Journal of the Operations Research
Society of Japan
Vol. 37, No. 4, December 1994

A QUEUEING SYSTEM WITH A SETUP TIME FOR SWITCHING OF THE SERVICE DISTRIBUTION

Koji Yamada
IBM Japan, Ltd.

Shoichi Nishimura
Science University of Tokyo

(Received February 15, 1993; Revised December 20, 1993)

Abstract A controlled $M/G/1$ type queueing system with a setup time for switching of the service distribution is considered. At first, customers are served by a regular service time. When the number of customers in the system exceeds m, the service time is switched to a high speed service time with a setup time. High speed services continue until the end of the busy period. We propose a simple algorithm for the calculation of the mean number of customers in the system by using a normalizing condition and a boundary condition. Moreover, explicit formulas of the probability mass function are derived when the regular service distribution is exponential or constant.

1 Introduction

In this paper, we consider an $M/G/1$ type queueing system with a setup time for switching of the service distribution. At first, customers are served by a *regular service time*. When the number of customers in the system exceeds m, the decision maker switches the service time to a *high speed service time*. Such a switching time is called a *setup time*. High speed services are continued until the end of the busy period.

This model has a close relation with the *N-Policy* model which has been known as a traditional optimal control model of $M/G/1$ queues. *Heyman* [4] provided the optimality of N-Policy under given cost structures. On the other hand, from standpoints of communication engineering, *Nishigaya, Mukumoto and Fukuda* [6] introduced this switching queueing model with applications in packet communication systems, where a fundamental matrix of absorbing transition states is used to obtain the mean number of customers. In our algorithm, a normalizing condition and a boundary condition are derived. Using these two conditions, we obtain the Z-transform of the probability mass function of the number of customers in the system. If the regular service distribution is exponential or constant, explicit expressions for the probability mass function are obtained.

Many queueing models such as polling, setup, machine breakdown and machine maintenance models, etc., have a characteristic that a server may stop service during an occasional interval. Recent researches (*Doshi* [2] and *Takagi* [8], among them) have unified these models as *Vacation Models*, where the time that server stops service is regarded as a vacation time. In our model, the setup time for switching is also considered as a vacation time.

The model is described in Section 2. In Section 3, we first obtain the Z-transform of the number of customers in the system with respect to the embedded Markov chain, and then derive the mean number of customers in the system. In Section 4, defining the supplementary series generated from the LST of the regular service distribution, we obtain the unknown parameters contained in two main results of the previous section. Also, explicit formulas of the probability mass function are derived when the regular service distribution is exponential or constant. Then, as a result, we propose an algorithm for calculation of the mean number of customers in the system. Finally, numerical calculations of the mean number of customers in the system are investigated in Section 5.

2 The Model and Notations

In this section, we shall define a controlled $M/G/1$ type queue with a setup time for switching of the service distribution. Suppose that the arrival process is a Poisson stream with rate λ. Initially, customers are served by a regular service time X_R. If the system is crowded, the decision maker changes the regular service time to the high speed one. That is, when the number of the customers in the system is equal to or greater than m at the completion of the regular service, the decision maker switches the regular service time X_R to the high speed service time X_H. In order to make this change, a vacation time X_V is needed as a setup time. It is supposed that decisions are made only at the completion of the service. When the system becomes empty, the service distribution is switched to the regular service distribution immediately. That is, a customer who arrives during an idle time of the system is served regularly without a vacation.

It is natural that the decision maker requires the high speed service when the system is crowded. *Heyman* [4] introduced the *N-Policy* by which the server is activated when there are N customers waiting for service and is deactivated when there is no customer in the system. As having been shown in inventory theory, the optimality of the N-Policy is proved under certain cost structures (*Heyman and Sobel* [5]), which includes a dormant cost, a running cost, a start-up cost, a shut-down cost and holding cost in $M/G/1$ type queue (*Heyman* [4], *Sobel* [7] and *Bell* [1]). In this paper, we consider the regular service time, the high speed time and the vacation time. Instead of a start-up cost and a shut-down cost, the vacation time is incurred when the service time is changed from the regular one to the high speed one.

