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A SEARCH GAME WITH TRAVELING COST ON A TREE

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Abstract There is a rooted tree. A person selects a vertex except the root, hides in it and stays there. The searcher is at the root and then examines each vertex until he finds the hider, traveling along edges of the tree. Associated with the examination are a traveling cost dependent on the distance from the last vertex examined and a fixed examination cost. The searcher wishes to minimize the expected cost of finding the hider, whereas the hider wishes to maximize it. The problem is formulated as a two-person zero-sum game and it is solved.

1. Introduction

The purpose of this paper is to analyse search problems on a finite graph. These problems occur, in a system which can be regarded as a network, when we must find some parts which cause the breakdown of the system, or in a building, when we must patrol and find someone which threatens the security.

In this paper the graph is a rooted tree. The hider (called HD) chooses one of all vertices except the root and stays there. At the beginning the searcher (called SR) is at the root. After SR chooses an ordering of the vertices except the root, SR examines each vertex in that order until SR finds HD, traveling along edges. It costs an amount when SR moves from a vertex to an adjacent one and also when SR checks a vertex. While HD wishes to maximize the sum of the traveling costs and the examination costs which are required to find HD, SR wishes to minimize it. We have a two-person zero-sum game and solve it. HD has a unique optimal strategy. An optimal strategy for SR is to mix orderings which correspond to traveling-salesman pathes.

Gluss[5] found a Bayes solution approximately when the graph is linear and the root is a terminal point. Kikuta[7] found a maximin solution in the same graph. The model in this note is a generalization of that in [7]. Arguments in [7] are applicable for solving the game. But in the process of doing so, we must need to show a new kind of inefficiency of some strategies of SR, under which SR goes to other branches of the tree before SR examines a branch throughly. It is not trivial to show it since those strategies may be optimal in cases where we consider Bayes solutions (See Kikuta[6]) and since the non-sequential formulation of strategies of SR makes it difficult to apply the Bellman's principle of optimality (See Bellman[2] and also Ferguson[3], Kikuta/ Ruckle[8]). In order to show it we find properties of the probability distribution which is the unique optimal strategy for HD (See (3.1), (3.2) and Property 3.1 below). It seems that these, in turn, reveal important things in the model and also make the proofs simpler than those in [7](For example, compare Lemma 6.7 below with Lemma 8 of [7]).

Gal[4] analysed a maximin solution in which only traveling costs are taken into consideration. If the examination costs of all vertices are positive HD's optimal strategy will distribute positive probabilities on all vertices. If the examination costs of all vertices converge to zero, it converges to the strategy given in [4]. Theorem 5.1 is a generalization of the result by Gal in this point. Lossner/Wegener[9] analysed a Bayes solution in a very general model and they gave critical numbers and conditions in order to check what is the next point to be examined. Ruckle[12] analysed a maximin solution in which only traveling costs are taken into consideration and the root is not specified. Ahlswede/Wegener[1] states models with traveling costs. Nakai[10] is a survey on the search theory and Ruckle[13] states search games with overlooking probabilities while we do not assume the overlooking probabilities. [11] is a special issue on the search theory.

2. The Model and Notation.

A *graph* (or *undirected graph*) G is an ordered pair (V,E) in which $V \equiv \{0,1,2,\ldots,n\}$ is a finite set of *vertices*, and E is a finite set of pairs of different vertices, (i,j), called *edges*. If $(i,j) \in E$, we say i and j are *adjacent*. A *path* between Vertices i_0 and i_s is a finite sequence of distinct edges of the form $(i_0,i_1),(i_1,i_2),\ldots,(i_{s-1},i_s)$. This path is denoted as $(i_0,i_1,\ldots,i_s)$. If $i_0 = i_s$ then this path is called a *cycle*. A *simple* path between i and j is a path between i and j with no repeated vertices. G is said to be *connected* if for any $i,j \in V$, there is a path between i and j.

Throughout this paper we assume a graph $G \equiv (V,E)$ is a *rooted tree*, i.e., it is connected, it has n edges, it has no cycle, and Vertex 0 is designated the *root*. For $i,j \in V$ such that $i \neq j$, i is called an *ancestor* of j if there is a simple path between 0 and j such that (i,k) is an edge on this path for some $k \in V$. j is called a *descendant* of i if i is an ancestor of j. j is called a *child* of i if j is a descendant of and adjacent to i. For $i \in V$, let D_i, K_i, and A_i for $i \neq 0$ be the sets of all descendants, all children and all ancestors of i respectively. We let $D \equiv D_0$. Define the set of *leaves* by $L \equiv \{i \in V : K_i = \emptyset\}$. For any $j \in D$, there is uniquely $i \in A_j$ such that $j \in K_i$. So we let $i \equiv a(j)$ and write as $e(j) \equiv (a(j),j) \in E$. Let $V_i \equiv \{i\} \cup D_i$. For $Y \subseteq K_i$, Let $e(Y) \equiv \cup_{j \in Y}\{e(j)\}$ and $V_Y \equiv \cup_{j \in Y} V_j$. Clearly $E = e(D)$. For $i \in V$ and $Y \subseteq D_i$, define $D_{(i;Y)} \equiv Y \cup \{j \in D_i : j \in A_y$ for some $y \in Y\}$. Define a tree with i as its root by $G_{(i;Y)} \equiv (D_{(i;Y)} \cup \{i\}, e(D_{(i,Y)}))$. In this paper for a nonnegative-valued function g on D, we let $g(Y) \equiv \Sigma_{i \in Y}\, g(i)$, where $Y \subseteq D$. We let $g(Y) = 0$ if $Y = \emptyset$. For a finite set X, |X| is the cardinality of X. The following is a relation between $G_{(i;D_i)}$ and $G_{(i;V_j)}$, the proof of which is easy and omitted.

Property 2.1. $D_i = V_{K_i} = \cup_{j \in K_i} V_j$, $e(V_j) = e(D_j) \cup \{e(j)\}$ and $e(D_i) = \cup_{j \in K_i} e(V_j)$.

Each edge e(j) ($j \in D$) is associated with a positive number d(j), called the *weight* of e(j). The *length* of a path is the sum of the weights of all the edges in the path. For $i,j \in V$, we define d(i,j) by the length of the simple path between i and j. Clearly $d(a(j),j) = d(j)$ for $j \in D$.

Define a game on G. Player 1 (the hider, abbreviated as HD) hides among one of all vertices in D, and stays there. Player 2 (the searcher, abbreviated as SR) examines each vertex until SR finds HD, traveling along edges. It is assumed that at the beginning of the search SR is at 0, and that SR travels along the simple path between i and j when $(i,j) \notin E$ and SR examines i after having examined j. Associated with the examination of $i \in D$ is the examination cost that consists of two parts:(I) a traveling cost $d(j,i) > 0$ of examining i after having examined j, and (II) an examination cost $c(i) > 0$. There is not a probability of overlooking HD, given that the right vertex is searched. For convenience, we let $d(i,i) = 0$ for all $i \in D$. Before searching (hiding resp.), SR (HD resp.) must determine a strategy so as to make the cost of finding HD as small (large resp.) as possible.

A (*pure*) *strategy* for HD is expressed by an element, say i, of D, which means HD determines on hiding in i. D is the set of all strategies for HD. For $Y \subseteq D$, let $\Sigma(Y)$ be the set of all permutations on Y. A strategy for SR is an element in $\Sigma \equiv \Sigma(D)$. Thus under $\sigma \in \Sigma(D)$, SR examines Vertices $\sigma(1), \sigma(2), \ldots, \sigma(n)$ in this order. This is often expressed as $\sigma = [\sigma(1), \ldots, \sigma(n)]$. For convenience we let $\sigma(n+1) = \sigma(0) = 0$.

For a strategy pair $(i,\sigma) \in D \times \Sigma$, the cost of finding HD, written as $f(i,\sigma)$, is :

(2.1) $$f(i,\sigma) = \sum_{x=1}^{\sigma^{-1}(i)} \{d(\sigma(x),\sigma(x-1)) + c(\sigma(x))\} .$$

Letting payoffs for HD and SR be $f(i,\sigma)$ and $-f(i,\sigma)$ respectively, we have a finite, two-person zero-sum game, denoted by $(f;D,\Sigma)$. We see that the payoff matrix of this game does not always have a saddle point, by checking the case of $n = 2$. So let $(f;P,Q)$ be the mixed extension of $(f;D,\Sigma)$ and we call it a game G just as we denote the graph. The elements of P and Q are called (*mixed*) strategies, and expressed as $p = (p(1),\ldots,p(n)) \in P$ and $q = \{q(\sigma)\} \in Q$, where $p(i)$ is the probability that HD chooses $i \in D$, and $q(\sigma)$ is the probability that SR chooses $\sigma \in \Sigma$. Thus

(2.2) $$p(D) = 1,\ p(i) \ge 0 \text{ for all } i \in D, \text{ and } q(\Sigma) = 1,\ q(\sigma) \ge 0 \text{ for all } \sigma \in \Sigma.$$

For a strategy pair $(p,q) \in P \times Q$, $f(p,q)$ is the expected cost of finding HD. In the same way we can define a game on $G_{(i;Y)}$ ($i \in V$, $Y \subseteq D_i$), in which at the beginning of the search SR is at i, and Y is the set of pure strategies of HD. Call the mixed extension of this game a game $G_{(i;Y)}$. Our problem is to solve the game G. For $j \in D$, we write as

