Journal of the Operations Research Society of Japan
Vol. 38, No. 1, March 1995

OPTIMAL STOPPING PROBLEM WITH SEVERAL SEARCH AREAS

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(Received April 5, 1993; Revised October 28, 1994)

Abstract The present paper deals with an optimal stopping problem with several possible search areas in which travel costs are assumed among the areas. In terms of the future availability of an offer once obtained and passed up, the following two cases are considered: (1) it becomes instantly and forever unavailable and (2) it remains forever available, called a no recall model and a recall model, respectively. The main results obtained here are as follows: 1. Both models have a reservation value property, and the reservation values are nondecreasing in the number of periods, t, remaining up to deadline and converge as $t \to \infty$; 2. Their limits in both models do not always become the same, which coincide in conventional optimal stopping problems; 3. In the recall model, there may exist double critical points w_* and $w^*(w_* < w^*)$ in terms of the present offer w in the sense that, if $w < w_*$, then the optimal next search area is i, if $w_* \leq w \leq w^*$, then $j \neq i$, and if $w^* < w$, then again i; and 4. Suppose the travel cost is independent of the starting search area. Then, in the recall model, the reservation value is independent of both the remaining periods and the current search area. Furthermore, in this case, the reservation values in both models converge to the same value as $t \to \infty$.

1. Introduction

Suppose that a piece of land must be disposed of by a certain day in the future (deadline) [5]. In order to find buyers in a different city every day, some cost (search cost) must be paid. Offers for the land vary with the buyer, and these offers are assumed to be mutually independent random variables having a known distribution function (offer distribution function). Now postulate that all offers up to the day before the deadline are not sufficiently large and the deadline has come without the asset being sold. Then it must be sold for the amount offered by a buyer that will appear at the deadline, however small it may be, so that this situation must be said to be quite risky. Taking the avoidance of such risk in the deadline into consideration, the owner of the land must determine a decision rule to sell the asset for as high a price as possible up to the deadline.

The reasonable decision rule will have the following structure. Now suppose a buyer has just appeared and offered a price w on a certain day before the deadline. Then, it must be determined whether to sell the asset to him at that price, or not to sell and continue the search for another buyer by paying the search cost. The decision can be characterized by a critical price such that, if the price is greater than it, sell the asset, if not, then don't sell. Usually the critical price is called a ***reservation value***. Of course, the critical price depends on the number of days t that remain up to the deadline. Then, the objective here is to find the sequence of t-dependent critical prices so as to maximize the expected net profit, the expected selling price minus the total expected search costs.

Many different models of these types of stochastic sequential decision processes [1–16], usually called an optimal stopping problem or a search problem, have been posed and investigated so far; however, in all of them, only one search area in which offerers* are searched for has been assumed. In the present paper, we will present a model with several possible search areas and examine the properties of its optimal decision rule. In this model, it goes without saying that travel costs are assumed between two cities, so the optimal decision rule must be prescribed with taking into consideration the travel costs accumulated every time the searcher moves from a certain search area to another.

2. Model

Consider the following discrete-time optimal stopping problem with a finite planning horizon. First, for convenience, let points in time be numbered backward from the final point in time of the planning horizon as $0, 1, \cdots$ and so on, equally spaced, where an interval between two successive points in time, say time t and time $t-1$, is called a period t. Suppose there exist $N \geq 1$ possible search areas, and let the set of them be $\mathcal{S} = \{1, 2, \ldots, N\}$. When the searcher moves from search area i to j, a travel cost $d_{ij} \geq 0$ is incurred with $d_{ii} = 0$. If paying $s_i \geq 0$ in search area i, then an offer can be obtained. Below, for all $i, j \in \mathcal{S}$, define

$$c_{ij} = d_{ij} + s_j, \tag{3.1}$$

called a travel and search cost. An offer w obtained in search area j is a random variable having a known distribution $F_j(w)$ with a finite expectation μ_j where, for $0 < a_j < b_j < \infty$, let $F_j(w) = 0$ for $w < a_j$, $0 < F_j(w) < 1$ for $a_j \leq w < b_j$, and $F_j(w) = 1$ for $b_j \leq w$. Sequentially obtained offers $w, w', \cdots$ are assumed to be stochastically independent. Here postulate that one of the offers obtained during the given planning horizon must be necessarily accepted. Throughout the paper, let us introduce a per-period discount factor $\beta \in (0, 1]$ and assume

$$\beta\mu_j - c_{ij} > 0 \tag{3.2}$$

for all $i, j \in \mathcal{S}$, the natural assumption implying that, provided that it has been decided to travel from a certain search area i to j and make the search there, the expected present value $\beta\mu_j$ of an offer obtained in search area j at least recovers the travel and search cost c_{ij} paid.

The objective here is to maximize the expected present discounted net value, the expected offer accepted minus the total expected travel and search cost. In this case, the optimal decision rule achieving the objective consists of the following two rules: *optimal stopping rule*, prescribing how to stop the search by accepting an offer and *optimal selection rule*, stating, if continuing the search, whether or not to conduct the search by staying in the current search area or, if not, which search area to move to.

In the present paper, we will also investigate the specialized case that travel cost d_{ij} is independent of the starting search area $i \in \mathcal{S}$. Now, when considering an optimal stopping problem, the following three cases are usually discussed in terms of future availability of an offer once obtained and passed up: (1) it becomes instantly and forever unavailable, (2) it remains forever available, and (3) it will be stochastically unavailable in the future, called a no recall model, a recall model, and an uncertain recall model [3,7,9] , respectively. In the present paper, only the first two models will be examined.

3. Examples

Below, let us give three concrete examples to which the model in the previous section will be well applied.

*In general, a person who offers price w is referred to as *offerer* and the price w as *offer* w.

- *Asset selling problem*

Consider again the asset selling problem stated in the previous section. The land owner travels to find a buyer among the cities starting from his city of residence. The search cost in each city may depend on its area, population, and so on; the travel costs may be proportional to the distance between two cities, and the distribution function of amounts offered by buyers may vary with the economic power of each city. In this case, the following two points must be decided each day over the whole planning horizon. First, seeing the amount offered by a buyer that has just appeared in a city, whether to stop the search by accepting it or to continue the search, and second, if continuing, whether to make the search in the current city or to move on to another city, and if moving, which city to go to.

- *Fishing grounds selection problem*

Suppose several promising fishing grounds have been found in the North Pacific by means of satellite observation, and a captain of a fishing fleet, provided that he is now in one of them, is considering which of these places would prove best. The moving costs from each fishing ground to the others are known. After conducting a trial catch at a place for a certain cost, the future catch that is expected if a full-scale catch is made there is revealed after a short time and let the future expected catch be assumed to be a random variable having a known distribution function, varying from place to place. Then, the captain must decide the following two points. First, knowing the future expected catch at a certain fishing ground, whether to conduct full-scale fishing at the fishing ground or not, and second, if not, which fishing ground to move to. If taking into consideration the unexpected movement of each fishing ground itself and the change of the future expected catch there due to the ever-changing marine conditions such as the current, the water temperature, the volume of plankton, and so on, the decision that must be made will become more challenging and difficult.

- *Technology selection problem*

Another example is the technology selection problem [16]. Suppose the research department of a certain production company has been assigned the task of finding a new and less expensive production process to produce some product. Several substitutable technologies are being considered for this. The long-run profits that will be yielded by a production process for each technology are uncertain and will not be known until development work, that will spend money, is completed. The long-run profits that will be revealed after development work is a random variable having a known distribution function, depending on the technology. If it was judged that a production process for a certain technology will not yield a sufficiently large long-run profit, then development work must be shifted to another technology. Then, a shifting cost is incurred, which may depend on the current technology and the one that is shifted to. It goes without saying that the technologies correspond to the search areas in our model. What must be decided in this problem are the following two points. First, knowing the long-run profit of the production process for a certain technology, whether to employ it or not, and second, if not, to which technology the development work should be shifted.