Let $R(x)$, $H(x)$ and $V(x)$ be the distribution functions and denote $R^*(s)$, $H^*(s)$ and $V^*(s)$ as the Laplace-Stieltjes transforms (LST) of X_R, X_H and X_V, respectively. We assume that the arrival process, service times and vacation times are independent. Let r_n be the probability mass function that n customers arrive during the regular service time X_R. We define $\tilde{r}(z)$ as a generating function of r_n. They are

$$\begin{aligned} r_n &\triangleq \int_0^\infty e^{-\lambda x}\frac{(\lambda x)^n}{n!}dR(x), \\ \tilde{r}(z) &\triangleq \sum_{n=0}^\infty z^n r_n = R^*(\lambda-\lambda z). \end{aligned}$$

Similarly, $h_n, \tilde{h}(z), v_n, \tilde{v}(z), (v\otimes h)_n$ and $(v \tilde{\otimes} h)(z)$ are defined as

$$\begin{aligned} h_n &\triangleq \int_0^\infty e^{-\lambda x}\frac{(\lambda x)^n}{n!}dH(x), \\ \tilde{h}(z) &\triangleq \sum_{n=0}^\infty z^n h_n = H^*(\lambda-\lambda z), \\ v_n &\triangleq \int_0^\infty e^{-\lambda x}\frac{(\lambda x)^n}{n!}dV(x), \\ \tilde{v}(z) &\triangleq \sum_{n=0}^\infty z^n v_n = V^*(\lambda-\lambda z), \\ (v\otimes h)_n &\triangleq \sum_{k=0}^n v_k h_{n-k}, \\ (v\tilde{\otimes} h)(z) &\triangleq \sum_{n=0}^\infty z^n \sum_{k=0}^n v_k h_{n-k} = \tilde{v}(z)\tilde{h}(z), \end{aligned} \tag{2.1}$$

where the notation "$\otimes$" represents a convolution.

As traffic intensities, we put

$$\begin{aligned}\rho_R &\triangleq \lambda E[X_R],\\ \rho_V &\triangleq \lambda E[X_V],\\ \rho_H &\triangleq \lambda E[X_H].\end{aligned}$$

It is well-known that some important statistical values can be obtained by Z-transforms as

$$\begin{aligned}\rho_R &= \left.\frac{d}{dz}\tilde{r}(z)\right|_{z=1},\\ \lambda^2 E[X_R^2] &= \left.\frac{d^2}{dz^2}\tilde{r}(z)\right|_{z=1}.\\ r_n &= \frac{(-\lambda)^n}{n!}\cdot\left.\frac{d^n}{ds^n}R^*(s)\right|_{s=\lambda} \quad (n\geq 0).\end{aligned} \tag{2.2}$$

For X_H and X_V, the statistical values of $\rho_H, \rho_V, \lambda^2 E[X_H^2], \lambda^2 E[X_V^2], h_n$ and v_n can be also obtained in the same way.

If $m=\infty$, then the process is the same as $M/G/1$. If $m<\infty$, the stability condition is $\rho_H<1$. We make the following assumption.

Assumption 2.1

$$0<m<\infty \qquad \textit{and} \qquad \rho_H<1$$

□

3 The Embedded Markov Chain Approach

In this section, our setup queueing model is analyzed as the *embedded Markov chain.* Our purpose is to get the steady-state probability mass function of the number of customers in the system. We use the notations as follows:

P_0 : the probability that the system is empty at the service completion,

P_n^R : the probability that there are n customers in the system at the regular service completion,

P_n^H : the probability that there are n customers in the system at the high speed service completion,

$$P_n \triangleq \begin{cases} P_0 & (n=0)\\ P_n^R + P_n^H & (n\geq 1)\end{cases}$$

: the probability that there are n customers in the system at the service completion,

and

$$E[L] \triangleq \sum_{n=0}^{\infty} nP_n,$$

: the mean number of customers in the system.

By *Burke's theorem* and *Poisson arrivals see time averages (PASTA) property* (*Wolff* [9]), the steady-state probability is equal to the probability of the system at any time (pp.7-8 in *Takagi* [8]). Z-transforms of these probability mass functions are defined as

$$\tilde{P^R}(z) \triangleq \sum_{n=1}^{\infty} z^n P_n^R,$$
$$\tilde{P^H}(z) \triangleq \sum_{n=1}^{\infty} z^n P_n^H,$$
$$\phi(z) \triangleq \sum_{n=1}^{m-1} z^n P_n^R \qquad (\text{ in the case of } m \geq 3 \text{ }),$$
$$\tilde{P}(z) \triangleq P_0 + \tilde{P^R}(z) + \tilde{P^H}(z).$$

It should be noted that $\tilde{P}(1) = 1$ but $\tilde{P^R}(1) < 1$, $\tilde{P^H}(1) < 1$ and $\phi(1) < 1$.