(2.3) $$w(j) \equiv 2d(j) + c(j).$$

$w(j)$ is the cost of the return trip which starts at $a(j)$ and examines j. For $i \in V$ and $Y \subseteq D_i$, we let $w(i;Y) \equiv w(Y)+2d(D_{(i;Y)} \backslash Y)$. This is the cost of the most efficient return trip which starts at i and examines all vertices in Y. For $i \in L$, let $v_i \equiv d(i)+c(i)$. This is the value of the game on a tree $(\{a(j),j\},\{e(j)\})$ with $a(j)$ as its root. Define inductively by,

(2.4) $$v_i \equiv d(i)+c(i) + \frac{w(D_i)v_{K_i}}{c(i)+w(D_i)} \text{ for } i \in D$$

where $v_{K_i} \equiv 0$ (if $i \in L$), and, for $Y \subseteq K_i$, $i \in V$,

(2.5) $$v_Y \equiv \frac{1}{w(V_Y)} \sum_{j \in Y} w(V_j)\{v_j + \sum_{k(<j) \in Y} w(V_k)\}.$$

Property 2.2. $v_Y < w(V_Y)$ for $Y\ (\neq \emptyset) \subseteq K_i$, $i \in V \backslash L$.
Proof: If $v_j < w(V_j)$ for all $j \in Y$, then by (2.5),

$$v_Y < \frac{1}{w(V_Y)} \sum_{j \in Y} w(V_j)\{w(V_j) + \sum_{k(<j) \in Y} w(V_k)\} < \frac{1}{w(V_Y)} w(V_Y)^2 = w(V_Y).$$

In particular, $v_{K_i} < w(D_i)$. Then by (2.4),

$$v_i < w(i) + \frac{w(D_i)w(D_i)}{c(i)+w(D_i)} < w(i) + w(D_i) = w(V_i).$$

If $j \in L$ then $v_j = d(j)+c(j) < w(j) = w(V_j)$. Since the graph G is finite we can start at every i such that $K_i \subseteq L$ and apply the above argument inductively. After a finite number of steps we arrive at 0. Then we have the desired result. ♦

3. A Strategy for the Hider

In this section we define a strategy for HD and analyse its properties in the game $G \equiv (f;P,Q)$.

Define $p^* \in P$ inductively as follows: For $j \in D \setminus K_0$, let

$$\text{(3.1)} \qquad \frac{p^*(a(j))}{c(a(j))} = \frac{p^*(V_j)}{w(V_j)}$$

For $j, k \in K_0$, let

$$\text{(3.2)} \qquad \frac{p^*(V_j)}{w(V_j)} = \frac{p^*(V_k)}{w(V_k)}.$$

Suppose SR is at i. The left hand side of (3.1) is the probability per unit cost when he examines i, while the right hand side is the probability per unit cost when he returns to i after examining all vertices in V_j. By (2.2), (3.1) and (3.2), $p^* \in P$ is defined completely and uniquely (See Appendix). It is easy to see $p^*(i) \to 1/n$ for all $i \in D$ if $c(j) \to +\infty$ for all $j \in D$, and $p^*(i) \to 0$ for all $i \in D \setminus L$ if $c(j) \to 0$ for all $j \in D$.

The following proposition gives basic properties of p^*, which are used in the proof of Theorem 5.1. Property 3.1(v) is a generalization of Property 3.1(iv). But Property 3.1(iv) is interesting since it states that HD should distribute more probabilities to vertices which locate more far from the root.

Property 3.1. (i) For $i \in D$, $\frac{p^*(i)}{c(i)} = \frac{p^*(D_i)}{w(D_i)}$.

(ii) For $i \in K_0$, $p^*(V_i) = \frac{w(V_i)}{w(D)}$.

(iii) For $i \in D \setminus K_0$, $\frac{p^*(a(i))}{c(a(i))} < \frac{p^*(i)}{c(i)}$.

(iv) For $i \in V$, let $Y \subseteq D_i$ and suppose $D_{(i;Y)} = Y$. Then
$\frac{p^*(Y \cup \{i\})}{w(Y \cup \{i\})} \le \frac{p^*(V_i)}{w(V_i)}$ if $i \neq 0$, and $\frac{p^*(Y)}{w(Y)} \le \frac{p^*(D_i)}{w(D_i)}$.

(v) For $i \in V$, let $Y \subseteq D_i$. Then

$$\frac{p^*(Y\cup\{i\})}{w(a(i);Y\cup\{i\})} \le \frac{p^*(V_i)}{w(V_i)} \text{ if } i \neq 0, \text{ and } \frac{p^*(Y)}{w(i;Y)} \le \frac{p^*(D_i)}{w(D_i)}.$$

Proof:(i) From (3.1), $p^*(i)/c(i) = p^*(V_j)/w(V_j)$ for all $j\in K_i$. From this, $p^*(i)/c(i) = \sum_{j\in K_i} p^*(V_j)/\sum_{j\in K_i} w(V_j) = p^*(D_i)/w(D_i)$.

(ii) (3.2) and $\sum_{i\in K_0} p^*(V_i) = 1$.

(iii) By Property 3.1(i), $(\frac{p^*(a(j))}{c(a(j))})/(\frac{p^*(j)}{c(j)}) = 1 - 2d(j)/w(V_j) < 1$, since $D_j = V_j\backslash\{j\}$.

(iv) Since p^* is defined inductively in (3.1) and (3.2), and because of inductive relation expressed in Property 2.1, we can use the induction. First suppose $K_i \subseteq L$. By (3.1) $p^*(j)/w(j) = p^*(i)/c(i)$ for all $j\in K_i$ ($= D_i$). This implies $p^*(Y)/w(Y) = p^*(D_i)/w(D_i)$ for all $Y \subseteq K_i$. Further,

$$\begin{aligned}p^*(V_i)w(Y\cup\{i\})-p^*(Y\cup\{i\})w(V_i) &= (p^*(D_i)+p^*(i))(w(Y)+w(i)) \\ &\quad -(p^*(Y)+p^*(i))(w(D_i)+w(i)) \\ &= w(Y)p^*(i)+w(i)p^*(D_i)-w(D_i)p^*(i)-w(i)p^*(Y) \\ &= c(i)p^*(Y)+w(i)p^*(D_i)-c(i)p^*(D_i)-w(i)p^*(Y) \\ &= (w(i)-c(i))(p^*(D_i)-p^*(Y)) \ge 0\end{aligned}$$

by (3.1) and Property 3.1(i). Hence the property holds if $K_i \subseteq L$. Next, assume for any $j\in K_i$, the property holds for all Y such that $Y\subseteq D_j$. Then suppose $Y \subseteq D_i$. If $Y\cap V_j \neq \emptyset$, $(Y\cap V_j, e(Y\cap D_j))$ is a rooted tree with j as its root. By the induction hypothesis, we have $p^*(Y\cap V_j)/w(Y\cap V_j) \le p^*(V_j)/w(V_j)$. From this, $p^*(Y\cap V_j) \le p^*(V_j)w(Y\cap V_j)/w(V_j)$. Add these together for $j\in K_i$ with $Y\cap V_j \neq \emptyset$, noting that $p^*(V_j)/w(V_j) = p^*(D_i)/w(D_i)$ for all $j\in K_i$. Then $p^*(Y) \le p^*(D_i)w(Y)/w(D_i)$. This, combined with Property 3.1(i) implies $p^*(Y\cup\{i\})/p^*(V_i) \le (w(Y)+c(i))/(w(D_i)+c(i))$ since $D_i = V_i\backslash\{i\}$. The right hand side of this is less than or equal to $w(Y\cup\{i\})/w(V_i)$ since $Y\cup\{i\} \subseteq V_i$. Hence the property holds. Since the graph G is finite we can start at every i such that $K_i \subseteq L$ and apply the above argument inductively. After a finite number of steps we arrive at 0.

(v) If $Y = D_{(i;Y)}$ it reduces to Property 3.1(iv). Suppose $Y \neq D_{(i;Y)}$. By Property 3.1(iv), $p^*(D_{(i;Y)}\cup\{i\})/p^*(V_i) \le w(D_{(i;Y)}\cup\{i\})/w(V_i)$, from which,

$$\text{(3.3)} \qquad \frac{p^*(Y\cup\{i\})}{p^*(V_i)} \le \frac{w(D_{(i;Y)}\cup\{i\})}{w(V_i)} - \frac{p^*(D_{(i;Y)}\backslash Y)}{p^*(V_i)}.$$

Further,

$$\text{(3.4)} \qquad \frac{w(i;Y)+w(i)}{w(V_i)} - \{\frac{w(D_{(i;Y)}\cup\{i\})}{w(V_i)} - \frac{p^*(D_{(i;Y)}\backslash Y)}{p^*(V_i)}\} = \frac{p^*(D_{(i;Y)}\backslash Y)}{p^*(V_i)} - \frac{c(D_{(i;Y)}\backslash Y)}{w(V_i)}$$

$$= \frac{1}{p^*(V_i)} \Sigma_{j\in D_{(i;Y)}\backslash Y} \{p^*(j) - \frac{c(j)p^*(V_i)}{w(V_i)}\}.$$

Here, for $j\in D_{(i;Y)}\backslash Y$,

$$\text{(3.5)} \qquad \frac{p^*(V_i)}{w(V_i)} \leq \frac{p^*(i)}{c(i)} \leq \frac{p^*(j)}{c(j)},$$

by (3.1), the relation of $j \in D_i$, and Property 3.1(iii). (3.3), (3.4) and (3.5) imply the first inequality. The second inequality follows in the same way. ♦

The next lemma follows from Property 3.1 and it is used in the next section.

