pr\{N(t) \leq x \mid N(0) = 0\} = \Phi\Big[\frac{x - \mu t}{\sigma\sqrt{t}}\Big], \tag{41}$$

where $\mu(> 0)$ and $\sigma(> 0)$ are the instantaneous mean and standard deviation and where $\Phi[\cdot]$ is a standard normal distribution function. The cumulative demand process is the following (μ, σ)-Brownian motion process:

$$N(t) = \mu t + \sigma B(t), \tag{42}$$

where the stochastic process $\{B(t), t \geq 0\}$ is the one-dimensional standard Brownian motion process. It should be noted that Poisson distribution can be approximated by the normal one if $\sigma/2\mu \ll 1$. Furthermore, the corresponding distribution of stock-exhaustion time is the following inverse Gaussian distribution:

$$F_1(t;\mu) = \Phi\Big[\frac{-Q + \mu t}{\sigma\sqrt{t}}\Big] + \exp\Big(\frac{2\mu Q}{\sigma^2}\Big)\Phi\Big[\frac{-Q - \mu t}{\sigma\sqrt{t}}\Big], \tag{43}$$

Unfortunately, since the hazard rate for the inverse Gaussian distribution is not always a monotone function of t, we can not directly apply Theorems 3.2 and 3.4 to obtain the optimal ordering time. Following Chhikara and Folks [5], we have the following result for the Brownian motion demand.

Lemma 4.3 For the inverse Gaussian distribution $F_1(t;\mu)$ given in Eq.(43), define

$$t_m = -\frac{3\sigma^2}{2\mu^2} + \frac{Q}{\mu}\Big(1 + \frac{9\sigma^4}{4Q^2\mu^2}\Big)^{1/2} \tag{44}$$

and

$$t_n = \frac{2Q^2}{3\sigma^2}. \tag{45}$$

Then:

(i) $F_1(t;\mu)$ is IHR if $0 \leq t \leq t_m$.

(ii) $F_1(t;\mu)$ is DHR if $t_n < t$.

(iii) The hazard rate for $F_1(t;\mu)$ is unimodal if $t_m < t \leq t_n$.

Since t_m is the point at which the mode of $F_1(t;\mu)$ occurs, it is clear that the hazard rate is increasing for $0 \leq t \leq t_m$. Also, by calculating $d(\log r(t))/dt$, we obtain that the hazard rate is decreasing for $t > t_n$. For the case of (iii), the equation $d(\log r(t))/dt = 0$ can have at most one root for $t > t_m$ and $r(t)$ attains its maximum value at t_e satisfying the following equation:

$$r(t) = \frac{\mu^2}{2\sigma^2} + \frac{3}{2t} - \frac{Q^2}{2\sigma^2 t^2}, \quad \text{for } t \in (t_m, t_n]. \tag{46}$$

Thus, if there exists $t_e \in (t_m, t_n]$, the distribution function $F_1(t;\mu)$ is IHR for $0 \leq t \leq t_e$, otherwise, for $0 \leq t \leq t_m$. For details, see Chhikara and Folks [5].

Now we define t_c satisfying $q_c(t_0) = 0$. Unifying Lemma 4.3 and Theorem 3.2 generates the following result for the optimal ordering time under the Gaussian cumulative demand.

Corollary 4.4 (1) Suppose that there exists a unique $t_e \in (t_m, t_n]$ under the assumption from (A-1) to (A-3). (i) If $q_c(t_e) > 0$ and $q_c(0) < 0$, then the optimal ordering time is ${t_0}^* = t_c$ or ${t_0}^* \to \infty$. (ii) If $q_c(0) \geq 0$, then ${t_0}^* = 0$ or ${t_0}^* \to \infty$. (iii) If $q_c(t_e) \leq 0$, then ${t_0}^* \to \infty$.

(2) Suppose that there does not exist $t_e \in (t_m, t_n]$ under the assumption from (A-1) to (A-3). (i) If $q_c(t_n) > 0$ and $q_c(0) < 0$, then the optimal ordering time is ${t_0}^* = t_c$ or ${t_0}^* \to \infty$. (ii) If $q_c(0) \geq 0$, then ${t_0}^* = 0$ or ${t_0}^* \to \infty$. (iii) If $q_c(t_n) \leq 0$, then ${t_0}^* \to \infty$.

The case of expected total discounted cost is omitted for brevity. Notice that the procedure to calculate the optimal ordering time for the Gaussian cumulative demand process may be heuristic. However, it is easy to seek the optimal solution by using computers.