We provide the next lemma which is used for calculating $\tilde{P}(z)$ and $E[L]$ in later analyses.

Lemma 3.1 *Let $\tilde{h}(z)$ be a Z-transform as defined in (2.1). If $\tilde{x}(z)$ is a Z-transform such that* $\lim_{z\uparrow 1} \tilde{x}(z) = 0$, *then*

$$\lim_{z\uparrow 1} \frac{\tilde{x}(z)}{z - \tilde{h}(z)} = \frac{\tilde{x}'(1)}{1 - \rho_H},$$
$$\lim_{z\uparrow 1} \left(\frac{\tilde{x}(z)}{z - \tilde{h}(z)}\right)' = \lim_{z\uparrow 1} \frac{\tilde{x}'(z)\left(z - \tilde{h}(z)\right) - \tilde{x}(z)\left(1 - \tilde{h}'(z)\right)}{\left(z - \tilde{h}(z)\right)^2}$$
$$= \frac{\tilde{x}''(1)(1 - \rho_H) + \tilde{x}'(1)\lambda^2 E[X_H^2]}{2(1 - \rho_H)^2},$$

where the notation $'$ represents the differential operation.

Proof : It is obvious from the direct calculation of L'Hôspital's rule. □

3.1 The cases of $m = 1$ and $m = 2$

We can get explicitly the Z-transform of stationary probability mass function of the number of customers in the system in the cases of $m = 1$ and $m = 2$.

In the case of $m = 1$, equilibrium equations of the embedded Markov chain are derived as follows:

$$P_0 = r_0 P_0 + v_0 h_0 P_1^R + h_0 P_1^H,$$
$$P_n^R = r_n P_0 \qquad (n \geq 1),$$
$$P_n^H = \sum_{k=1}^{n+1} h_{n+1-k} P_k^H + \sum_{k=1}^{n+1} (v \otimes h)_{n+1-k} P_k^R \qquad (n \geq 1).$$

From these equations, the next proposition is obtained.

Proposition 3.1 *In the case of $m = 1$, the Z-transform $\tilde{P}(z)$ and the mean number of customers in the system $E[L]$ are*

$$\tilde{P}(z) = P_0 \cdot \left[\frac{\tilde{v}(z)\tilde{h}(z)\tilde{r}(z) - z - r_0(\tilde{v}(z)\tilde{h}(z) - z)}{z - \tilde{h}(z)} + \tilde{r}(z) + 1 - r_0\right],$$

$$E[L] = P_0 \cdot \frac{\left[\begin{array}{c} (1 - r_0)((1 - \rho_H)(\lambda^2 E[X_V^2] + 2\rho_V \rho_H) + \rho_V \lambda^2 E[X_H^2]) \\ +(1 - \rho_H)(\lambda^2 E[X_R^2] + 2\rho_R(1 + \rho_V)) \\ +\rho_R \lambda^2 E[X_H^2] \end{array}\right]}{2(1 - \rho_H)^2}, \qquad (3.1)$$

where

$$P_0 = \frac{1-\rho_H}{1-\rho_H+(1-r_0)\rho_V+\rho_R}.$$

□

Similarly, in the case of $m = 2$, the equilibrium equations of the embedded Markov chain can be given by

$$\begin{aligned} P_0 &= r_0P_0 + r_0P_1^R + h_0P_1^H, \\ P_n^R &= r_nP_0 + r_nP_1^R \qquad (n \geq 1), \\ P_n^H &= \sum_{k=1}^{n+1} h_{n+1-k}P_k^H + \sum_{k=2}^{n+1}(v \otimes h)_{n+1-k}P_k^R \qquad (n \geq 1). \end{aligned}$$

Then we have the following proposition.