Lemma 3.2. (i) Let $i \in V_j$ for $j \in K_0$. Under $\sigma \in \Sigma$, suppose SR passes through i and then comes back to i again after checked all vertices in Y ($\subseteq D_j$). Suppose X ($\subseteq$ D) is the set of vertices which he passes through in this trip. Assume $Y \cap A_i = \emptyset$. Then

$$\frac{p^*(Y)}{w(Y)+2d(X)} \leq \frac{p^*(i)}{c(i)}.$$

(ii) Let $i, j \in K_0$. Let $Y \subseteq D_i$ and $y \in V_j$. Then

$$\frac{p^*(Y)}{w(0;Y \cup \{i\})-c(i)} \leq \frac{p^*(Y \cup \{i\})}{w(0;Y \cup \{i\})} \leq \frac{p^*(y)}{c(y)}.$$

Proof: (i) *Case 1.* Assume $Y \subseteq D_i$. Then $w(Y)+2d(X) = w(i;Y)$. By Property 3.1(i) and the second inequality of Property 3.1(v), we have the desired result.
Case 2. Assume $Y \not\subseteq D_i$ and $Y \neq D_i$. By the definition of Y, $Y \cap D_i = \emptyset$. There is k ($\neq$i)$\in V_j$ such that $Y \cup \{i\} \subseteq D_k$ and $w(Y)+2d(X) = w(k;Y) + 2d(k,i)$. $i \neq j$ since $Y \not\subseteq D_i$. Also $i \neq k$ by the definition of k. Since $k \in A_i$, we have $k \notin Y$. Applying *Case 1* with Property 3.1(iii),

$$\frac{p^*(i)}{c(i)} \geq \frac{p^*(k)}{c(k)} \geq \frac{p^*(Y)}{w(k;Y)} \geq \frac{p^*(Y)}{w(k;Y)+2d(k,i)}.$$

(ii) By Property 3.1(v), (3.2), (3.1), and Property 3.1(iii),

$$\frac{p^*(Y \cup \{i\})}{w(0;Y \cup \{i\})} \leq \frac{p^*(V_i)}{w(V_i)} = \frac{p^*(V_j)}{w(V_j)} \leq \frac{p^*(j)}{c(j)} \leq \frac{p^*(y)}{c(y)},$$

Further, by Property 3.1(v) and Property 3.1(i),

$$\frac{p^*(Y \cup \{i\})}{w(0;Y \cup \{i\})} - \frac{p^*(Y)}{w(0;Y \cup \{i\})-c(i)} = B\{p^*(i)w(i;Y)-c(i)p^*(Y)+p^*(i)(w(i)-c(i))\}$$

$$\geq Bp^*(i)(w(i)-c(i)) \geq 0,$$

where $B \equiv 1/\{w(0;Y \cup \{i\})(w(0;Y \cup \{i\})-c(i))\}$. ♦

4. Construction of a Strategy for the Searcher

In this section we define a strategy of SR. A part of this kind of strategy has been already discussed in [8] in which a strong assumption is imposed on a game.

Let $i\in D$ and $j\in K_i$. By any $\sigma\in\Sigma(D_i)$, V_j is partitioned into $D_\sigma(j,1),\ldots,D_\sigma(j,r_j)$, where $D_\sigma(j,h)$ is the set of vertices which SR examines in his *h*th visit to the branch V_j. Of course, he may examine vertices which are not in V_j before he examines vertices in $D_\sigma(j,1)$, depending on σ. We call $\sigma\in\Sigma(D_i)$ *partially distance-efficient* (abbreviated as PDE) if for $j\in K_i$, and for $y = 1,\ldots,r_j$, SR examines each vertex in $D_\sigma(j,y)$ when he passes by it for the first time in his *y*th visit to $G_{(j;D_j)}$. $\Sigma_{PDE}(D_i)$ is the set of all PDE-strategies $\sigma\in\Sigma(D_i)$. We call $\sigma\in\Sigma(D_i)$ *sub-distance-efficient* (abbreviated as SDE) if for $j\in K_i$, and for $y = 2,\ldots,r_j$, $D_\sigma(j,y)$ includes no ancestor of vertices in $D_\sigma(j,1)\cup\ldots\cup D_\sigma(j,y-1)$. $\Sigma_{SDE}(D_i)$ is the set of all SDE-strategies $\sigma\in\Sigma(D_i)$. We say that $\sigma\in\Sigma(D_i)$ is *distance-efficient* (abbreviated as DE) if SR travels along each edge in E at most twice under σ. $\Sigma_{DE}(D_i)$ is the set of all DE-strategies $\sigma\in\Sigma(D_i)$. $\Sigma_{DE}(D_i)\subseteq\Sigma_{SDE}(D_i)$. If $\sigma\in\Sigma_{DE}(D_i)$, SR visits $G_{(j;D_j)}$ only once for all $j\in K_i$. Unless otherwise specified, Σ_{SDE}, Σ_{PDE}, Σ_{DE} mean $\Sigma_{SDE}(D)$, $\Sigma_{PDE}(D)$, $\Sigma_{DE}(D)$ respectively. Next we define sets of permutations inductively as follows. Let $Y \equiv \{y_1,\ldots,y_m\}\subseteq D$ where $y_1<\ldots<y_m$. First let $\Sigma(y_1,y_2)\equiv\{[y_1,y_2],[y_2,y_1]\}$. Assuming $\Sigma(y_1,\ldots,y_{m-1})$ is defined, define $\Sigma(y_1,\ldots,y_{m-1},y_m)\equiv\{[y_m,\sigma(y_1),\ldots,\sigma(y_{m-1})]: \sigma\in\Sigma(y_1,\ldots,y_{m-1})\}\cup\{[\sigma(y_1),\ldots,\sigma(y_{m-1}),y_m]: \sigma\in\Sigma(y_1,\ldots,y_{m-1})\}$. Then $\Sigma(y_1,\ldots,y_m)\subseteq\Sigma(Y)$. In particular if $Y = K_i$, $i\in V$, we write as $\Sigma_i\equiv\Sigma(y_1,\ldots,y_m)$. Next, define for $h = 2,\ldots,m$,

$$r(y_h>) = \frac{v_{\{y_1,\ldots,y_{h-1}\}} - w(V_{y_h}) + v_{y_h}}{w(V_{\{y_1,\ldots,y_h\}})} \quad\text{and } r(y_h<) = 1-r(y_h>).$$

For $i\in V$, suppose $K_i = \{y_1,\ldots,y_m\}$ where $y_1<\ldots<y_m$. Then define for $\sigma\in\Sigma_i$,

$$r(i;\sigma)\equiv \prod_{\sigma^{-1}(y_{h-1})<\sigma^{-1}(y_h)} r(y_h>) \prod_{\sigma^{-1}(y_{h-1})>\sigma^{-1}(y_h)} r(y_h<).$$

Next, for $i\in D\backslash L$, $\tau(i<)$ means that SR examines i first and then examines all vertices in D_i. $\tau(i>)$ means that SR examines all vertices in D_i and then examines i. Define for $i\in D\backslash L$, $\Sigma\{i,D_i\}\equiv\{\tau(i<),\tau(i>)\}$. For $i\in D\backslash L$ let

$$r(\tau(i<))\equiv\frac{v_i-v_{K_i}-d(i)}{c(i)} \quad\text{and } r(\tau(i>))\equiv 1-r(\tau(i<)).$$

Property 4.1. (i) $r(y_h>) = \dfrac{v_{\{y_1,\ldots,y_h\}} - v_{y_h}}{w(V_{\{y_1,\ldots,y_{h-1}\}})}$ and $r(y_h<) = \dfrac{v_{\{y_1,\ldots,y_h\}} - v_{\{y_1,\ldots,y_{h-1}\}}}{w(V_{y_h})}$

(ii) For $i\in V$, $\sum_{\sigma\in\Sigma_i} r(i;\sigma) = 1$, and $r(i;\sigma)>0$ for all $\sigma\in\Sigma_i$.

(iii) $0<r(\tau(i<))<1$.

(iv) For $i\in V$ and $j\in K_i$, $\sum_{k\in K_i\backslash\{j\}} w(V_k) \sum_{\sigma\in\Sigma_i:\sigma^{-1}(k)<\sigma^{-1}(j)} r(i;\sigma) = v_{K_i} - v_j$.