Next, in order to obtain the optimal order quantity, we have following theorems.

Theorem 4.5 The long-run average cost as a function of the ordering time and the order quantity is given by Eq.(3) and the corresponding expected cost for one cycle and the mean time of one cycle under the assumption of the Brownian motion demand are

$$\begin{aligned}\phi(t_0, Q) &= h(\frac{Q^2}{2\mu} + \frac{\sigma^2 Q}{2\mu^2}) - hQ(t_0 + L_2)\bar{F}_1(t_0 + L_2; \mu) + \frac{hQ^2}{\mu} F_2(t_0 + L_2; \mu) \\ &\quad + c_2 Q + \{(c_1 - c_2)Q + k(L_1 - L_2 - t_0)\} F_1(t_0; \mu) \\ &\quad + k(t_0 + L_2) F_1(t_0 + L_2; \mu) + \frac{kQ}{\mu}\big\{F_2(t_0 + L_2; \mu) - F_2(t_0; \mu)\big\}, \qquad (47)\end{aligned}$$

$$\begin{aligned}T(t_0, Q) &= (L_1 - L_2 - t_0) F_1(t_0; \mu) + (t_0 + L_2) F_1(t_0 + L_2; \mu) \\ &\quad + \frac{Q}{\mu}\big\{F_2(t_0 + L_2; \mu) + \bar{F}_2(t_0; \mu)\big\}, \qquad (48)\end{aligned}$$

where

$$F_2(t; \mu) = \Phi\big[\frac{Q - \mu t}{\sigma\sqrt{t}}\big] + \exp(\frac{2\mu Q}{\sigma^2})\Phi\big[\frac{-Q - \mu t}{\sigma\sqrt{t}}\big]. \qquad (49)$$

Theorem 4.6 The expected total discounted cost as a function of the ordering time and the order quantity is given by Eq.(9). Then, we have

$$\begin{aligned}\pi(t_0, Q) &= \frac{hQ}{\beta}\big\{(1 - e^{Q\lambda}) + e^{-\beta(t_0 + L_2)}\bar{F}_1(t_0 + L_2; \mu) - e^{Q\lambda}\bar{F}_1(t_0 + L_2; \theta)\big\} \\ &\quad - \frac{h}{\beta}\big\{\mu(1 - e^{Q\lambda})/\beta - Qe^{Q\lambda}\big\} + c_1 Q e^{Q\lambda - \beta L_1} F_1(t_0; \theta) + c_2 Q e^{-\beta(t_0 + L_2)}\bar{F}_1(t_0; \mu) \\ &\quad + \frac{k}{\beta}(1 - e^{-\beta L_1}) e^{Q\lambda} F_1(t_0; \theta) + \frac{k}{\beta} e^{Q\lambda}\big\{F_1(t_0 + L_2; \theta) - F_1(t_0; \theta)\big\}, \qquad (50)\end{aligned}$$

$$\bar{\delta}(t_0, Q) = 1 - e^{Q\lambda - \beta L_1} F_1(t_0; \theta) - e^{Q\lambda}\bar{F}_1(t_0 + L_2; \theta)$$

$$-e^{Q\lambda-\beta(t_0+L_2)}\Big\{F_1(t_0+L_2;\theta)-F_1(t_0;\theta)\Big\}, \tag{51}$$

where

$$\lambda=\frac{\mu-\theta}{\sigma^2}, \tag{52}$$

$$\theta=(\mu^2+2\sigma^2\beta)^{1/2}. \tag{53}$$

Proofs of Theorems 4.5 and 4.6 are also presented in Appendix. The behavior of the expected costs above for the order quantity under the condition of the Brownian motion demand is also shown in Figs. 2 and 3. We can graphically recognize for various parameters that the expected cost functions are strictly convex in Q.

In the following section, we numerically calculate the optimal inventory policy $(t_0{}^*, Q^*)$ for the Brownian motion demand and examine its sensitivity for some model parameters.

5. Numerical Examples

The optimal control policies for joint optimization of ordering time and order quantity for the expected cost criteria are numerically discussed. Especially, we focus on the case of Brownian motion demand since the Poisson process is approximated by a Brownian motion process for sufficiently large intensity parameters.

Table 1. Dependence of the optimal inventory policy on μ and σ: the long-run average cost $C(t_0, Q)$.