Proposition 3.2 *In the case of $m = 2$, the Z-transform $\tilde{P}(z)$ and the mean number of customers in the system are*

$$\begin{aligned} \tilde{P}(z) &= \frac{P_0}{1-r_1}\cdot\left[\frac{\tilde{r}(z)\tilde{v}(z)\tilde{h}(z) - z + r_1z(1-\tilde{v}(z)\tilde{h}(z)) + r_0(z-\tilde{v}(z)\tilde{h}(z))}{z-\tilde{h}(z)}\right. \\ &\qquad \left. + \tilde{r}(z) + 1 - r_0 - r_1\right], \\ E[L] &= P_0\cdot\frac{\left[\begin{array}{l}(1-r_0-r_1)((1-\rho_H)(\lambda^2E[X_V^2]+2\rho_V\rho_H)+\rho_V\lambda^2E[X_H^2]) \\ \quad +(1-\rho_H)(\lambda^2E[X_R^2]+2\rho_R(1+\rho_V)) \\ \quad +\rho_R\lambda^2E[X_H^2] - r_1(2(\rho_H+\rho_V)(1-\rho_H)+\lambda^2E[X_H^2])\end{array}\right]}{2(1-r_1)(1-\rho_H)^2}, \end{aligned} \tag{3.2}$$

where

$$P_0 = \frac{(1-r_1)(1-\rho_H)}{(1-r_0-r_1)\rho_V+\rho_R-\rho_H+1-r_1}.$$

□

We note that r_0 and r_1 in Proposition 3.1 and Proposition 3.2 can be obtained by (2.2).

3.2 The case of a general m

We consider the case of $m \geq 3$, where the derivation of $\tilde{P}(z)$ is more difficult than that in the case of $m < 3$. Equilibrium equations of the embedded Markov chain are derived as follows:

$$P_0 = r_0P_0 + r_0P_1^R + h_0P_1^H, \tag{3.3}$$

$$P_n^R = r_nP_0 + \sum_{k=1}^{n+1} r_{n+1-k}P_k^R \qquad (1 \leq n \leq m-2), \tag{3.4}$$

$$P_n^R = r_nP_0 + \sum_{k=1}^{m-1} r_{n+1-k}P_k^R \qquad (m-1 \leq n), \tag{3.5}$$

$$P_n^H = \sum_{k=1}^{n+1} h_{n+1-k}P_k^H \qquad (1 \leq n \leq m-2), \tag{3.6}$$

$$P_n^H = \sum_{k=1}^{n+1} h_{n+1-k}P_k^H + \sum_{k=m}^{n+1}(v\otimes h)_{n+1-k}P_k^R \qquad (m-1 \leq n). \tag{3.7}$$

In this section, we shall consider $\tilde{P}(z)$ under given Z-transform $\phi(z)$ i.e. $P_1^R, \cdots, P_{m-1}^R$. These values will be obtained in the next section. At first, the Z-transforms of P_n^R and P_n^H are obtained by using equilibrium equations from (3.3) to (3.7).

Lemma 3.2 *The Z-transform of P_n^R is*

$$\tilde{P}^R(z) = (\tilde{r}(z) - r_0)\, P_0 - r_0 P_1^R + \frac{\tilde{r}(z)\phi(z)}{z}. \tag{3.8}$$

Proof: From (3.4) and (3.5), we have

$$\begin{aligned}
\sum_{n=1}^{m-2} z^n P_n^R &= \sum_{n=1}^{m-2} z^n r_n P_0 + \sum_{n=1}^{m-2} z^n \sum_{k=1}^{n+1} r_{n+1-k} P_k^R, \\
\sum_{n=m-1}^{\infty} z^n P_n^R &= \sum_{n=m-1}^{\infty} z^n r_n P_0 + \sum_{n=m-1}^{\infty} z^n \sum_{k=1}^{m-1} r_{n+1-k} P_k^R.
\end{aligned}$$

Adding the above two equations, we get

$$\begin{aligned}
\sum_{n=1}^{\infty} z^n P_n^R &= \sum_{n=1}^{\infty} z^n r_n P_0 + \sum_{n=1}^{m-2} z^n \sum_{k=1}^{n+1} r_{n+1-k} P_k^R + \sum_{n=m-1}^{\infty} z^n \sum_{k=1}^{m-1} r_{n+1-k} P_k^R \\
&= \sum_{n=1}^{\infty} z^n r_n P_0 + \sum_{k=1}^{m-1} P_k^R \sum_{n=k-1}^{\infty} z^n r_{n+1-k} - r_0 P_1^R.
\end{aligned}$$