Proof:(i) By (2.5) for $Y = \{y_1,\ldots,y_h\}$ and $Y = \{y_1,\ldots,y_{h-1}\}$ respectively, we have

$$v_{\{y_1,\ldots,y_h\}}w(V_{\{y_1,\ldots,y_h\}}) - v_{\{y_1,\ldots,y_{h-1}\}}w(V_{\{y_1,\ldots,y_{h-1}\}}) = w(V_{y_h})\{v_{y_h} + w(V_{\{y_1,\ldots,y_{h-1}\}})\}.$$

From this, we have the desired result, using $w(V_{\{y_1,\ldots,y_{h-1}\}})+w(V_{y_h}) = w(V_{\{y_1,\ldots,y_h\}})$.
(ii) By Property 2.2, $r(i;\sigma) > 0$. We let $K_i = \{y_1,\ldots,y_m\}$ and $y_1 < \ldots < y_m$. When $m = 2$, $r(i;[y_1,y_2]) = r(y_2>)$ and $r(i;[y_2,y_1]) = r(y_2<)$. Hence it holds. Assume it holds for $i,\ldots,m-1$. Let $\Sigma' \equiv \{\sigma\in\Sigma_i:\sigma(y_1) = y_m\}$ and $\Sigma'' \equiv \{\sigma\in\Sigma_i:\sigma(y_m) = y_m\}$. Then $\Sigma_i = \Sigma'\cup\Sigma''$.

$$\sum_{\sigma\in\Sigma_i} r(i;\sigma) = \sum_{\sigma\in\Sigma'} r(i;\sigma) + \sum_{\sigma\in\Sigma''} r(i;\sigma)$$

$$= r(y_m<) \sum_{\sigma\in\Sigma(y_1,\ldots,y_{m-1})} r(i;\sigma) + r(y_m>) \sum_{\sigma\in\Sigma(y_1,\ldots,y_{m-1})} r(i;\sigma) = r(y_m<) + r(y_m>) = 1$$

by the induction hypothesis.
(iii) By (2.3) and (2.4), $v_i-v_{K(i)}-d(i) = c(i)-c(i)v_{K_i}/(c(i)+w(D_i)) < c(i)$, which implies $r(\tau(i<)) < 1$. By Property 2.2, $v_{K_i} < w(D_i)$, which implies $c(i)- c(i)v_{K_i}/(c(i)+w(D_i)) > 0$. Hence $r(\tau(i<)) > 0$.
(iv) Suppose $j < k$. Then by Propert 4.1(ii), $\sum_{\sigma\in\Sigma_i:\sigma^{-1}(k)<\sigma^{-1}(j)} r(i;\sigma) = r(k<)$, and if $j > k$ then $\sum_{\sigma\in\Sigma_i:\sigma^{-1}(k)<\sigma^{-1}(j)} r(i;\sigma) = r(j>)$. Hence

$$\sum_{k\in K_i\setminus\{j\}:k>j} w(V_k) \sum_{\sigma\in\Sigma_i:\sigma^{-1}(k)<\sigma^{-1}(j)} r(i;\sigma) = \sum_{k\in K_i\setminus\{j\}:k>j} w(V_k)r(k<) + \sum_{k\in K_i\setminus\{j\}:k<j} w(V_k)r(j>)$$

$$= v_{K_i}-v_{\{k:k\leq j\}}+v_{\{k:k\leq j\}}-v_j = v_{K_i} - v_j. \quad \blacklozenge$$

We consider an element in Σ_{DE} such that :

(I:Search in $G_{(a(i);V_i)}$) Suppose SR is at a(i) and $i\notin L$. When he examines V_i, he examines in the order of an element in $\Sigma\{i,D_i\}$. I.e., either he examines i first and then examines D_i throughly or he examines D_i throughly and then examines i.
(II:Search in $G_{(i;D_i)}$) Suppose SR is at i and $i\notin L$. When he examines D_i, he examines $\{V_j\}_{j\in K_i}$ in the order of an element in Σ_i, where he examines each V_j throughly before moving to other branches.

It is easy to see any element in Σ that satisfy (I) for all $i\in D\setminus L$ and (II) for all $i\in V\setminus L$ belongs to Σ_{DE}. So let $\Sigma^* \equiv \Sigma^*(D)$ be the set of all elements in Σ_{DE} that satisfy (I) for all $i\in D\setminus L$ and (II) for all $i\in V\setminus L$. In the same way we can define $\Sigma^*(D_i)$ ($i\in V\setminus L$) and $\Sigma^*(V_i)$ ($i\in D\setminus L$). Each element in Σ^* can be specified by selecting one in $\Sigma\{i,D_i\}$ at each $i\in D\setminus L$ and one in Σ_i at each $i\in V\setminus L$. Hence express $\sigma\in\Sigma^*$ as

$$\sigma = \sigma_0 \prod_{i\in D\setminus L} (\sigma_i\tau_i),$$

where $\sigma_i \in \Sigma_i$ and $\tau_i \in \Sigma\{i,D_i\}$. If $|K_0| > 1$ then

$$\sigma = \sigma_0 \prod_{j\in K_0} \prod_{i\in V_j\setminus L} (\sigma_i\tau_i) = \sigma_0 \prod_{j\in K_0} \tau_j\sigma^j,$$

where $\sigma^j \equiv \prod_{i\in V_j\setminus L} \sigma_i \prod_{i\in D_j\setminus L} \tau_i \in \Sigma^*(D_j)$. If $|K_0| = 1$, let $K_0 = \{1\}$. Then

$$\sigma = \sigma_1\tau_1 \prod_{i\in D_1\setminus L} (\sigma_i\tau_i) = \sigma_1\tau_1 \prod_{j\in K_1} \tau_j\sigma^j.$$

Define $q^*\in Q$ as follows.

$$q^*(\sigma) = \prod_{i\in V\setminus L} r(i;\sigma_i) \prod_{i\in D\setminus L} r(\tau_i) \text{ if } \sigma\in\Sigma^*, \text{ and } q^*(\sigma) = 0 \text{ if } \sigma\notin\Sigma^*.$$

Property 4.2. $q^*(\Sigma^*) = 1$.
Proof: If $K_0 \subseteq L$ then $q^*(\sigma) = r(0;\sigma)$ for all $\sigma\in\Sigma^*$. By Property 4.1(ii) Property 4.2 holds. Next, if $K_0 =\{1\}$ and $K_1 \subseteq L$ then $\sigma\in\Sigma^*$ is expressed as $\sigma = \tau_1\sigma_1$ where $\tau_1\in\Sigma\{1,D_1\}$ and $\sigma_1\in\Sigma_1$. Then $q^*(\Sigma^*) = r(\tau(1<))\Sigma r(1;\sigma_1)+r(\tau(1>))\Sigma r(1;\sigma_1) = r(\tau(1<))+r(\tau(1>)) = 1$ by Property 4.1(ii) and definition. Suppose $|K_0| > 1$ and assume the property holds for any game $G_{(0;V_j)}$ where $j\in K_0$. Then

$$q^*(\sigma) = r(0;\sigma_0) \prod_{j\in K_0} \prod_{i\in V_j\setminus L} (r(i;\sigma_i)r(\tau_i)) = r(0;\sigma_0) \prod_{j\in K_0} q^*(\sigma^j),$$

where $\sigma^j\in\Sigma^*(V_j)$ and $q^*(\sigma^j)$ means the distribution in the game $G_{(0;V_j)}$. Hence

$$q^*(\Sigma^*) = \sum_{\sigma\in\Sigma^*} r(0;\sigma_0) \prod_{j\in K_0} q^*(\sigma^j) = \sum_{\rho\in\Sigma_0} r(0;\rho) \sum_{\sigma\in\Sigma^*:\sigma_0=\rho} \prod_{j\in K_0} q^*(\sigma^j)$$

$$= \sum_{\rho\in\Sigma_0} r(0;\rho) \prod_{j\in K_0} \sum_{\rho^j\in\Sigma^*(V_j)} q^*(\rho^j) = \sum_{\rho\in\Sigma_0} r(0;\rho) = 1,$$

by the assumption. If $|K_0| = 1$, let $K_0 = \{1\}$. Assume the property holds for the game $G_{(1;D_1)}$. Then

$$q^*(\Sigma^*) = \sum_{\sigma\in\Sigma^*} r(\tau_1) \prod_{i\in V_1\setminus L} r(i;\sigma_i) \prod_{i\in D_1\setminus L} r(\tau_i) = r(\tau(1<)) \sum_{\sigma\in\Sigma^*:\tau_1=\tau(1<)} \prod_{i\in V_1\setminus L} r(i;\sigma_i) \prod_{i\in D_1\setminus L} r(\tau_i)$$

$$+ r(\tau(1>)) \sum_{\sigma\in\Sigma^*:\tau_1=\tau(1>)} \prod_{i\in V_1\setminus L} r(i;\sigma_i) \prod_{i\in D_1\setminus L} r(\tau_i)$$

$$= r(\tau(1<)) \sum_{\sigma\in\Sigma^*(D_1)} \prod_{i\in V_1\setminus L} r(i;\sigma_i) \prod_{i\in D_1\setminus L} r(\tau_i) + r(\tau(1>)) \sum_{\sigma\in\Sigma^*(D_1)} \prod_{i\in V_1\setminus L} r(i;\sigma_i) \prod_{i\in D_1\setminus L} r(\tau_i)$$

$$= r(\tau(1<)) + r(\tau(1>)) = 1,$$

by the assumption and definitions. Since the graph G is finite we can start at every i such that $K_i \subseteq L$ and apply the above argument inductively. After a finite number of steps we arrive at 0. ♦

Remark 4.3. In this section we have constructed a strategy for SR, and it is based on a fixed order of vertices, i.e., 1,2,...,n. This is seen in the definition of $\Sigma_i \equiv \Sigma(y_1,\ldots,y_m)$ where $i\in V$ and $y_1 <\ldots< y_m$. Suppose $\tau\in\Sigma(K_i)$. Clearly we can define $\Sigma(\tau(y_1),\ldots,\tau(y_m))$ which is different from Σ_i unless τ is the identity permutation. So formally we can consider $\prod_{i\in V\setminus L} |K_i|!$ strategies for SR.