$[k=\$30, h=\$7, c_1=\$2, c_2=\$1, L_1=2, L_2=5,]$

	$\sigma=0.5$			$\sigma=0.8$		
μ	Q^*	t_0^*	$C(t_0^*, Q^*)$	Q^*	t_0^*	$C(t_0^*, Q^*)$
0.4	1.810	38.158	15.660	1.657	104.810	17.995
0.6	2.089	25.973	17.278	1.968	43.051	18.707
0.8	2.278	11.557	18.643	2.177	31.334	19.636
1.0	2.416	10.855	19.786	2.327	17.676	20.527
1.2	2.518	10.188	20.754	2.438	16.161	21.332

First, we examine the numerical characteristics of optimal inventory policy when the drift and variance parameters are changed. Tables 1 and 2 show the dependence of the optimal inventory control policy $(t_0{}^*, Q^*)$ and its associated expected cost on the infinitesimal parameters. Numerical examples illustrate that the optimal order quantity increases as the drift parameter and the variance parameter become larger and smaller, respectively. In addition, both expected costs, $C(t_0{}^*, Q^*)$ and $V(t_0{}^*, Q^*)$, increase as the drift and variance parameters become larger. The increasing drift parameter implies a rise of the average demand, and the increasing variance parameter means that an uncertainty on demand is more remarkable. If the demand for items increases, one should make a more satisfactory order quantity ready, and larger ordering and inventory holding costs will be anticipated. The increasing variance parameter gives a rise to the increase of costs and has an effect to order moderately. Thus, the results will satisfy our intuition.

Table 2. Dependence of the optimal inventory policy on μ and σ: the expected total discounted cost $V(t_0, Q)$.

$[k = \$30, h = \$7, c_1 = \$2, c_2 = \$1, L_1 = 2, L_2 = 5, \beta = 0.05]$

	$\sigma = 0.5$			$\sigma = 0.8$		
μ	Q^*	t_0^*	$V(t_0^*, Q^*)$	Q^*	t_0^*	$V(t_0^*, Q^*)$
0.4	1.776	40.652	302.901	1.676	71.452	350.313
0.6	2.052	19.349	334.579	1.958	42.554	365.111
0.8	2.248	18.193	361.768	2.162	27.612	383.320
1.0	2.393	17.183	384.743	2.314	17.650	400.908
1.2	2.504	43.262	404.325	2.431	17.182	416.968

Second, the sensitivity analyses are carried out for the lead time parameters. In Tables 3 and 4, the optimal order quantity and the minimum expected costs increase, as the lead time for expedited order becomes larger. It seems to be natural that the required time to deliver items influences the order quantities and the corresponding costs. It is, however, really surprising that the lead time for regular order exerts hardly influence the optimal order quantity and the expected costs. This fact tells us that the lead time for expedited order is a governing factor for the optimal inventory policies.

Table 3. Dependence of the optimal inventory policy on the lead times: the long-run average cost $C(t_0, Q)$.

$[k = \$30, h = \$7, c_1 = \$2, c_2 = \$1, \mu = 1.2, \sigma = 0.5]$

	$L_2 = 5$			$L_2 = 7$		
L_1	Q^*	t_0^*	$C(t_0^*, Q^*)$	Q^*	t_0^*	$C(t_0^*, Q^*)$
0.8	1.920	10.042	16.566	1.920	10.051	16.566
1.6	2.373	10.275	19.738	2.373	10.303	19.738
2.4	2.634	10.063	21.568	2.634	10.123	21.568
3.2	2.810	9.935	22.801	2.810	9.974	22.801
4.0	2.939	8.645	23.702	2.938	9.244	23.702

Next, we consider the sensitivity for the cost parameters. Tables 5-8 demonstrate the dependence of the optimal inventory policy on cost parameters. From Tables 5 and 6, the increase of ordering cost for expedited order makes the order quantity and the expected costs decrease and increase, respectively. On the other hand, one observes that the change of ordering cost for a regular order does not influence both the optimal order quantity and the expected costs. This also shows that only the cost parameter for expedited order is sensitive to the optimal policy. The results for the inventory holding and the shortage costs are straightforward. The increase of holding cost gives the effect of making the order quantity decrease, and the increasing shortage cost gives a quite contrary tendency. It is intuitively obvious that the expected costs increase as these cost parameters become larger.

Finally, the dependence of the optimal inventory policy on the discount factor is presented in Table 9. The decrease of order quantity for the increasing discount factor is contrary to the earlier results on the optimal stock level in Dohi *et al.* [6]. This clearly results from the difference of model structure.