We obtain

$$\begin{aligned}
\sum_{k=1}^{\infty} z^k P_k^R &= (\tilde{r}(z) - r_0)\, P_0 - r_0 P_1^R + \frac{\tilde{r}(z)}{z} \sum_{k=1}^{m-1} z^k P_k^R, \\
\tilde{P}^R(z) &= (\tilde{r}(z) - r_0)\, P_0 - r_0 P_1^R + \frac{\tilde{r}(z)\phi(z)}{z}.
\end{aligned}$$

□

Lemma 3.3 *The Z-transform of P_n^H is*

$$\begin{aligned}
\tilde{P}^H(z) &= \frac{1}{z - \tilde{h}(z)} \Big[r_0 \left(P_0 + P_1^R\right) \left(z - \tilde{v}(z)\tilde{h}(z)\right) + P_0 \left(\tilde{r}(z)\tilde{v}(z)\tilde{h}(z) - z\right) \\
&\quad + \frac{\tilde{v}(z)\tilde{h}(z)\phi(z)}{z} (\tilde{r}(z) - z) \Big].
\end{aligned}$$

Proof: Formulas (3.6) and (3.7) yield

$$\begin{aligned}
\sum_{n=1}^{m-2} z^n P_n^H &= \sum_{n=1}^{m-2} z^n \sum_{k=1}^{n+1} h_{n+1-k} P_k^H, \\
\sum_{n=m-1}^{\infty} z^n P_n^H &= \sum_{n=m-1}^{\infty} z^n \sum_{k=1}^{n+1} h_{n+1-k} P_k^H + \sum_{n=m-1}^{\infty} z^n \sum_{k=m}^{n+1} (v \otimes h)_{n+1-k} P_k^R.
\end{aligned}$$

Then we get

$$\begin{aligned}
\sum_{n=1}^{\infty} z^n P_n^H &= \sum_{n=1}^{\infty} z^n \sum_{k=1}^{n+1} h_{n+1-k} P_k^H + \sum_{n=m-1}^{\infty} z^n \sum_{k=m}^{n+1} (v \otimes h)_{n+1-k} P_k^R \\
&= \sum_{k=1}^{\infty} P_k^H \sum_{n=k-1}^{\infty} z^n h_{n+1-k} - h_0 P_1^H
\end{aligned}$$

$$+\sum_{k=m}^{\infty} P_k^R \sum_{n=k-1}^{\infty} z^n (v \otimes h)_{n+1-k},$$

$$\tilde{P}^H(z) = \frac{\tilde{h}(z)}{z}\tilde{P}^H(z) - h_0 P_1^H + \frac{\tilde{v}(z)\tilde{h}(z)}{z}\sum_{n=m}^{\infty} z^n P_n^R,$$

$$\left(1 - \frac{\tilde{h}(z)}{z}\right)\tilde{P}^H(z) = -h_0 P_1^H + \frac{\tilde{v}(z)\tilde{h}(z)}{z}\sum_{n=m}^{\infty} z^n P_n^R. \quad (3.9)$$

On the other hand, from definition of $\phi(z)$ and (3.8), we have

$$\sum_{n=m}^{\infty} z^n P_n^R = \tilde{P}^R(z) - \phi(z)$$

$$= (\tilde{r}(z) - r_0) P_0 - r_0 P_1^R + \left(\frac{\tilde{r}(z)}{z} - 1\right)\sum_{n=1}^{m-1} z^n P_n^R. \quad (3.10)$$

From (3.9) and (3.10), we have

$$\left(1 - \frac{\tilde{h}(z)}{z}\right)\tilde{P}^H(z) = -h_0 P_1^H$$

$$+\frac{\tilde{v}(z)\tilde{h}(z)}{z}\left[(\tilde{r}(z) - r_0) P_0 - r_0 P_1^R + \left(\frac{\tilde{r}(z)}{z} - 1\right)\sum_{n=1}^{m-1} z^n P_n^R\right].$$