4. Preliminaries

For any real number x and any $i, j \in \mathcal{S}$, define

$$K_{ij}(x) = \beta \int_0^\infty \max\{w, x\} dF_j(w) - x - c_{ij} \tag{4.1}$$

$$= \beta \int_0^\infty \max\{w - x, 0\} dF_j(w) - (1-\beta)x - c_{ij} \tag{4.2}$$

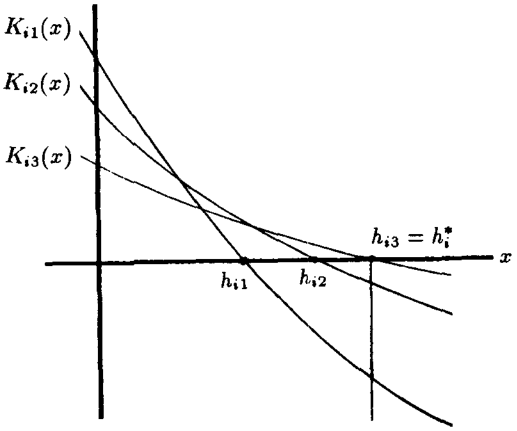

Figure 1
Relationship of $K_{ij}(y)$ and h_{ij}

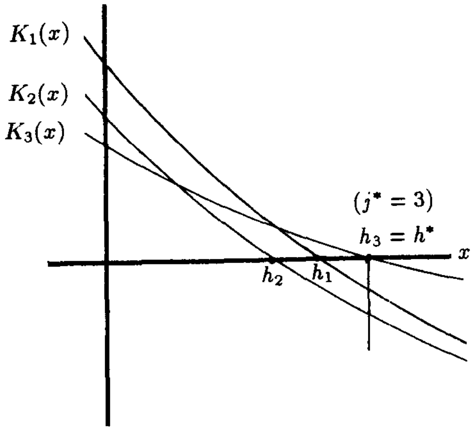

Figure 2
Relationship of $K_j(y)$ and h_j

where
$$K_{ij}(x) = \begin{cases} \beta\mu_j - x - c_{ij}, & x \le a_j, \\ -(1-\beta)x - c_{ij} \le 0, & b_j \le x. \end{cases} \tag{4.3}$$

Therefore, it follows that
$$\lim_{x\to-\infty} K_{ij}(x) = \infty, \tag{4.4}$$
$$\lim_{x\to\infty} K_{ij}(x) = \begin{cases} -\infty & \text{if} \quad \beta < 1, \\ -c_{ij} & \text{if} \quad \beta = 1. \end{cases} \tag{4.5}$$

Let the minimum solution of the equation $K_{ij}(x) = 0$, if it exists, be denoted by h_{ij} (Figure 1), the maximum of all h_{ij} for $j \in \mathcal{S}$ by h_i^*, and the i maximizing h_i^* on $\mathcal{S}$ by i^*, and let $h^* = h_{i^*}^*$. That is,
$$h_{ij} = \min\{x \mid K_{ij}(x) = 0\}, \qquad h_i^* = \max_{j\in\mathcal{S}} h_{ij}, \qquad h^* = h_{i^*}^* = \max_{i\in\mathcal{S}} h_i^*. \tag{4.6}$$

Lemma 1

(a) *$K_{ij}(x)$ is nonincreasing in x and strictly decreasing in $x < b_j$.*

(b) *$K_{ij}(x) + x$ is nondecreasing in x,*

(c) *h_{ij} exists for all i, j where $b_j \ge h_{ij} \ge \beta\mu_j - c_{ij} > 0$ for all i and j.*

(d) 1. *If $(1-\beta)^2 + c_{ij}^2 \neq 0$, then h_{ij} is the unique solution of $K_{ij}(x) = 0$ and $0 < h_{ij} < b_j$.*
  2. *If $(1-\beta)^2 + c_{ij}^2 = 0$, then $h_{ij} = b_j$*

(e) *$|K_{ij}(y) + y - K_{ij}(x) - x| \le \beta|y - x|$ for any x and y,*

Proof: The inequalities below will be used in this proof. For any x and y
$$\begin{aligned} K_{ij}(x) &= \beta\int_x^\infty (w - x)dF_j(w) - (1-\beta)x - c_{ij} \\ &\ge \beta\int_y^\infty (w - x)dF_j(w) - (1-\beta)x - c_{ij}, \end{aligned}$$

by use of which we can get

$$\begin{aligned} K_{ij}(y) - K_{ij}(x) &\leq \Big(\beta \int_y^\infty (w-y)dF_j(w) - (1-\beta)y - c_{ij}\Big) \\ &\qquad -\Big(\beta \int_y^\infty (w-x)dF_j(w) - (1-\beta)x - c_{ij}\Big) \\ &= \beta \int_y^\infty (x-y)dF(w) + (1-\beta)(x-y) \\ &= \beta(x-y)(1-F(y)) + (1-\beta)(x-y) \\ &= (y-x)(\beta F(y) - 1) \ \cdots (1^*). \end{aligned}$$

Similarly we have

$$K_{ij}(y) - K_{ij}(x) \geq (y-x)(\beta F(x) - 1) \ \cdots (2^*).$$

(a) The nonincreasingness is clear from (4.2). For any $x < y < b_j$, we have $K_{ij}(y) - K_{ij}(x) < 0$ from (1^*). Hence, it follows that $K_{ij}(x)$ is strictly decreasing in $x < b_j$.

(b) Obvious from (4.1).

(c) The existence of h_{ij} is clear from (a), (4.4), and (4.5). It is obvious from $K_{ij}(b_j) \leq 0$ that $h_{ij} \leq b_j$. Since

$$0 = K_{ij}(h_{ij}) \geq \beta \int w dF_j(w) - h_{ij} - c_{ij} = \beta\mu_j - h_{ij} - c_{ij},$$

we get $h_{ij} \geq \beta\mu_j - c_{ij}$.

(d1) If $\beta < 1$, then, as easily seen from (4.2), $K_{ij}(x)$ is strictly decreasing in x with $K_{ij}(x) \to \infty$ $(-\infty)$ as $x \to -\infty$ (∞); hence, h_{ij} uniquely exists. If $\beta = 1$, then $c_{ij} > 0$ from $(1-\beta)^2 + c_{ij}^2 > 0$ and we see that $K_{ij}(x)$ is strictly decreasing in $x < b_j$ with $K_{ij}(x) \to \infty$ as $x \to -\infty$ and $K_{ij}(b_j) = -c_{ij} < 0$; hence, h_{ij} uniquely exists. The inequality $0 < h_{ij} < b_j$ is immediately obtained from $K_{ij}(b_j) < 0$ and $K_{ij}(0) = \beta\mu_j - c_{ij} > 0$ due to the assumption (3.2).

(d2) In this case, clearly $K_{ij}(x) > K_{ij}(b_j) = 0$ for $x < b_j$ and $K_{ij}(h_{ij}) = 0$ for $x \geq b_j$ form (4.3); hence, it follows by definition that $h_{ij} = b_j$.