Table 4. Dependence of the optimal inventory policy on the lead times: the expected total discounted cost $V(t_0, Q)$.

$[k = \$30, h = \$7, c_1 = \$2, c_2 = \$1, \mu = 1.2, \sigma = 0.5, \beta = 0.05]$

	$L_2 = 5$			$L_2 = 7$		
L_1	Q^*	t_0^*	$V(t_0^*, Q^*)$	Q^*	t_0^*	$V(t_0^*, Q^*)$
0.8	1.913	10.481	325.169	1.913	11.152	325.169
1.6	2.360	10.268	385.270	2.360	10.268	385.270
2.4	2.619	9.862	419.487	2.619	9.156	419.487
3.2	2.794	13.726	442.320	2.794	9.350	442.320
4.0	2.924	13.761	458.847	2.924	9.062	458.847

Table 5. Dependence of the optimal inventory policy on the ordering costs: the long-run average cost $C(t_0, Q)$.

$[k = \$30, h = \$7, L_1 = 2, L_2 = 5, \mu = 1.2, \sigma = 0.5]$

	$c_2 = 5$			$c_2 = 7$		
c_1	Q^*	t_0^*	$C(t_0^*, Q^*)$	Q^*	t_0^*	$C(t_0^*, Q^*)$
5	2.260	9.151	22.550	2.260	9.325	22.550
6	2.171	8.827	23.126	2.171	9.013	23.126
7	2.080	8.635	23.690	2.080	8.777	23.690
8	1.987	8.570	24.240	1.987	8.631	24.240
9	1.893	8.614	24.777	1.893	8.606	24.777

Table 6. Dependence of the optimal inventory policy on the ordering costs: the expected total discounted cost $V(t_0, Q)$.

$[k = \$30, h = \$7, L_1 = 2, L_2 = 5, \mu = 1.2, \sigma = 0.5, \beta = 0.05]$

	$c_2 = 5$			$c_2 = 7$		
c_1	Q^*	t_0^*	$V(t_0^*, Q^*)$	Q^*	t_0^*	$V(t_0^*, Q^*)$
5	2.310	10.370	436.966	2.310	10.370	436.966
6	2.242	10.507	447.570	2.242	15.605	447.570
7	2.171	10.266	458.022	2.171	10.266	458.022
8	2.099	10.115	468.310	2.099	10.790	468.310
9	2.024	10.557	478.424	2.024	10.557	478.424

Throughout the numerical experiments, we could observe no monotone tendencies for the optimal ordering time. In fact, the optimal ordering time shows a complex behavior for each model parameter. For a fixed order quantity, we obtained the optimal ordering time in Section 3. The analytical properties of optimal ordering time for the joint optimization problems are not simple as the case of fixed order quantity any longer.

6. Conclusion

We have integrated the sophisticated techniques used in reliability theory into the cyclic inventory control problem with an emergency order. We have analyzed properties of the

Table 7. Dependence of the optimal inventory policy on the inventory holding cost.

$[k = \$30, c_1 = \$2, c_2 = \$1, L_1 = 2, L_2 = 5, \mu = 1.2, \sigma = 0.5, \beta = 0.05]$

h	Q^*	t_0^*	$C(t_0^*, Q^*)$	Q^*	t_0*	$V(t_0^*, Q^*)$
3	3.993	5.607	16.722	4.570	61.947	316.499
4	3.787	6.591	17.997	3.755	56.966	346.059
5	3.235	12.746	19.097	3.208	14.329	369.319
6	2.829	11.315	19.998	2.809	9.318	388.345
7	2.518	10.188	20.754	2.503	9.064	404.325

Table 8. Dependence of the optimal inventory policy on the shortage cost.

$[h = \$7, c_1 = \$2, c_2 = \$1, L_1 = 2, L_2 = 5, \mu = 1.2, \sigma = 0.5, \beta = 0.05]$

k	Q^*	t_0^*	$C(t_0^*, Q^*)$	Q^*	t_0^*	$V(t_0^*, Q^*)$
15	1.328	8.601	12.427	1.347	10.043	244.310
20	1.763	8.553	15.469	1.771	10.668	303.053
25	2.156	9.047	18.221	2.154	10.683	355.921
30	2.518	10.188	20.754	2.503	9.064	404.325
35	2.855	11.428	23.114	2.828	9.277	449.191

Table 9. Dependence of the optimal inventory policy on the discount factor.