5. Solution of the Game

In this section we state that strategies defined in Sections 3 and 4 are optimal and give some numerical examples which illustrates optimal strategies. The proof of the theorem is in Section 6.

Theorem 5.1. The value of the game $G \equiv (f;P,Q)$ is v_{K_0}. p* is the unique optimal strategy for HD. q* is an optimal strategy for SR.

Example 5.2[1]**.** Let V = {0,1,2,...,9} and E = {(0,1),(1,2),(1,3),(3,4),(3,5),(0,6), (6,7) ,(6,8),(6,9)}.

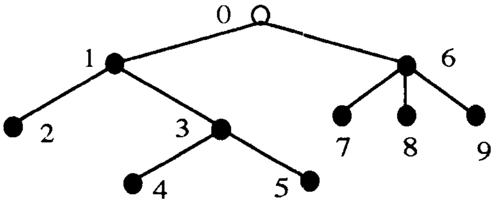

Figure 2.

By (3.1), (3.2) and Property 3.1 we have

$$\frac{p^*(1)}{c(1)} = \frac{p^*(2)}{w(2)} = \frac{p^*(3,4,5)}{w(3,4,5)},\ \frac{p^*(6)}{c(6)} = \frac{p^*(7)}{w(7)} = \frac{p^*(8)}{w(8)} = \frac{p^*(9)}{w(9)},$$

$$\frac{p^*(1,2,3,4,5)}{w(1,2,3,4,5)} = \frac{p^*(6,7,8,9)}{w(6,7,8,9)},\ \text{and}\ \frac{p^*(3)}{c(3)} = \frac{p^*(4)}{w(4)} = \frac{p^*(5)}{w(5)}.$$

From these and (2.3) we can calculate the optimal strategy for HD. An optimal strategy for SR is as follows:

$r(3;[4,5]) = \dfrac{z(4)+w(5)-z(5)}{w(45)}$, $r(\tau(3<)) = \dfrac{c(3)+w(45)-v(45)}{c(3)+w(45)}$, $r(1;[2,3]) = \dfrac{z(2)+w(345)-v(345)}{w(2345)}$, $r(\tau(1<)) = \dfrac{c(1)+w(2345)-v(2345)}{c(1)+w(2345)}$, $r(\tau(6<)) = \dfrac{c(6)+w(789)-v(789)}{c(6)+w(789)}$, $r(0;[1,6]) = \dfrac{v(12345)+w(6789)-v(6789)}{w(D)}$, and r(6;[7,8,9]), r(6;[8,7,9]), r(6;[9,7,8]), r(6;[9,8,7]) are given in the same way. If $c(i) \to 0$ for all i, then $p^*(i) \to 0$ for i = 1, 3, and 6. If $c(i) \to +\infty$ for all i then

[1] p*(2,3,4) and w(2,3,4) stand for p*({2,3,4}) and w({2,3,4}) respectively.

$p^*(i) \to 1/9$ for all i.

Example 5.3. Suppose G is m-ary and complete and its height is n (See pp.273-5 of [14]). Then G has $1 + m + m^2 + \ldots + m^n$ vertices and $m + m^2 + \ldots + m^n$ edges. Assume $c(i) = c > 0$ for all i and $d(i,j) = 1$ for all $(i,j)\in E$. By the symmetry of the tree we can assume HD assigns the same probabilities $p(i)$ to the vertices in the *i*th level. By (2.3), $mp(1) + m^2p(2) + \ldots + m^np(n) = 1$. From this and (3.1), (3.2), we have, after elementary caluculations,

$$p(n) = \{\prod_{h=1}^{n}(m + \frac{b}{1+m+m^2+\ldots+m^{h-1}})\}^{-1}, \text{ and } \frac{p(i)}{p(n)} = \prod_{h=1}^{n-i}(1 - \frac{1-b}{1+m+m^2+\ldots+m^{h-1}}),$$

where $i = 1,\ldots,n-1$ and $b \equiv c/(2+c)$. When $m = 1$, this coincides with (3.2) at p.367 of [7].

6. Proof of the Theorem.

In this section we prove Theorem 5.1.

Lemma 6.1. $f(i,q^*) = v_{K_0}$ for all $i\in D$.

Proof: Let $(i_0=0,i_1,\ldots,i_m,i_{m+1}=i)$ be the simple path between 0 and i. By considering the definition of q^* and the search procedures, (I) and (II), we have

$$\begin{aligned} f(i,q^*) &= d(0,i)+c(i)+w(D_i)r(\tau(i>))\sum_{\sigma\in\Sigma^*:\tau_i=\tau(i>)}\frac{q^*(\sigma)}{r(\tau(i>))} \\ &+ \sum_{h=1}^{m} c(i_h)r(\tau(i_h<))\sum_{\sigma\in\Sigma^*:\tau_{i_h}=\tau(i_h<)}\frac{q^*(\sigma)}{r(\tau(i_h<))} \\ &+ \sum_{h=0}^{m}\sum_{j\in K_{i_h}:j\neq i_{h+1}} w(V_j)\sum_{\mu\in\Sigma_{i_h}:\mu^{-1}(j)<\mu^{-1}(i_{h+1})} r(i_h;\mu)\sum_{\sigma\in\Sigma^*:\sigma_{i_h}=\mu}\frac{q^*(\sigma)}{r(i_h;\mu)} \\ &= d(0,i_m)+d(i)+c(i)+w(D_i)r(\tau(i>))+\sum_{h=1}^{m} c(i_h)r(\tau(i_h<))+\sum_{h=0}^{m}(v_{K_{i_h}}-v_{i_{h+1}}) \\ &= d(i)+c(i)+v_{K_0}-v_i+w(D_i)r(\tau(i>)) = v_{K_0} \end{aligned}$$

by (2.4) and Property 4.1(iv). ♦

Lemma 6.2. For any $\sigma\in\Sigma_{DE}$, $f(p^*,\sigma) = v_{K_0}$.

Proof: Assume $K_0 \subseteq L$. Then by Property 3.1(ii), $p^*(i) = w(i)/w(D)$ for all $i\in D$. So $f(p^*,\sigma) = \Sigma_{i\in D}\, p^*(i)f(i,\sigma) = (1/w(D))\, \Sigma_{i\in D}\, w(\sigma(i))\{w(\{\sigma(1),\ldots,\sigma(i-1)\})+d(\sigma(i))+c(\sigma(i))\} = v_{K_0}$ by (2.5). Next assume $|K_0| = 1$ and let $K_0 = \{1\}$. Assume the lemma holds for the game $G_{(1;D_1)}$. $p^*(1) + p^*(D_1) = 1$ and $p^*(1)/c(1) = p^*(D_1)/w(D_1)$, by (2.2) and Property 3.1(i). From these we have

$$p^*(1) = \frac{c(1)}{w(D_1)+c(1)}. \tag{6.1}$$

Let $K_1 = \{2,...,u\}$. Without loss of generality, suppose σ checks in the order of $G_{(1;V_2)}, \ldots, G_{(1;V_k)}, 1, G_{(1;V_{k+1})}, \ldots, G_{(1;V_u)}$. Since $\sigma \in \Sigma_{DE}$, $\sigma^{-1}(1) = |V_2|+\ldots+|V_k|+1$. Further,

$$f(p^*,\sigma) = p^*(1)\{d(1)+c(1)+\sum_{h=2}^{k} w(V_h)\} + \sum_{\sigma^{-1}(i)<\sigma^{-1}(1)} p^*(i)\{f(i,\sigma)-d(1)+d(1)\}$$

$$+ \sum_{\sigma^{-1}(i)>\sigma^{-1}(1)} p^*(i)\{f(i,\sigma)-d(1)-c(1)+d(1)+c(1)\}$$

$$= d(1)+c(1)p^*(1)+c(1)p^*(\{i:\sigma^{-1}(i)>\sigma^{-1}(1)\})+p^*(1)\sum_{h=2}^{k} w(V_h)+(1-p^*(1))v_{K_1},$$

since

$$\sum_{\sigma^{-1}(i)<\sigma^{-1}(1)} \frac{p^*(i)}{1-p^*(1)}\{f(i,\sigma)-d(1)\} + \sum_{\sigma^{-1}(i)>\sigma^{-1}(1)} \frac{p^*(i)}{1-p^*(1)}\{f(i,\sigma)-d(1)-c(1)\} = v_{K_1},$$

by the assumption. Hence, by (3.1), Property 3.1(i), (6.1), and (2.4),

$$f(p^*,\sigma) = d(1)+c(1)p^*(1)+p^*(1)\sum_{h=k+1}^{u} w(V_h)+p^*(1)\sum_{h=2}^{k} w(V_h)+(1-p^*(1))v_{K_1}$$

$$= d(1)+c(1)p^*(1)+w(D_1)p^*(1)+(1-p^*(1))v_{K_1}$$

$$= d(1)+c(1)p^*(1)+c(1)p^*(D_1)+(1-p^*(1))v_{K_1} = v_1 = v_{K_0}.$$

In particular if $K_1 \subseteq L$ the lemma holds. Assume $|K_0| > 1$ and let $K_0 = \{1,...,m\}$. Assume $\sigma \in \Sigma_{DE}$ examines in the order of $G_{(0;V_1)}, \ldots, G_{(0;V_m)}$. Then by the assumption and Property 3.1(ii), for $x = 1,...,m$,

$$\sum_{z \in V_x} p^*(z)f(z,\sigma) = \frac{w(V_x)}{w(D)}\{\sum_{h=1}^{x-1} w(V_h)+ v_x\}.$$