(e) From (1^*) and (2^*) we get

$$\beta(y-x)F_j(x) \leq K_{ij}(y) + y - K_{ij}(x) - x \leq \beta(y-x)F_j(y),$$

hence it follows that

$$\begin{aligned} &|K_{ij}(y) + y - K_{ij}(x) - x| \\ &\qquad \leq \max\{|\beta(y-x)F_j(y)|, |\beta(y-x)F_j(x)|\} \\ &\qquad = \beta|y-x| \max\{F_j(y), F_j(x)\} \\ &\qquad \leq \beta|y-x|. \ \blacksquare \end{aligned}$$

The following is clear from the fact that $K_{ij}(x)$ is nonincreasing in x (Figure 1):

$$\max_{j\in\mathcal{S}} K_{ij}(x) \begin{cases} \leq 0 & \text{if } x \geq h_i^*, \\ \geq 0 & \text{if } x \leq h_i^*, \end{cases} \tag{4.7}$$

hence

$$\max_{j\in\mathcal{S}} K_{ij}(h_i^*) = 0. \tag{4.8}$$

• *Case that d_{ij} is independent of i*

In this case, since c_{ij} becomes independent of i, so also are $K_{ij}(x)$ and h_{ij}; therefore, let $K_j(x) = K_{ij}(x)$ and $h_j = h_{ij}$. Then, let the smallest solution of equation $K_j(x) = 0$ be denoted by h_j, and the j maximizing h_j on $\mathcal{S}$ by j^* (Figure 2) where $h^* = h_j$. That is,

$$h^* = h_{j^*} = \max_{j \in \mathcal{S}} h_j. \tag{4.9}$$

Remark 1 (See Figure 2) The j^* coincides with the j attaining the maximum of the left hand side of

$$\max_{j \in \mathcal{S}} K_j(h^*) = 0.$$

5. No Recall Model

Let $u_t(w,i)$ denote the maximum expected present discounted net value starting from time t, in search area i, with a current offer w. Then, clearly $u_0(w,i) = w$, and we have

$$u_t(w,i) = \max\{w, U_t(i)\}, \quad t \geq 1, \tag{5.1}$$

where $U_t(i)$ is the maximum expected present discounted net value when continuing the search, expressed as

$$U_t(i) = \max_{j \in \mathcal{S}}\{\beta \int_0^\infty u_{t-1}(\xi, j) dF_j(\xi) - c_{ij}\} \tag{5.2}$$

where ξ is the value of an offer that will be obtained at the next point in time. Substituting (5.1) into (5.2) yields

$$U_t(i) = \max_{j \in \mathcal{S}}\{K_{ij}(U_{t-1}(j)) + U_{t-1}(j)\}, \quad t \geq 2, \tag{5.3}$$

where

$$U_1(i) = \max_{j \in \mathcal{S}}\{\beta\mu_j - c_{ij}\}. \tag{5.4}$$

Here we shall denote the j attaining the maximums of the right hand sides of (5.3) and (5.4) by $\nu_t(i)$. Then, supposing an offer w has been obtained at time t in search area i, we can prescribe the optimal decision rules as follows:

Optimal Stopping Rule *If $w > U_t(i)$, then stop the search by accepting the offer, or else continue the search, hence the $U_t(i)$ is a reservation value in the model.*

Optimal Selection Rule *If it is decided to continue the search, then the optimal search area of the next point in time is $\nu_t(i)$. Hence, if $\nu_t(i) = i$, then it is optimal to continue the search by staying in the current search area i.*

Let $k_n(t,i)$ denote an optimal search area at time $n = 0, 1, \cdots, t$, starting from time t in search area i. Then

$$k_n(t,i) = \nu_{n+1}(k_{n+1}(t,i)), \quad n = 0, 1, \cdots, t-1, \tag{5.5}$$

where $k_t(t,i) = i$.

Theorem 1

(a) $U_t(i) \leq h^*$ $(< \infty)$ *for all t and i,*

(b) $U_t(i)$ *is nondecreasing in t and converges as $t \to \infty$ to the limit $U(i)$, satisfying*

$$U(i) = \max_{j \in \mathcal{S}}\{K_{ij}(U(j)) + U(j)\}, \tag{5.6}$$

(c) $U(i) \geq h_{ii}$,
(d) *If* $\beta < 1$, *then* $U(i)$ *is the unique solution of the equation* (5.6).

Proof: (a) It is immediate from (5.4) and Lemma 1(c) that

$$U_1(i) \leq \max_{j \in \mathcal{S}} h_{ij} = h_i^* \leq h^*.$$

Suppose $U_{t-1}(i) \leq h^*$. Then, since $h^* \geq h_i^*$, we have from (5.3), Lemma 1(b), and (4.7)

$$U_t(i) \leq \max_{j \in \mathcal{S}}\{K_{ij}(h^*) + h^*\} = \max_{j \in \mathcal{S}} K_{ij}(h^*) + h^* \leq h^*.$$

(b) The monotonicity of $U_t(i)$ in t can be easily proven by induction starting with

$$U_2(i) \geq \max_{j \in \mathcal{S}}\{\beta \int_0^\infty \xi dF_j(\xi) - c_{ij}\} = \max_{j \in \mathcal{S}}\{\beta\mu_j - c_{ij}\} = U_1(i)$$

due to $u_1(\xi, j) \geq \xi$ for all ξ from (5.1). The convergency is obvious from this and (a). (5.6) is immediately obtained from (5.3).

(c) Rearranging (5.6) by transposing $U(i)$ from the left hand side to the right yields

$$0 = \max\{K_{ii}(U(i)),\ \max_{j \neq i}\{K_{ij}(U(j)) + U(j) - U(i)\}\},$$

from which it must follow that $K_{ii}(U(i)) \leq 0$, implying $h_{ii} \leq U(i)$.

(d) Suppose equation (5.6) has another finite solution $V(i)$, i.e.,

$$V(i) = \max_{j \in \mathcal{S}}\{K_{ij}(V(j)) + V(j)\}. \tag{5.7}$$

Let $\Delta = \max_{i \in \mathcal{S}}|U(i) - V(i)|$ where $0 < \Delta < \infty$. Then, using the general formula

$$|\max_j a(j) - \max_j b(j)| \leq \max_j |a(j) - b(j)|,$$

from (5.6), (5.7), and Lemma 1(e), we can immediately see

$$\begin{aligned} |U(i) - V(i)| &= |\max_{j \in \mathcal{S}}\{K_{ij}(U(j)) + U(j)\} - \max_{j \in \mathcal{S}}\{K_{ij}(V(j)) + V(j)\}| \\ &\leq \max_{j \in \mathcal{S}} |\{K_{ij}(U(j)) + U(j)\} - \{K_{ij}(V(j)) + V(j)\}| \\ &\leq \beta \max_{j \in \mathcal{S}} |U(j) - V(j)| \\ &= \beta\Delta, \end{aligned}$$

hence we have $\Delta \leq \beta\Delta$, yielding the contradiction $1 \leq \beta$. Consequently, the solution $U(i)$ must be unique. ■

By $\nu(i)$ we shall denote the j that maximizes the right hand side of (5.6).

- *Case that d_{ij} is independent of i*

In this case, c_{ij}, $U_t(i)$, $\nu_t(i)$, $k_n(t, i)$, $U(i)$, and $\nu(i)$ are all also independent of i, so that let us denote them by c_j, U_t, ν_t, $k_n(t)$, U, and ν, respectively. Then, U_t satisfies

$$U_t = U_{t-1} + \max_{j \in \mathcal{S}} K_j(U_{t-1}), \quad t \geq 1, \tag{5.8}$$

where $U_1 = \max_j\{\beta\mu_j - c_j\}$.