$[k = \$30, h = \$7, c_1 = \$2, c_2 = \$1, L_1 = 2, L_2 = 5, \mu = 1.2, \sigma = 0.5]$

β	Q^*	t_0^*	$V(t_0^*, Q^*)$
0.01	2.516	10.120	2064.530
0.05	2.503	43.262	404.325
0.10	2.480	37.378	196.973
0.15	2.450	17.944	127.989
0.20	2.414	15.829	93.597

optimal ordering time, and derived the expected cost criteria with respect to two types of demand processes. In addition, it has been numerically examined that the optimal control policy, which includes the optimal ordering time and the optimal order quantity, minimizing the expected cost criteria, were uniquely obtained for various model parameters.

In Section 3, the main reason why the optimal ordering time has a relatively simple form results from the assumption that the expedited order point is zero. In fact, it is noted that a general reorder point – reorder quantity policy in [11] could not be analytically derived. This problem remains to be solved. The results of this paper will, however, give an alternative direction to the inventory model with an emergency as well as a theoretical support to the practitioner.

Appendix

In this appendix, we derive the long-run average cost and the expected total discounted cost over an infinite time horizon for the Poisson process demand and the Brownian motion demand. Given the distribution functions of the stock-exhaustion time in Eqs.(32) and

(43), it is easy to calculate the other parts of the expected costs except for $\mathrm{E}[\int_0^\tau X(t)dt]$ and $\mathrm{E}[\int_0^\tau \exp(-\beta t)X(t)dt]$. Therefore, we give analytical expressions for two different demand distributions.

To deal with this problem we apply the first Dynkin formula (*e.g.* see [10, p. 297]. For an alternative derivation, see [14].). The first Dynkin formula provides the following expression:

$$\mathrm{E}[\int_0^\tau \exp(-\beta t)X(t)dt] = Q\mathrm{E}[\int_0^\tau \exp(-\beta t)dt] - \mathrm{E}[\int_0^\tau \exp(-\beta t)N(t)dt]. \tag{54}$$

For a bounded and well-defined function $g(\cdot)$ and a complex number β, we have

$$\mathrm{E}[\int_0^\tau \exp(-\beta t)g(N(t))dt] = U(0) - U(Q)\mathrm{E}[\exp(-\beta\tau)], \tag{55}$$

where

$$U(x) = (R_\beta g)(x) = \int_{-\infty}^{Q} G_\beta(x,y)g(y)dy. \tag{56}$$

R_β and $G_\beta(x,y)$ are called the *resolvent operator* and its *kernel* of $N(t)$, respectively, given $N(0)=x$. Note that the kernel $G_\beta(x,y)$ is equivalent to the Laplace–Stieltjes transform of the distribution function of $N(t)$.

When the demand process follows the Poisson process, we have

$$U(x) = \sum_{y=x}^{Q} \frac{g(y)}{\mu+\beta}(\frac{\mu}{\mu+\beta})^{y-x}. \tag{57}$$

Thus, by putting $g(y)=y$, we have

$$\mathrm{E}[\int_0^\tau \exp(-\beta t)N(t)dt] = U_p(1) - U_p(Q)\mathrm{E}[\exp(-\beta\tau)], \tag{58}$$

where

$$U_p(x) = \frac{1}{\beta}\{x + \frac{\mu}{\beta} - Q(\frac{\mu}{\mu+\beta})^{Q-x+1} - \frac{\mu}{\beta}(\frac{\mu}{\mu+\beta})^{Q-x}\}. \tag{59}$$

Finally, we have

$$\mathrm{E}[\exp(-\beta\tau)] = (\frac{\mu}{\mu+\beta})^{Q} \tag{60}$$

and

$$\mathrm{E}[\int_0^\tau N(t)dt] = \frac{Q(Q-1)}{2\mu}, \tag{61}$$

by applying the l'Hospital's theorem.

Next, let us consider the case of Brownian motion process. Applying the result in [10] gives

$$\mathrm{E}[\int_0^\tau \exp(-\beta t)N(t)dt] = \frac{\mu}{\beta^2}(1 - e^{\lambda Q}) - \frac{Q}{\beta}e^{-\lambda Q} \tag{62}$$

and

$$\mathrm{E}[\int_0^\tau N(t)dt] = \frac{Q^2}{2\mu} - \frac{\sigma^2 Q}{2\mu^2}. \tag{63}$$

By partially using the results above, we obtain Eqs.(33), (36), (47) and (50).

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