Adding these together for $x = 1,...,m$, we have, by (2.5) and Property 2.1,

$$f(p^*,\sigma) = \frac{1}{w(D)}\sum_{x=1}^{m} w(V_x)\{\sum_{h=1}^{x-1} w(V_h)+v_x\} = v_{K_0}.$$

Since the graph G is finite we can start at every i such that $K_i \subseteq L$ and apply the above argument inductively. After a finite number of steps we arrive at 0. ♦

From Lemmas 6.1 and 6.2 it suffices to prove that $f(p^*,\sigma) \geq v_{K_0}$ for all $\sigma \in \Sigma$, in order to show that p* is an optimal strategy for HD. Lemmas 6.3,6.4,6.5 and Corollary 6.6 are devoted to doing it. Lemma 6.7 proves that p* is an unique optimal strategy.

Lemma 6.3. Let $i \in D_j$ ($j \in K_0$). Under $\sigma \in \Sigma$, suppose SR passes through i, examines all vertices in Y ($\subseteq D_j$) and then comes back to i again and examine it. Assume $Y \cap A_i = \emptyset$. Change the order of examination of i so that he examines it just before he examines Y. Let the resulting strategy be τ. Then $f(p^*,\sigma) \geq f(p^*,\tau)$.

Proof: Suppose X is the set of vertices which he passes through in this trip. First assume the path defined by τ is the same as the path defined by σ. Then $f(p^*,\sigma)-f(p^*,\tau) \geq p^*(i)\{w(Y)+2d(X)\}-c(i)p^*(Y) \geq 0$ by Lemma 3.2(i). Next assume the path defined by τ is not the same. Then the traveling cost decreases by $d(y,i)+d(i,k)-d(y,k)$ by the change from σ to τ, where $y \in Y$, $\sigma^{-1}(y) = \sigma^{-1}(i)-1$, and $\sigma^{-1}(k) = \sigma^{-1}(i)+1$. Thus, $f(p^*,\sigma)-f(p^*,\tau) \geq p^*(i)\{w(Y)+2d(X)\}-c(i)p^*(Y)+(d(y,i)+d(i,k)-d(y,k))p^*(Z) \geq 0$, where $Z = \{j \in V:\sigma^{-1}(j) > \sigma^{-1}(i)\}$. ♦

Let $B(\sigma) \equiv \{(i,Y)$:as in Lemma 6.3$\}$. It is easy to see $|B(\sigma)|$ is finite and $|B(\tau)| \leq |B(\sigma)|-1$ (See Lemma A.2 in Appendix). Hence from Lemma 6.3, we see that for any $\sigma \in \Sigma$ there is some $\sigma\#$ such that $f(p^*,\sigma) \geq f(p^*,\sigma\#)$ and $|B(\sigma\#)|=0$, by changing the order of examination step by step and getting a sequence of permutations from σ to $\sigma\#$. Then $\sigma\# \in \Sigma_{PDE}$.

Example 5.2 (Continued). Let $\sigma^0 = [2,4,1,3,7,8,5,6,9] \in \Sigma \backslash \Sigma_{PDE}$. First apply Lemma 6.3 by letting $Y = \{4\}$, $X = \{3\}$, and $i = 1$. Then apply Lemma 6.3 twice. We have $\sigma^1 = [1,2,3,4,7,8,5,6,9] \in \Sigma_{PDE}$. Noting $p^*(1)/c(1) = p^*(2)/w(2)$ and $p^*(3)/c(3) = p^*(4)/w(4)$ we have $f(p^*,\sigma^0)-f(p^*,\sigma^1) = -c(1)p^*(4)+(2d(3)+w(4))p^*(1)+2d(3)p^*(3,7,8,5,6,9) > 0$ by Lemma 3.2(i).

Suppose $Y = \{\sigma(k),\sigma(k+1),\ldots,\sigma(k+k')\} \subseteq V$ for $\sigma \in \Sigma$. Define by $s(Y;\sigma)$ the cost of checking vertices in Y under σ, starting at $\sigma(k)$ and terminating at $\sigma(k+k')$, and including the examination costs of $\sigma(k)$ and $\sigma(k+k')$. Let $i = \sigma(k)$ and $j = \sigma(k+k')$. Define $s(i,j;\sigma) \equiv s(Y;\sigma) = c(Y) + \sum_{h=k}^{k+k'-1} d(\sigma(h),\sigma(h+1))$. For $Y, Z \subseteq V$, let $Y = \{\sigma(k),\ldots,\sigma(k+k')\}$ and $Z = \{\sigma(l),\ldots,\sigma(l+l')\}$. Suppose $Y \cap Z = \emptyset$. Then define $d_\sigma(Y,Z) \equiv d(\sigma(k+k'),\sigma(l))$. For $i \in V \backslash Y$, let $d_\sigma(i,Y) \equiv d(i,\sigma(k))$ and $d_\sigma(Y,i) \equiv d(\sigma(k+k'),i)$. For convenience, for any $Z \subseteq V$ and $x,y \in V \backslash Z$, we let $d_\sigma(x,Z) + s(Z;\sigma) + d_\sigma(Z,y) = d(x,y)$ whenever $Z = \emptyset$.

Lemma 6.4. Suppose $\sigma \in \Sigma_{PDE}$. Suppose $i,j \in V_1$, $1 \in K_0$, $i \in D_\sigma(1,u)$ ($u \geq 2$), and $j \in D_\sigma(1,1)$. Suppose $j \in K_i$. Change the order of examination of i, and transfer σ to τ so that $\tau^{-1}(i) = \tau^{-1}(j)-1$. Then $f(p^*,\sigma) \geq f(p^*,\tau)$.

Proof: Suppose $D_\sigma(1,1) = S \cup \{j\} \cup T$ where S and T are subsets of V which are examined before and after j under σ respectively. Suppose W is the set of all vertices which are examined before vertices in $D_\sigma(1,1)$ are done, and suppose X is the set of all vertices which are examined between vertices in $D_\sigma(1,1)$ and $D_\sigma(1,u)$. Suppose $Y \subseteq D_\sigma(1,u)$ is the set of all vertices which are examined before i. Thus SR examines in the order of $WS\{j\}TXY\{i\}$ ... We can suppose $D_j \cap S = \emptyset$ and $Y \cap D_i = \emptyset$ since $\sigma \in \Sigma_{PDE}$. Then under σ, SR will examine j after examined S and passed through i. After examined j and before examines i he will pass through 0 at least twice. Hence

$$(6.2) \qquad f(i,\sigma) - f(i,\tau) \geq w(0;X) + w(0;\{j\} \cup T \cup Y).$$

Next let $k \in \{j\} \cup T \cup X \cup Y$. Then under σ and τ SR will travel along the same path before he examines k. Hence

(6.3) $\quad f(k,\sigma) - f(k,\tau) = -c(i).$

Suppose $\sigma^{-1}(k) > \sigma^{-1}(i)$. Then in the same way,

(6.4)
$$f(k,\sigma)-f(k,\tau) = \begin{array}{l} d_\sigma(Y,i)+d(i,k_1)-d_\tau(Y,k_1) \geq 0 \text{ (if } Y \neq \emptyset), \\ d_\sigma(X,i)+d(i,k_1)-d_\tau(X,k_1) \geq 0 \text{ (if } Y = \emptyset). \end{array}$$

Here $\sigma^{-1}(k_1) = \sigma^{-1}(i)+1$. By (6.2), (6.3), and (6.4),

$$f(p^*,\sigma) - f(p^*,\tau) \geq p^*(i)w(0;\{j\} \cup T \cup Y) - c(i)p^*(\{j\} \cup T \cup Y)+p^*(i)w(0;X) - c(i)p^*(X).$$