Corollary 1 *$U = h^*$ and $\nu = j^*$.*

Proof: From Theorem 1(a,c) we have $h_i \leq U \leq h^*$ for all i, hence $h^* \leq U \leq h^*$, so that $U = h^*$. If $t \to \infty$ in (5.8), then we have $\max_j K_j(h^*) = 0$, hence the optimal search area ν of the next point in time is given by j maximizing $K_j(h^*)$ on $\mathcal{S}$, i.e., j^* (Remark 1), hence $\nu = j^*$ ■

Thus, if an infinite planning horizon is permitted, then the optimal stopping and selection rule can be described as follows:

Optimal Stopping and Selection Rule *Continue the search in search area j^* till an offer $w \geq h^*$ appears.*

6. Recall Model

Let $u_t(y, i)$ denote the maximum expected present discounted net value, starting from time t, in search area i, with the best offer y so far. Then, clearly $u_0(y, i) = y$, and we have

$$u_t(y, i) = \max\{y, U_t(y, i)\}, \quad t \geq 1, \tag{6.1}$$

in which $U_t(y, i)$ is the maximum expected present discounted net value when continuing the search, expressed by

$$U_t(y, i) = \max_{j \in \mathcal{S}}\{\beta \int_0^\infty u_{t-1}(\max\{\xi, y\}, j) dF_j(\xi) - c_{ij}\}, \quad t \geq 2, \tag{6.2}$$

where

$$U_1(y, i) = \max_{j \in \mathcal{S}} K_{ij}(y) + y. \tag{6.3}$$

Here we shall denote the j maximizing the right hand sides of (6.2) and (6.3) by $\nu_t(y, i)$. Then, supposing the searcher is in search area i at time t with the best offer y so far, we can prescribe the optimal decision rules as follows.

Optimal Stopping Rule *If $y > U_t(y, i)$, then stop the search by accepting the best offer y, or else continue the search.*

Optimal Selection Rule *If it is decided to continue the search, then the optimal search area of the next point in time is $\nu_t(y, i)$.*

Now let $Y_t(j|i)$ be a set of y for y in which the right hand sides of (6.2) and (6.3) are maximized by the search area $j = j$; this is the optimal next search area. It will be demonstrated in a numerical example in Section 7 that the set may be given by the union of exclusive intervals.

Let $k_n(t, \boldsymbol{y}, i)$ denote the optimal search areas at times $n = 0, 1, \cdots, t$, starting from time t in search area i where $\boldsymbol{y} = (y_1, y_2, \cdots, y_t)$ is the vector of the best offers y_n at times $n = 1, 2, \cdots, t$ with $y_t = y$ where $y_t \leq y_{t-1} \leq \cdots \leq y_1$. Then

$$k_n(t, \boldsymbol{y}, i) = \nu_{n+1}(y_{n+1}, k_{n+1}(t, \boldsymbol{y}, i)), \quad n = 0, 1, \cdots, t-1, \tag{6.4}$$

where $k_t(t, \boldsymbol{y}, i) = i$.

Lemma 2

(a) *For all $t \geq 1$, $U_t(y, i) - y$ is nonincreasing in y, diverges to ∞ as $y \to -\infty$, and becomes nonpositive for a sufficiently large y, hence the equation $U_t(y, i) - y = 0$ has a positive solution.*

(b) *$U_t(y,i)$ is nondecreasing in t for all y and i with the upper-bound* $\max\{y,h^*\}$*; hence, it converges as $t \to \infty$ to a limit $U(y,i)$ for all y and i, satisfying*

$$U(y,i) = \max_{j\in\mathcal{S}}\{\beta\int_0^\infty u(\max\{\xi,y\},j)dF_j(\xi) - c_{ij}\}. \tag{6.5}$$

Proof: (a) First, it is clear that $U_1(y,i) - y$ $(= \max_j K_{ij}(y))$ is nonincreasing in y from Lemma 1(a), diverges to ∞ as $y \to -\infty$ from (4.4), and becomes nonpositive for any sufficiently large y from (4.3). Next, assume that the assertions hold for $U_{t-1}(y,i) - y$; hence, $u_{t-1}(y,i) - y$ $(= \max\{0, U_t(y,i) - y\})$ is also nonincreasing in y and becomes 0 for any sufficiently large y. Then, rearranging $U_t(y,i) - y$ by substituting

$$u_{t-1}(\max\{\xi,y\},j) = \max\{\xi,y\} + (u_{t-1}(\max\{\xi,y\},j) - \max\{\xi,y\}),$$

we have

$$\begin{aligned} U_t(y,i) - y &= \max_{j\in\mathcal{S}}\{\beta\int_0^\infty \max\{\xi,y\}dF_j(\xi) - y - c_{ij} \\ &\qquad +\beta\int_0^\infty (u_{t-1}(\max\{\xi,y\},j) - \max\{\xi,y\})dF_j(\xi)\} \\ &= \max_{j\in\mathcal{S}}\{K_{ij}(y) + \beta\int_0^\infty (u_{t-1}(\max\{\xi,y\},j) - \max\{\xi,y\})dF_j(\xi)\}. \end{aligned}$$

The terms inside the braces, $K_{ij}(y)$ and

$$\beta\int_0^\infty (u_{t-1}(\max\{\xi,y\},j) - \max\{\xi,y\})dF_j(\xi),$$

are both nonincreasing in y and becomes nonpositive for any sufficiently large y from Lemma 1(a) and the induction hypothesis; hence, so also is $U_t(y,i) - y$. In addition to this, since $K_{ij}(y)$ diverges to ∞ as $y \to -\infty$, so also is $U_t(y,i) - y$. Thus, it follows that $U_t(y,i) - y = 0$ has a solution for all $t \geq 1$, which is positive because of

$$\begin{aligned} U_t(0,i) - 0 &= \max_{i\in\mathcal{S}}\{\beta\int_0^\infty u_{t-1}(\xi)dF_j(\xi) - c_{ij}\} \\ &\geq \max_{i\in\mathcal{S}}\{\beta\int_0^\infty \xi dF_j(\xi) - c_{ij}\} \\ &= \max_{i\in\mathcal{S}}\{\beta\mu_j - c_{ij}\} > 0. \end{aligned}$$

(b) First, the monotonicity of $U_t(y,i)$ in t can be easily verified by induction starting with

$$\begin{aligned} U_2(y,i) &\geq \max_{j\in\mathcal{S}}\{\beta\int_0^\infty \max\{\xi,y\}dF_j(\xi) - c_{ij}\} \\ &= \max_{j\in\mathcal{S}} K_{ij}(y) + y \\ &= U_1(y,i) \end{aligned}$$

due to $u_1(\max\{\xi,y\},j) \geq \max\{\xi,y\}$ from (6.1). Next, let us show that it has an upper-bound in y and i. If $y \leq h_i^*$, then

$$U_1(y,i) \leq \max_{j\in\mathcal{S}} K_{ij}(h_i^*) + h_i^* = h_i^*$$

from Lemma 1(b) and (4.8). If $y \geq h_i^*$, then $U_1(y,i) \leq y$ from (6.3) and (4.7). Hence, for any y we have $U_1(y,i) \leq \max\{y,h_i^*\} \leq \max\{y,h^*\}$. Assume that $U_{t-1}(y,i) \leq \max\{y,h^*\}$.