Also, $h_0 P_1^H$ is removed by using (3.3). Hence

$$\left(z - \tilde{h}(z)\right)\tilde{P}^H(z) = z\left(r_0 P_0 + r_0 P_1^R - P_0\right)$$

$$+\tilde{v}(z)\tilde{h}(z)\left[(\tilde{r}(z) - r_0) P_0 - r_0 P_1^R + \left(\frac{\tilde{r}(z)}{z} - 1\right)\phi(z)\right],$$

$$\tilde{P}^H(z) = \frac{1}{z - \tilde{h}(z)}\left[r_0\left(P_0 + P_1^R\right)\left(z - \tilde{v}(z)\tilde{h}(z)\right) + P_0\left(\tilde{r}(z)\tilde{v}(z)\tilde{h}(z) - z\right)\right.$$

$$\left. + \frac{\tilde{v}(z)\tilde{h}(z)\phi(z)}{z}(\tilde{r}(z) - z)\right].$$

□

In the following theorem, the Z-transform of $\{P_n\}_{n=0}^{\infty}$ is given.

Theorem 3.1 *The Z-transform of P_n is*

$$\tilde{P}(z) = (1 + \tilde{r}(z)) P_0 - r_0\left(P_0 + P_1^R\right) + \frac{\tilde{r}(z)}{z}\phi(z)$$

$$+\frac{1}{z - \tilde{h}(z)}\left[r_0\left(P_0 + P_1^R\right)\left(z - \tilde{v}(z)\tilde{h}(z)\right)\right.$$

$$\left. + P_0\left(\tilde{r}(z)\tilde{v}(z)\tilde{h}(z) - z\right) + \frac{\tilde{v}(z)\tilde{h}(z)\phi(z)}{z}(\tilde{r}(z) - z)\right]. \quad (3.11)$$

Proof: From Lemma 3.2 and Lemma 3.3, this theorem is proved with

$$\tilde{P}(z) = P_0 + \tilde{P}^R(z) + \tilde{P}^H(z).$$

□

Lemma 3.4 (The normalizing condition) *Using $P_0 + P_1^R$ and $P_n^R (1 \le n \le m-1)$, we have*

$$P_0 = \frac{1 - \rho_H + r_0\left(P_0 + P_1^R\right)\rho_V - \phi(1)(\rho_R - \rho_H)}{1 + \rho_R + \rho_V - \rho_H}. \quad (3.12)$$

Proof: From the normalizing condition given by (3.11), we have

$$\begin{aligned} 1 &= \lim_{z\uparrow 1} \tilde{P}(z) \\ &= P_0\left(1+\frac{\rho_R+\rho_V}{1-\rho_H}\right) - r_0\left(P_0+P_1^R\right)\frac{\rho_V}{1-\rho_H} + \phi(1)\left(1-\frac{1-\rho_R}{1-\rho_H}\right), \\ P_0 &= \frac{1-\rho_H + r_0\left(P_0+P_1^R\right)\rho_V - \phi(1)(\rho_R-\rho_H)}{1+\rho_R+\rho_V-\rho_H}. \end{aligned}$$

□

As the main result of this section, the mean number of customers in the system is obtained.

Theorem 3.2 *The mean number of customers in the system $E[L]$ is*

$$\begin{aligned} E[L] &= \frac{P_0 - r_0\left(P_0+P_1^R\right)}{2(1-\rho_H)^2}\left[(1-\rho_H)\left(\lambda^2 E[X_V^2] + 2\rho_V\rho_H\right) + \rho_V\lambda^2 E[X_H^2]\right] \\ &\quad + \frac{P_0}{2(1-\rho_H)^2}\left[(1-\rho_H)\left(\lambda^2 E[X_R^2] + 2\rho_R(\rho_V+1)\right) + \rho_R\lambda^2 E[X_H^2]\right] \\ &\quad + \frac{\phi(1)}{2(1-\rho_H)^2}\left[\begin{array}{c} \lambda^2 E[X_R^2](1-\rho_H) + 2(1-\rho_H)(\rho_R-1)\rho_V \\ -\lambda^2 E[X_H^2](1-\rho_R) \end{array}\right] \\ &\quad + \frac{\rho_R-\rho_H}{1-\rho_H}\phi'(1). \end{aligned}$$