If $1 \notin \{j\} \cup T \cup Y$,

(6.5)
$$\frac{p^*(i)}{c(i)} \geq \frac{p^*(1)}{c(1)} \geq \frac{p^*(\{j\} \cup T \cup Y)}{w(1;\{j\} \cup T \cup Y)} \geq \frac{p^*(\{j\} \cup T \cup Y)}{w(0;\{j\} \cup T \cup Y)},$$

by Property 3.1(iii) and Lemma 3.2(i). If $1 \in \{j\} \cup T \cup Y$, then as the second line of the proof of Lemma 3.2(ii), we have the same. On the other hand, for $y \in K_0$ let $X_{y,h}$ is the set of vertices that SR examines in his yth visit to V_y in the search of X. Thus $X \cap V_y = \cup_h X_{y,h}$. Then $d_\sigma(0,X)+s(X;\sigma)+d_\sigma(X,0) = \sum_{y \in K_0} \sum_h \{d_\sigma(0,X_{y,h})+s(X_{y,h};\sigma) +d_\sigma(X_{y,h},0)\}$ and $d_\sigma(0,X_{y,h})+s(X_{y,h};\sigma)+d_\sigma(X_{y,h},0) \geq w(0;X_{y,h})$ for all h and $y \in K_0$, and, by Lemma 3.2(ii)

$$\frac{p^*(i)}{c(i)} \geq \frac{p^*(X_{y,h})}{w(0;X_{y,h})} \geq \frac{p^*(X_{y,h})}{d_\sigma(0,X_{y,h})+s(X_{y,h};\sigma)+d_\sigma(X_{y,h},0)}$$

for $y \in K_0$. This and (6.5) imply $f(p^*,\sigma) - f(p^*,\tau) \geq 0$. ♦

In Lemma 6.4 if $\tau \notin \Sigma_{PDE}$, then after applying the operation of Lemma 6.3 to τ some times, we get μ such that $|B(\mu)|=0$. This operation does not generate a new couple (i,j) as in Lemma 6.4, since $Y \cup \{i\} \subseteq D_j$, $j \in K_0$ in Lemma 6.3. So from Lemmas 6.3 and 6.4 we see that for any $\sigma \in \Sigma_{PDE}$ there is some $\sigma\# \in \Sigma_{SDE} \cap \Sigma_{PDE}$ such that $f(p^*,\sigma) \geq f(p^*,\sigma\#)$, by changing the order of examination step by step, applying Lemmas 6.3 and 6.4, alternately if necessary, and getting a sequence of permutations starting at σ and arriving at $\sigma\#$.

Example 5.2 (Continued). $\sigma^1 = [1,2,3,4,7,8,5,6,9] \in \Sigma_{PDE} \backslash \Sigma_{SDE}$. Let $j = 7$, $i = 6$, $D_{\sigma^1}(6,1) = \{7,8\}$, $D_{\sigma^1}(6,2) = \{6,9\}$, $S = \emptyset$, $T = \{8\}$, $Y = \emptyset$, $X = \{5\}$, and $W = \{1,2,3,4\}$. Applying Lemma 6.4, we have $\sigma^2 = [1,2,3,4,6,7,8,5,9] \in \Sigma_{SDE} \cap \Sigma_{PDE}$. $f(p^*,\sigma^1) - f(p^*,\sigma^2) = (2d(0,5)+2d(7) +2d(0,8)+c(7,8,5))p^*(6)-c(6)p^*(7,8,5) = w(7,8)p^*(6)-c(6)p^*(7,8)+(2d(0,3)+w(5))p^*(6)-c(6)p^*(5) > 0$ by Lemma 3.2.

Lemma 6.5. Suppose $\sigma\in\Sigma_{SDE}\cap\Sigma_{PDE}$. Let $i\in D$ and $U = \cup_{j\in K}V_j$, where $K \subseteq K_i$. Suppose under σ SR examines vertices in the order of ... i Z U W. Here $Z,W \subseteq D$. Assume the restriction of σ to U belongs to $\Sigma_{DE}(U)$. Define $\tau\in\Sigma$ so that under τ SR examines in the order of ... i U Z W. Then $\tau\in\Sigma_{SDE}\cap\Sigma_{PDE}$ and $f(p^*,\sigma) \geq f(p^*,\tau)$.

Proof: By the definition of U, $U \subseteq D_i$. Further $A_i\cap Z = \emptyset$. Hence $\tau\in\Sigma_{SDE}\cap\Sigma_{PDE}$. Let $k\in Z$. Then $f(k,\sigma) = ...+d_\sigma(i,Z)+s(z,k;\sigma)$ and $f(k,\tau) = ...+w(U)+d_\sigma(i,Z)+s(z,k;\sigma)$, where z is the first vertex in Z which is examined under σ. Hence

$$(6.6)\qquad f(k,\sigma) - f(k,\tau) = - w(U).$$

Let $k\in U$. Then $f(k,\sigma) = ... +d_\sigma(i,Z)+s(Z;\sigma)+d_\sigma(Z,i)+d(i,u) +s(u,k;\sigma)$, and $f(k,\tau) = ...+d(i,u)+s(u,k;\sigma)$, where u is the first vertex in U which is examined under σ. Hence

$$(6.7)\qquad f(k,\sigma) - f(k,\tau) = d_\sigma(i,Z)+s(Z;\sigma)+d_\sigma(Z,i).$$

Let $k\in W$. Then $f(k,\sigma) = ... +d_\sigma(i,Z)+s(Z;\sigma)+d_\sigma(Z,i)+w(U)+d_\sigma(i,W) +s(w,k;\sigma)$, and $f(k,\tau) = ... +w(U)+d_\sigma(i,Z)+s(Z;\sigma)+d_\sigma(Z,w)+s(w,k;\sigma)$, where w is the first vertex in W which is examined under σ. Hence

$$(6.8)\qquad f(k,\sigma) - f(k,\tau) = d_\sigma(Z,i)+d_\sigma(i,W)-d_\sigma(Z,w) \geq 0.$$

From (6.6), (6.7), and (6.8), we have

$$(6.9)\qquad f(p^*,\sigma)-f(p^*,\tau) \geq -w(U)p^*(Z)+p^*(U)\{d_\sigma(i,Z)+s(Z;\sigma)+d_\sigma(Z,i)\}.$$

There is $t\in V\backslash Z$ such that $Z \subseteq D_t$ and SR passes through t when he examines Z under σ. Suppose $t \neq 0$. Since $\sigma\in\Sigma_{PDE}\cap\Sigma_{SDE}$, applying Lemma 3.2(i) we have

$$d_\sigma(i,Z)+s(Z;\sigma)+d_\sigma(Z,i) = 2d(i,t)+d_\sigma(t,Z)+s(Z;\sigma)+d_\sigma(Z,t)$$

$$\geq d_\sigma(t,Z)+s(Z;\sigma)+d_\sigma(Z,t) \geq w(t;Z) \geq \frac{p^*(Z)c(t)}{p^*(t)}.$$

From this and (6.9) we have

$$f(p^*,\sigma) - f(p^*,\tau) \geq - w(U)p^*(Z) + p^*(U)\frac{p^*(Z)c(t)}{p^*(t)} = C\{\frac{p^*(U)}{w(U)} - \frac{p^*(t)}{c(t)}\} = C\{\frac{p^*(i)}{c(i)} - \frac{p^*(t)}{c(t)}\} \geq 0,$$

since $p^*(U)/w(U) = p^*(i)/c(i)$ by (3.1), where $C \equiv w(U)p^*(Z)c(t)/p^*(t)$. Suppose $t = 0$. Suppose $i\in V_x$ for $x\in K_0$. Let $Z \equiv Z_1\cup\ldots\cup Z_k$ where SR examines in the order of $iZ_10Z_20\ldots0Z_k$. Here 0 means SR goes back to 0. Then $d_\sigma(i,Z)+s(Z;\sigma)+d_\sigma(Z,i) \geq d_\sigma(i,Z_1)+s(Z_1;\sigma)+d_\sigma(Z_1,i)+$

$\sum_{h=2}^{k} w(0;Z_h)$, and $p^*(Z) = \sum_{h=1}^{k} p^*(Z_h)$. Hence $f(p^*,\sigma)-f(p^*,\tau) \geq \sum_{h=2}^{k} \{-w(U)p^*(Z_h)+p^*(U)w(0;Z_h)\}-w(U)p^*(Z_1)+p^*(U)\{d_\sigma(i,Z_1)+s(Z_1;\sigma)+d_\sigma(Z_1,i)\}$. Apply Property 3.1(iii), Lemma 3.2 and the same argument as in $t \neq 0$. ♦

Example 5.2 (Continued). Let $i = 3$, $U = \{5\}$, $Z = \{4,6,7,8\}$, and $W = \{9\}$. Applying Lemma 6.5, we have $\sigma^3 = [1,2,3,5,4,6,7,8,9] \in \Sigma_{DE}$. $f(p^*,\sigma^2)-f(p^*,\sigma^3) = -w(5)p^*(4,6,7,8)+(2d(0,3)+w(4,6,7,8))p^*(5)+(2d(0,3)+2d(6))p^*(9) > 0$.