Then, since $u_{t-1}(\max\{\xi, y\}, j) \le \max\{\max\{\xi, y\}, \max\{y, h^*\}\} = \max\{\xi, \max\{y, h^*\}\}$ for all ξ, y and j, we have for any y

$$\begin{aligned} U_t(y,i) &\le \max_{j\in S}\{\beta \int_0^\infty \max\{\xi, \max\{y, h^*\}\} dF_j(\xi) - c_{ij}\} \\ &= \max_{j\in S}\{K_{ij}(\max\{y, h^*\}) + \max\{y, h^*\}\} \\ &= \max_{j\in S} K_{ij}(\max\{y, h^*\}) + \max\{y, h^*\} \\ &\le \max\{y, h^*\}. \end{aligned}$$

from (4.7) because of $\max\{y, h^*\} \ge h^* \ge h_i^*$ for all y. Thus, it follows by induction that $U_t(y,i)$ has an upper-bound $\max\{y, h^*\}$; hence, it converges to the limit $U(y,i)$ as $t \to \infty$ for all y and i. (6.5) is immediately obtained from (6.2). ■

By $\nu(y,i)$ we shall denote the j that attains the maximum of the right hand side of (6.5).

Theorem 2
(a) *There exists the minimum solution $z_t(i)$ of $U_t(y,i) - y = 0$, which is positive, i.e.,*

$$z_t(i) = \min\{z \mid U_t(z,i) - z = 0\}. \tag{6.6}$$

(b) *$z_t(i)$ is nondecreasing in t and converges as $t \to \infty$ to a finite number $z(i)$,*
(c) *$u_t(y,i) = U_t(y,i) > y$ for $y < z_t(i)$ and $u_t(y,i) = y \ge U_t(y,i)$ for $z_t(i) \le y$,*
(d) *$z_1(i) = h_i^*$ and $h_i^* \le z_t(i) \le h^*$ for all t, i, hence $z_t(i^*) = h^*$ for all t,*

Proof: (a) Obvious from Lemma 2(a).
(b) The monotonicity of $z_t(i)$ in t is clear from the fact that $U_t(y,i) - y$ is nondecreasing in t from Lemma 2(b). That $z_t(i)$ is upper-bounded in t is also evident from the fact that $U_t(y,i)$ is upper-bounded in t from Lemma 2(b); hence, $z_t(i)$ converges as $t \to \infty$.
(c) Immediate from Lemma 2(a).
(d) That $z_1(i)$ is given by h_i^* is clear from (6.3) and (4.7). Assume the assertion in the theorem is true for $t-1$. Here note that $U_t(y,i) \ge \max_j K_{ij}(y) + y$ due to $u_{t-1}(\max\{\xi, y\}, j) \ge \max\{\xi, y\}$, hence $U_t(y,i) \ge y$ for $y \le h_i^*$ from (4.7). Thus we have

$$U_t(h_i^*, i) - h_i^* \ge 0 \cdots (1^*).$$

Since $\max\{\xi, h^*\} \ge h^* \ge z_{t-1}(j)$ for all ξ and j by the assumption, we have

$$u_{t-1}(\max\{\xi, h^*\}, j) = \max\{\xi, h^*\}$$

for all ξ from (c). Therefore, from (6.2) we have $U_t(h^*, i) = \max_j K_{ij}(h^*) + h^* \le h^*$ due to (4.7). Thus we have

$$U_t(h^*, i) - h^* \le 0 \cdots (2^*).$$

Accordingly, it follows from (1^*), (2^*), and Lemma 2(a) that $h_i^* \le z_t(i) \le h^*$ (Figure 3). ■

It follows from Theorem 2(c) that the optimal stopping rule can be restated as follows.

Optimal Stopping Rule *If $y \ge z_t(i)$, then stop the search by accepting the current best offer y, or else continue the search, hence the $z_t(i)$ is a reservation value in the model.*

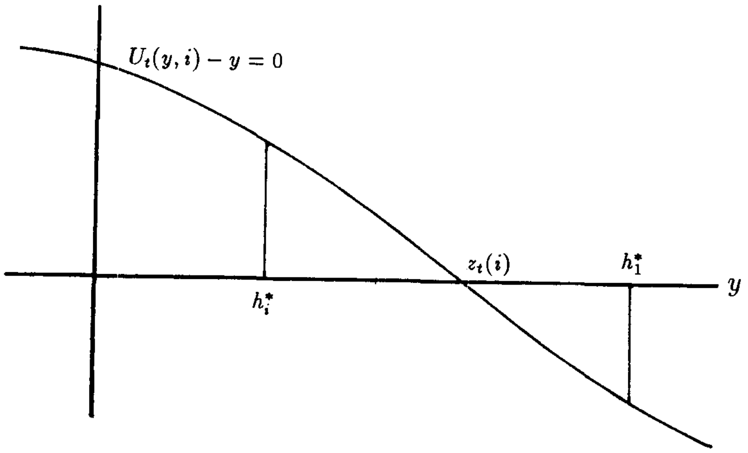

Figure 3
$U_t(h_i^*, i) - h_i^* \geq 0$ and $U_t(h^*, i) - h^* \leq 0$

- *Case that d_{ij} is independent of i*

In this case, c_{ij}, $U_t(y,i)$, $u_t(y,i)$, $z_t(i)$, $\nu_t(y,i)$, $k_n(t,\boldsymbol{y},i)$, $U(y,i)$, $\nu(y,i)$, and $Y_t(j|i)$ are all also independent of i, so let us represent them by c_j, $U_t(y)$, $u_t(y)$, z_t, $\nu_t(y)$, $k_n(t,\boldsymbol{y})$, $U(y)$, $\nu(y)$, and $Y_t(j)$, respectively, where

$$U(y) = \max_{j\in\mathcal{S}}\{\beta\int_0^\infty u(\max\{\xi,y\})dF_j(\xi) - c_j\} \tag{6.7}$$

Corollary 2
(a) $z_t = h^*$ *for all* t.
(b) $u_t(y) = U_t(y) > y$ *if* $y < h^*$, *and* $u_t(y) = y \geq U_t(y)$ *if* $h^* \leq y$.
(c) $u(y) = U(y) > y$ *if* $y < h^*$, *and* $u(y) = y \geq U(y)$ *if* $h^* \leq y$.

Proof: (a) Clear because of $h^* = \max_i h_i^* \leq z_t \leq h^*$ from Theorem 2(d). (b) Evident from (a) and Theorem 2(c). (c) Obvious from (b). ■

Now, note that (6.7) can be expressed as

$$U(y) = \max_{j\in\mathcal{S}}\{\beta\int_0^\infty (u(\max\{\xi,y\})I(h^* \geq \xi) + u(\max\{\xi,y\})I(\xi > h^*))dF_j(\xi) - c_j\}$$

where $I(\cdot)$ is the indicator function, i.e., $I(S) = 1$ if the statement S is true, or else $I(S) = 0$. Then, from Corollary 2(c), if $y \leq h^*$, then $u(\max\{\xi,y\}) = \max\{\xi,y\} = \xi$ for $\xi \geq h^*$, hence, the above expression becomes

$$U(y) = \max_{j\in\mathcal{S}}\{\beta\int_0^\infty (U(\max\{\xi,y\})I(h^* \geq \xi) + \xi I(\xi > h^*))dF_j(\xi) - c_j\}, \quad y \leq h^*. \tag{6.8}$$

Theorem 3 If $\beta < 1$, then $U(y) = h^*$ and $\nu(y) = j^*$ for $y \leq h^*$.

Proof: It will suffice to prove the following two points: (1) If the right hand side of (6.8) is rearranged by substituting $U(y) = h^*$, $y \leq h^*$, then the resultant expression becomes equal to h^* and (2) equation (6.7) has an unique solution. First, let us prove (1). Suppose $y \leq h^*$. Then

$$\begin{aligned}\text{r.h.s. of (6.8)} &= \max_{j\in\mathcal{S}}\{\beta\int_0^\infty (h^*I(h^*\geq\xi)+\xi I(\xi\geq h^*))dF_j(\xi)-c_j\}\\ &= \max_{j\in\mathcal{S}}\{\beta\int_0^\infty \max\{\xi,h^*\}dF_j(\xi)-c_j\}\\ &= \max_{j\in\mathcal{S}} K_j(h^*)+h^* = h^*.\end{aligned} \tag{6.9}$$

Next, let us prove (2). Suppose equation (6.7) has another finite solution $V(y)$, $y\leq h^*$ such that $V(y)>y$ for $y<h^*$ and $V(y)\leq y$ for $h^*\leq y$. Then

$$V(y)=\max_{j\in\mathcal{S}}\{\beta\int_0^\infty (V(\max\{\xi,y\})I(h^*\geq\xi)+\xi I(\xi>h^*))dF_j(\xi)-c_j\},\quad y\leq h^*. \tag{6.10}$$

Let $\varDelta=\sup_{\leq h^*}|U(y)-V(y)|$ where $0<\varDelta<\infty$. Then, using the same method as in the proof of Theorem 1(d), we immediately obtain from (6.8) and (6.10)

$$\begin{aligned}|U(y)-V(y)| &\leq \beta\max_{j\in\mathcal{S}}\int_0^\infty |U(\max\{\xi,y\})-V(\max\{\xi,y\})|I(h^*\geq\xi)dF_j(\xi)\\ &= \beta\max_{j\in\mathcal{S}}\int_0^{h^*}|U(\max\{\xi,y\})-V(\max\{\xi,y\})|dF_j(\xi)\\ &\leq \beta\varDelta\max_{j\in\mathcal{S}}F_j(h^*)\\ &\leq \beta\varDelta,\end{aligned}$$

from which we have $\varDelta\leq\beta\varDelta$, yielding the contradiction $1\leq\beta$. Thus (6.7) must have an unique solution.

Suppose $y\leq h^*$. Then, we have $U(y)=\max_{j\in\mathcal{S}}K_j(h^*)+h^*$ from (6.9), hence it must be $\nu(y)=j^*$ (Remark 1). ■

7. Numerical Examples

Here consider again the asset selling problem of Section 3 where there are three possible cities $i=1(\bigcirc),2(\circledcirc),3(\odot)$ which are search areas, i.e., $\mathcal{S}=\{1,2,3\}$, and amounts offered by buyers that will appear in each city are random variables having triangle distributions such as those in Figure 4. Let the travel costs between two cities be $d_{ij}=1.0$ for $i\neq j$ and $d_{ii}=0.0$. Let the discount factor $\beta=0.98$.

7.1 No Recall Model

Let the search cost in each city be $s_1=3.0$, $s_2=5.0$, and $s_3=11.0$, respectively. Then, h_{ij} and h_i^* are obtained as shown in Table 1, hence $h^*=54.65$ and $i^*=2$.

If the search is made in a city i on a day t, then the optimal reservation value $U_t(i)$ and the optimal next search city $\nu_t(i)$ are calculated as shown in Table 2 (Figure 5)

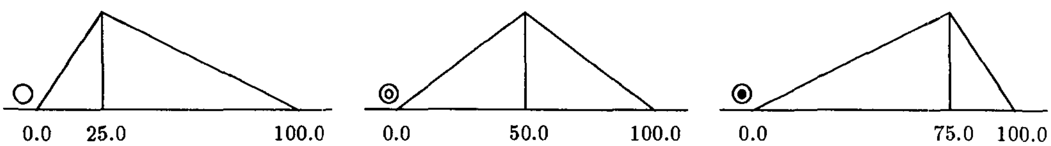

Figure 4
Offer distribution functions of three cities

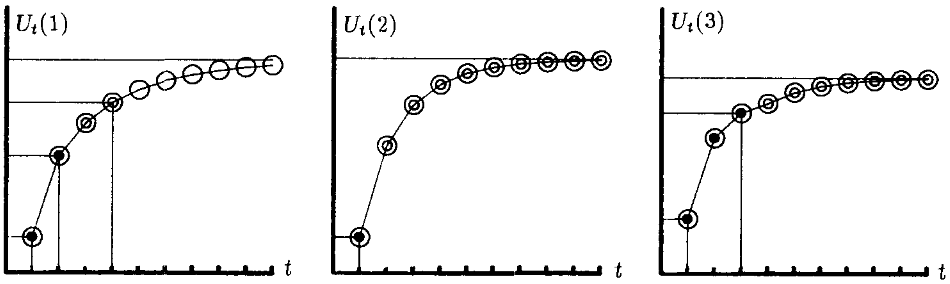

Figure 5
Reservation values of three cities

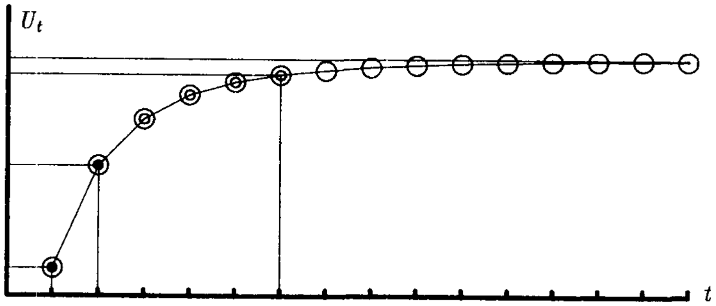

Figure 6
Reservation values U_t
(Case of travel cost d_{ij} being independent of i)

Table 3 shows the path of the optimal search cities, provided that the search starts from day $t = 10$ in each city $i = 1,2,3$ and amounts offered by buyers that appear on days $t = 10, 9, \cdots, 1$ are all lower than the corresponding reservation values $U_t(i)$. The table shows the following three different scenarios: 1. If starting from city 1, then it is optimal to stay in the city up to day 4, then go to city 2 staying there up to day 1, and finally go to city 3; 2. If city 2, then it is optimal to stay in the city up to day 1 and finally go to city 3; 3. If city 3, then it is optimal to go to city 2 on the next day staying there up to day 1 and finally go to city 3.

- *Case that d_{ij} is independent of i*

For example, let $d_{ij} = 0$ for all i, j and $s_1 = 2.9$, $s_2 = 5.0$, and $s_3 = 11.0$. Then, we have $h_1 = 54.89$, $h_2 = 54.65$, and $h_3 = 52.45$, hence $h^* = 54.89$ and $j^* = 1$. In this case, the optimal reservation value U_t and the optimal next search city ν_t are calculated as shown in Table 4 (Figure 6).

7.2 Recall Model

In this model, for the convenience of numerical calculations, let us transform the price distribution functions $f(w)$ into a discrete distribution functions $g(w_n)$ as follows. First, the interval [0,100] is divided into $N = 500$ subintervals, equally spaced. Then, let the $N + 1$ points $w_n = (100/N) \times n$, $n = 0, 1, \ldots, N$, and $g(w_n) = f(w_n)/S$ with $S = \sum_{n=0}^{N} f(w_n)$. Let the search costs in the three cities be $s_1 = 3.0$, $s_2 = 5.0$, and $s_3 = 20.0$. In this case, h_{ij}

and h_i^* are obtained as shown in Table 5, hence $h^* = 65.40$ and $i^* = 1$.

Furthermore we obtain $z_t(1) = h_1^* = 65.4$, $z_t(2) = h_2^* = 61.2$, and $z_t(3) = h_3^* = 40.2$ for $t = 1, 2, 3, 4$ (Note here that, although $z_t(i)$ is independent of t in this example, it does not always become so in other examples.) This means that the optimal city becomes as follows. For example, if the search has been made in city $i = 1$ on day $t = 4$ and the highest price y_4 offered by buyers up to the day is greater than $z_4(1) = 65.4$, then it is optimal to stop the search by selling the asset to the buyer who offered the highest amount, or else continue the search.