Corollary 6.6. For $\sigma \in \Sigma_{PDE} \cap \Sigma_{SDE}$, there exists $\sigma^\# \in \Sigma_{DE}$ such that $f(p^*,\sigma) \geq f(p^*,\sigma\#)$.
Proof: Let $\sigma \in \Sigma_{PDE} \cap \Sigma_{SDE}$. Let $i \in D$ and $D_i \subseteq L$. Applying Lemma 6.5 to every $j \in D_i$, get $\tau \in \Sigma_{PDE} \cap \Sigma_{SDE}$ in which SR examines in the order of $\cdots iD_i \cdots$. Apply this operation to every $i \in D$ such that $D_i \subseteq L$. Then consider $i \in D$ such that the restriction of the search to D_i belongs to $\Sigma_{DE}(D_i)$. Clearly this operation can be continued inductively and at each step the expected cost does not increase. Since the graph is finite this operation ends after a finite number of steps, when the resulting strategy, say $\sigma\#$, of SR belongs to Σ_{DE}. ♦

Lemma 6.7. If $p \in P$ is optimal, then $p = p^*$.
Proof: Since p is optimal, for any $\sigma \in \Sigma^*$, $f(p,\sigma) \geq v_{K_0}$. But from Lemma 6.1, $f(p,q) = \sum_{\sigma \in \Sigma^*} f(p,\sigma) = v_{K_0}$. This implies

(6.10) $$f(p,\sigma) = v_{K_0} \text{ for all } \sigma \in \Sigma^*.$$

Suppose $K_0 = \{1,\ldots,m\}$ and $\sigma \in \Sigma^*$ indicates that SR examines $G_{(a(m);V_m)}$ last. Suppose $\sigma' \in \Sigma^*$ is a strategy such that it indicates that SR examines $G_{(a(m);V_m)}$ first and the other parts of σ' are the same as in σ. Since $f(p,\sigma) = f(p,\sigma')$ by (6.10), we have

(6.11) $$\sum_{x=1}^{m} \sum_{z \in V_x} p(z)f(z,\sigma) = \sum_{x=1}^{m} \sum_{z \in V_x} p(z)f(z,\sigma').$$

Since $f(z,\sigma') = f(z,\sigma)+w(V_m)$ for all $z \in V_{\{1,\ldots,m-1\}}$ and $f(z,\sigma') = f(z,\sigma)-w(V_{\{1,\ldots,m-1\}})$ for all $z \in V_m$, we have, from (6.11), $p(V_m) = w(V_m)/w(D)$. But, noting Remark 4.3 and considering other strategies for SR, consequently we must have

(6.12) $$p(V_x) = w(V_x)/w(D) \text{ for all } x \in K_0.$$

From this, $p(V_x)w(V_y) = p(V_y)w(V_x)$ for all $x,y \in K_0$. Next let $K_i = \{j_1,\ldots,j_u\}$, $i \in D$. Applying the same argument we have

(6.13) $$p(V_y)w(V_x) = p(V_x)w(V_y).$$

Suppose $\sigma \in \Sigma^*$ indicates that SR examines all vertices in D_i just after he examines i. Let $\sigma' \in \Sigma^*$

be a strategy such that it indicates that SR examines i just after he examines all vertices in D_i and the other parts are the same as in σ. Since $f(p,\sigma) = f(p,\sigma')$ by (6.16), we have

$$p(i)f(i,\sigma) + \sum_{z \in D_i} p(z)f(z,\sigma) = p(i)f(i,\sigma') + \sum_{z \in D_i} p(z)f(z,\sigma'). \tag{6.14}$$

Since $f(z,\sigma') = f(z,\sigma)-c(i)$ for all $z \in D_i$ and $f(i,\sigma') = f(i,\sigma)+w(D_i)$, we have, from (6.14),

$$w(D_i)p(i) = c(i)p(D_i). \tag{6.15}$$

Since $D_i = V_{\{j_1,\ldots,j_u\}}$ and $p(D_i) = p(V_{\{j_1,\ldots,j_u\}})$, (6.19) and (6.21) imply

$$w(V_{j_x})p(i) = c(i)p(V_{j_x}). \tag{6.16}$$

But, (6.12) and (6.16) coincide with (3.1) and (3.2). Consequently we must have $p = p^*$. ♦

7. Final Remarks.

We have solved a serach game on a rooted tree with traveling costs. The hider has a unique optimal strategy which is given recursively. The searcher has many optimal strategies if there are many traveling-salesman pathes. A mixture of any ordering corresponding to traveling-salesman pathes is an optimal strategy. (3.1), (3.2) and Property 3.1(iii) and (v) were important to prove the theorem.

A generalization of the model is a game on a graph with cycles. As we see by checking a game on a circle, the hider's strategy is very different from that in a game on a tree.

(3.1) is a recursive relation on the hider's probability distribution. We could discuss more on relations between the sequential search and the model in this note, by noting the Bellman's principle of optimality ([2] , [3] and [8]).

Assume the uniform distribution, $p^u \in P$, on D as an a priori distribution. So consider a one decision-maker problem: Minimize $f(p^u,\sigma)$ subject to $\sigma \in \Sigma$. Let σ^u be a solution of this problem. It is an exercise to show that $\sigma^u \in \Sigma_{DE}$. If $p^* = p^u$ then $f(p^u,\sigma^u)$ is equal to the value of the game. This is the case only when Vertex 0 has n children, each of which has no child. If p^* is more different from p^u, $f(p^u,\sigma^u)$ will be smaller than the value of the game. This argument is a generalization of that at p. 381 of [7].

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Appendix

Proposition A.1. There exists uniquely $p^*\in P$ that satisfies (2.2), (3.1) and (3.2).

Proof: Define for all $i\in D$,

$$p^*(i) = \frac{c(i)}{w(D)} \prod_{h=1}^{m} \frac{w(V_{i_h})}{w(D_{i_h})+c(i_h)} \quad \text{(A.1)}$$

where $(i_0=0,i_1,\ldots,i_m=i)$ is the simple path between 0 and i. Take $i\in D$ such that $K_i \subseteq L$. For any $j\in K_i$, $D_j = \emptyset$ and let $(0,i_1,...,i,j)$ is the simple path between 0 and j. By (A.1)

$$p^*(j) = \frac{c(j)}{w(D)} \left(\prod_{h=1}^{m} \frac{w(V_{i_h})}{w(D_{i_h})+c(i_h)}\right) \frac{w(j)}{c(j)} = \frac{w(j)}{w(D)} \prod_{h=1}^{m} \frac{w(V_{i_h})}{w(D_{i_h})+c(i_h)} .$$

From this and (A.1),

$$p^*(V_i) = p^*(i)+\sum_{j\in K_i} p^*(j) = \frac{w(V_i)}{w(D)} \prod_{h=1}^{m-1} \frac{w(V_{i_h})}{w(D_{i_h})+c(i_h)} \quad \text{(A.2)}$$

Next suppose for $k\in D$, $p^*(V_i)$ is given by (A.2) for all $i\in K_k$. Let $(i_0=0,i_1,\ldots,i_{m-1} = k, i_m=i)$ be the simple path between 0 and i. By (A.1) and (A.2), noting $w(D_{i_{m-1}})+c(i_{m-1}) = \Sigma_{i\in K_k}w(V_i) + c(k)$, we have $p^*(V_k) = p^*(k)+\Sigma_{i\in K_k} p^*(V_i) = (w(V_k)/w(D)) \prod_{h=1}^{m-2} (w(V_{i_h})/(w(D_{i_h})+c(i_h)))$, which means $p^*(V_k)$ is also given by (A.2), replacing i by k. So starting at every i such that $K_i \subseteq L$ and considering inductively, we have (A.2) for all $i\in D$. In particular if $i\in K_0$, then m = 1 and we have $p^*(V_i)/w(V_i) = 1/w(D)$, which implies (3.2). Let $i\in K_k$ and $(i_0=0,i_1,\ldots,i_{m-1} = k, i_m=i)$ be the simple path between 0 and i. By (A.1) for k and (A.2) for i, we have $p^*(k)/c(k) =$

$p^*(V_i)/w(V) = (1/w(D)) \prod_{h=1}^{m-1} (w(V_{i_h})/(w(D_{i_h})+c(i_h)))$, which is (3.1). By (A.1), $p^*(i) > 0$ for all $i \in D$. Further $p^*(D) = \Sigma_{i \in K_0} p^*(V_i) = \Sigma_{i \in K_0} w(V_i)/w(D) = 1$. Hence we have (2.2).

Conversely suppose $p^* \in P$ satisfies (2.2), (3.1) and (3.2). Let $j \in K_i$ and $i \in D$. By (3.1), $p^*(D_i) = p^*(i)w(D_i)/c(i)$ (See the proof of Property 3.1(i)). Hence $p^*(V_i) = p^*(i)+p^*(D_i) = (c(i)+w(D_i))p^*(i)/c(i)$. From this, $p^*(i)/c(i) = p^*(V_i)/(c(i)+w(D_i))$, from which, combined with (3.1), we have

$$\text{(A.3)} \qquad \frac{p^*(V_j)}{w(V_j)} = \frac{p^*(V_i)}{c(i)+w(D_i)}.$$

Suppose $(i_0=0, i_1, \ldots, i_m=i)$ is the simple path between 0 and i. Applying (A.3) to each edge on this path, we have a representation of (A.2). From (A.2) and (3.1), we have (A.1). ♦

Lemma A.2. $|B(\tau)| \leq |B(\sigma)| - 1$.

Proof: Assume (l,Z) has been yielded newly by the transformation from σ to τ in the notation of Lemma 6.3. This occurs only when the pathes which SR uses have changed. Let $\sigma^{-1}(y) = \sigma^{-1}(i)-1$ and $\sigma^{-1}(k) = \sigma^{-1}(i)+1$. So $\sigma^{-1}(l) > \sigma^{-1}(k)$. Then (l,Z) has been yielded only when both of the simple pathes between i and y and between i and k do not pass through l and the simple path between y and k does pass through l. But in this case, there will be a cycle which passes through i and l, contradicting G is a tree. ♦

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