Table 6 shows the set $Y_t(j|i)$ for y in which a given city j becomes the optimal next search city if the search is made in a city i on a day t. It should be remarked in the table that there exist ***double critical points*** on the highest price y_t in the following sense. For example, when the search is made in city $i = 1$ on day $t = 2$, if $y_2 \leq 34.8$ or $42.8 \leq y_2 \leq 65.4$, then the optimal next search city is city $j = 3$, and if $34.8 \leq y_2 \leq 42.8$, then it is $j = 1$. That is, we have $Y_2(3|2) = [0.0, 34.8] \cup [42.8, 65.4]$, the union of the two exclusive intervals. This implies the following. Suppose the highest price up to the previous day (i.e., day $t = 3$) is $y_3 = 30.0$. In this case, if the price of day 2 is $w = 20.0$, then the next optimal search city is city 3 due to $y_2 = \max\{y_3, 20.0\} = 30.0 < 34.8$, if $w = 35.0$, then city 1 due to $34.8 < y_2 = \max\{y_3, 35.0\} = 35.0 < 42.8$, and if $w = 45.0$, then ***again*** city 3 due to $42.8 < y_2 = \max\{y_3, 45.0\} = 45.0 < 65.4$.

From Table 6 we can show all the possible paths of the optimal search cities (Figure 7) up to day 0, provided that the search starts from day 4 in city $i = 1$. For example, the following paths of the optimal search cities can be obtained:

$$\text{If } \ (y_4, y_3, y_2, y_1) = \begin{cases} (40.0,\ 43.0,\ 45.0,\ 50.0), & \text{then } 1 \to 2 \to 2 \to 2 \to 3, \\ (41.7,\ 42.0,\ 42.7,\ 60.0), & \text{then } 1 \to 1 \to 1 \to 1 \to 3, \\ (41.0,\ 47.0,\ \ -,\ \ -), & \text{then } 1 \to 2 \to 3 \to \textit{stop}. \end{cases}$$

- *Case that d_{ij} is independent of i*

For example, let $d_{ij} = 0$ for all i, j, and let $s_1 = 3.0$, $s_2 = 7.0$, and $s_3 = 15.0$. Then, we have $h_1 = 54.50$, $h_2 = 50.33$, and $h_3 = 46.56$, hence, $h^* = 54.50$ and $j^* = 1$. Accordingly, it follows that, if the highest price so far is greater than 54.50, then it is optimal to stop the search by selling the asset to the buyer that offered the amount, or else continue the search. Table 7 shows the set $Y_t(j)$ for y in which a given city j becomes the optimal next search city.

Also in this case, we can draw the same tree diagram as shown in Figure 7 and trace the paths of the optimal search cities. For example, we have

$$\text{If } \ (y_4, y_3, y_2, y_1) = \begin{cases} (10.0,\ 20.0,\ 25.0,\ 45.0), & \text{then } 3 \to 1 \to 2 \to 1, \\ (10.0,\ 20.0,\ 27.0,\ 30.0), & \text{then } 3 \to 1 \to 1 \to 2. \end{cases}$$

8. Summary of Conclusions

1. In the no recall model, the reservation value $U_t(i)$ is nondecreasing in t with $U_t(i) \leq h^*$ for all t, i and converges to a finite number $U(i)$ as $t \to \infty$ with $h_{ii} \leq U(i) \leq h^*$ for all i (Theorem 1(a,b,c)). If $\beta < 1$, then the $U(i)$ is given by the unique solution of equation (5.6) (Theorem 1(d)). In the recall model, the reservation value is given by the minimum solution $z_t(i)$ of $U_t(y, i) - y = 0$ with $z_1(i) = h_i^*$ and $h_i^* \leq z_t(i) \leq h^*$ for all t, i, hence $z_t(i^*) = h^*$ for all t, i.e., the reservation value of search area i^* is independent of time t (Theorem 2(a,d)). The reservation value $z_t(i)$ is nondecreasing in t and converges as $t \to \infty$ to a finite number $z(i)$ (Theorem 2(b)).

The above conclusions tell us that, in both models, the optimal stopping rule has a *reservation value property* and, if the searcher wants to behave optimally, then the more periods that remain up to the deadline, the higher the price he searches for.

2. In the recall model, it was demonstrated by a numerical example that there may exist double critical points w_* and w^* ($w_* < w^*$) in terms of present offer w in the following sense. If $w < w_*$, then it is optimal to go to search area i, if $w_* \leq w \leq w^*$, then other search area j, and if $w^* < w$, then again search area i.

3. Suppose the travel cost is independent of the starting search area i. In this case, of course, so also are the reservation value and the optimal next search area. Then, in the recall model, the reservation value becomes equal to h^* for all t, i, that is, the optimal stopping rule is time-independent as well as area-independent (Corollary 2(a)). This implies that whatever point in time the search process starts from, the optimal stopping rule at that point is the same as the one at time 1 when the search process terminates at the next point in time. In other words, whatever planning horizon remains, it is optimal to behave, in terms of stopping decision, *as if* there remains only one period of planning horizon. This property is usually called a *myopic property* In the no recall model, as the planning horizon tends to infinity, the reservation value and the optimal search area converge to h^* and j^*. This implies that in the limiting planning horizon the optimal decision rule becomes the same as the one in the recall model.

Table 1

$h^* = 54.6$, $i^* = 2$

i	h_{i1}	h_{i2}	h_{i3}	h_i^*
1	54.55	52.40	50.92	54.55
2	51.31	54.65	50.92	54.65
3	51.31	52.40	52.45	52.45

Table 2

Reservation values $U_t(i)$ and optimal next search cities $\nu_t(i)$

t	10	9	8	7	6	5	4	3	2	1
$U_t(1)$	54.27	54.16	54.00	53.77	53.44	52.96	52.28	51.18	49.45	45.17
$\nu_t(1)$	1	1	1	1	1	1	2	2	3	3
$U_t(2)$	54.60	54.57	54.51	54.40	54.21	53.88	53.28	52.18	50.02	45.17
$\nu_t(2)$	2	2	2	2	2	2	2	2	2	3
$U_t(3)$	53.60	53.57	53.51	53.40	53.21	52.88	52.28	51.76	50.45	46.17
$\nu_t(3)$	2	2	2	2	2	2	2	3	3	3

Table 3

Optimal search cities $k_n(t, i)$, $n = 10, 9, \ldots, 0$, starting from day $t = 10$ in city $i = 1, 2, 3$

n	10	9	8	7	6	5	4	3	2	1	0
starting city	1 →	1 →	1 →	1 →	1 →	1 →	1 →	2 →	2 →	2 →	3
starting city	2 →	2 →	2 →	2 →	2 →	2 →	2 →	2 →	2 →	2 →	3
starting city	3 →	2 →	2 →	2 →	2 →	2 →	2 →	2 →	2 →	2 →	3

Table 4

Reservation values U_t

(Case of travel cost d_{ij} being independent of i)

t	10	9	8	7	6	5	4	3	2	1
U_t	54.73	54.66	54.57	54.43	54.25	53.94	53.39	52.39	50.45	46.17
ν_t	1	1	1	1	2	2	2	2	3	3

Table 5

$h^* = 65.4,\ i^* = 1$

i	h_{i1}	h_{i2}	h_{i3}	h_i^*
1	54.80	57.40	65.40	65.40
2	46.00	54.80	61.20	61.20
3	21.20	30.00	40.20	40.20

Table 6

Set $Y_t(j|i)$ for y in which a given city j is the optimal next search city.

For example, $Y_3(3|1) = [0.0,\ 34.8] \cup [42.8,\ 65.4]$

$t=1$	$i=1$	$y_1 \in [0.0,\ 65.4] \to j=3$	$y_1 \in [65.4,\ \infty] \to stop$		
	$i=2$	$y_1 \in [0.0,\ 61.2] \to j=3$	$y_1 \in [61.2,\ \infty] \to stop$		
	$i=3$	$y_1 \in [0.0,\ 40.2] \to j=3$	$y_1 \in [40.2,\ \infty] \to stop$		
$t=2$	$i=1$	$y_2 \in [0.0,\ 34.8] \to j=3$	$y_2 \in [34.8,\ 42.8] \to j=1$	$y_2 \in [42.8,\ 65.4] \to j=3$	$y_2 \in [65.4,\ \infty] \to stop$
	$i=2$	$y_2 \in [0.0,\ 46.8] \to j=2$	$y_2 \in [46.8,\ 61.2] \to j=3$	$y_2 \in [61.2,\ \infty] \to stop$	
	$i=3$	$y_2 \in [0.0,\ 40.2] \to j=3$	$y_2 \in [40.2,\ \infty] \to stop$		
$t=3$	$i=1$	$y_3 \in [0.0,\ 39.2] \to j=2$	$y_3 \in [39.2,\ 42.8] \to j=1$	$y_3 \in [42.8,\ 65.4] \to j=3$	$y_3 \in [65.4,\ \infty] \to stop$
	$i=2$	$y_3 \in [0.0,\ 46.8] \to j=2$	$y_3 \in [46.8,\ 61.2] \to j=3$	$y_3 \in [61.2,\ \infty] \to stop$	
	$i=3$	$y_3 \in [0.0,\ 40.2] \to j=3$	$y_3 \in [40.2,\ \infty] \to stop$		
$t=4$	$i=1$	$y_4 \in [0.0,\ 41.6] \to j=2$	$y_4 \in [41.6,\ 42.8] \to j=1$	$y_4 \in [42.8,\ 65.4] \to j=3$	$y_4 \in [65.4,\ \infty] \to stop$
	$i=2$	$y_4 \in [0.0,\ 46.8] \to j=2$	$y_4 \in [46.8,\ 61.2] \to j=3$	$y_4 \in [61.2,\ \infty] \to stop$	
	$i=3$	$y_4 \in [0.0,\ 40.2] \to j=3$	$y_4 \in [40.2,\ \infty] \to stop$		

Table 7

Set $Y_t(j)$ for y in which a given city j is the optimal next city

(Case of travel cost d_{ij} being independent of j)

$t=1$	$y_1 \in [0.0,\ 19.8] \to j=3$	$y_1 \in [19.8,\ 43.2] \to j=2$	$y_1 \in [43.2,\ 54.5] \to j=1$	$y_1 \in [54.5,\ \infty] \to stop$
$t=2$	$y_2 \in [0.0,\ 26.8] \to j=2$	$y_2 \in [26.8,\ 54.5] \to j=1$	$y_2 \in [54.5,\ \infty] \to stop$	
$t=3$	$y_3 \in [0.0,\ 54.5] \to j=1$	$y_3 \in [54.5,\ \infty] \to stop$		
$t=4$	$y_4 \in [0.0,\ 54.5] \to j=3$	$y_4 \in [54.5,\ \infty] \to stop$		

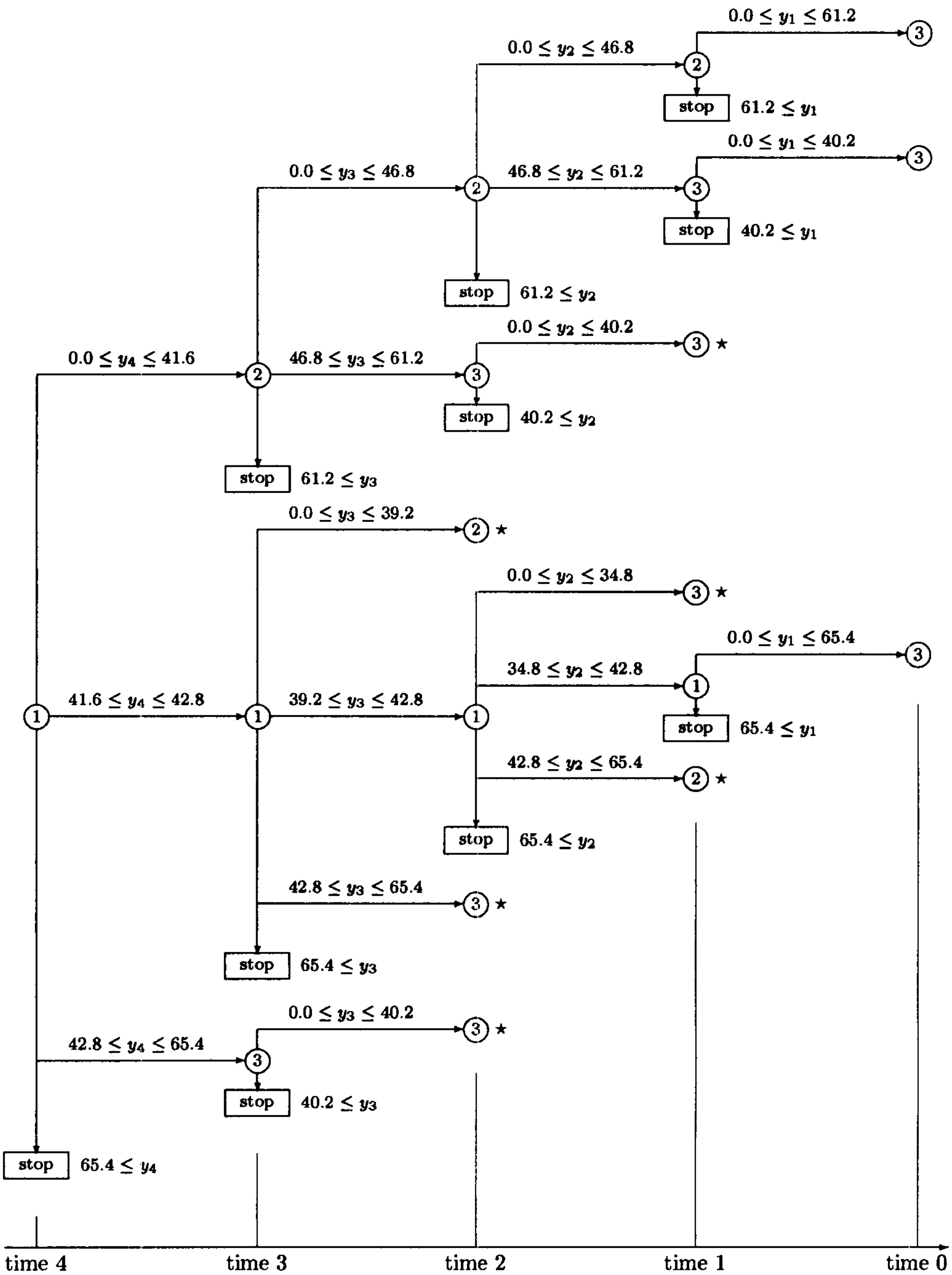

Figure 7 (⋆ the same as above)

The path of the optimal search cities, provided that the search starts from day $t = 4$ in city $i = 1$

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