Proof: From (3.11), we have

$$\begin{aligned} E[L] &= \lim_{z\uparrow 1}\frac{d}{dz}\tilde{P}(z) \\ &= \rho_R P_0 + \rho_R\phi(1) + \phi'(1) - \phi(1) \\ &\quad + \frac{r_0\left(P_0+P_1^R\right)}{2(1-\rho_H)^2}\left[-(1-\rho_H)(\lambda^2 E[X_V^2] + 2\rho_V\rho_H) - \rho_V\lambda^2 E[X_H^2]\right] \\ &\quad + \frac{P_0}{2(1-\rho_H)^2}\left[(1-\rho_H)\left(\begin{array}{c} \lambda^2 E[X_R^2] + \lambda^2 E[X_V^2] + \lambda^2 E[X_H^2] \\ +2(\rho_R\rho_V + \rho_V\rho_H + \rho_H\rho_R) \end{array}\right)\right. \\ &\qquad \left. + (\rho_R+\rho_V+\rho_H-1)\lambda^2 E[X_H^2]\right] \\ &\quad + (\phi(1)(\rho_V+\rho_H-1) + \phi'(1))\frac{\rho_R-1}{1-\rho_H} \\ &\quad + \frac{\phi(1)}{2(1-\rho_H)^2}\left(\lambda^2 E[X_R^2](1-\rho_H) + \lambda^2 E[X_H^2](\rho_R-1)\right) \\ &= \frac{P_0 - r_0\left(P_0+P_1^R\right)}{2(1-\rho_H)^2}\left[(1-\rho_H)(\lambda^2 E[X_V^2] + 2\rho_V\rho_H) + \rho_V\lambda^2 E[X_H^2]\right] \\ &\quad + \frac{P_0}{2(1-\rho_H)^2}\left[(1-\rho_H)\left(\lambda^2 E[X_R^2] + 2\rho_R(\rho_V+\rho_H)\right) + \rho_R\lambda^2 E[X_H^2]\right] \\ &\quad + \rho_R P_0 + \rho_R\phi(1) + \phi'(1) - \phi(1) \\ &\quad + (\phi(1)(\rho_V+\rho_H-1) + \phi'(1))\frac{\rho_R-1}{1-\rho_H} \end{aligned}$$

$$+\frac{\phi(1)}{2(1-\rho_H)^2}\left(\lambda^2E[X_R^2](1-\rho_H)+\lambda^2E[X_H^2](\rho_R-1)\right)$$

$$= \frac{P_0-r_0\left(P_0+P_1^R\right)}{2(1-\rho_H)^2}\left[(1-\rho_H)\left(\lambda^2E[X_V^2]+2\rho_V\rho_H\right)+\rho_V\lambda^2E[X_H^2]\right]$$

$$+\frac{P_0}{2(1-\rho_H)^2}\left[(1-\rho_H)\left(\lambda^2E[X_R^2]+2\rho_R(\rho_V+1)\right)+\rho_R\lambda^2E[X_H^2]\right]$$

$$+\frac{\phi(1)}{2(1-\rho_H)^2}\left[\begin{array}{c}\lambda^2E[X_R^2](1-\rho_H)+2(1-\rho_H)(\rho_R-1)\rho_V\\ -\lambda^2E[X_H^2](1-\rho_R)\end{array}\right]$$

$$+\frac{\rho_R-\rho_H}{1-\rho_H}\phi'(1).$$

□

4 The Mean Number of Customers in the System

In the previous section for $m=1$ and $m=2$, $\tilde{P}(z)$ is obtained. In the case of $m\geq 3$, we derive $P^R(z)$, $P^H(z)$ and $\tilde{P}(z)$ in Lemma 3.2, Lemma 3.3 and Theorem 3.1, respectively, where parameters r_0, ρ_R, ρ_V, ρ_H, $\lambda^2E[X_R^2]$, $\lambda^2E[X_V^2]$ and $\lambda^2E[X_H^2]$ are given and parameters $P_0, P_1^R, \phi(1)$ and $\phi'(1)$ are unknown. In this section, the derivation of these unknown parameters is considered mainly. Since P_0 and P_n^R satisfy homogeneous linear equations, the problem is to obtain their coefficients. We introduce a supplementary series f_n which is defined by the solution of the recursive equation. Using the normalizing condition and the boundary condition, an algorithm of the computation $E[L]$ for $m\geq 3$ is discussed.

4.1 The Definitions and Analyses for Supplementary Series

First of all, we give some definitions of series.

Definition 4.1 *The series $\{y_n\}_{n=1}^{\infty}$ is defined from $\{r_n\}_{n=0}^{\infty}$ as follows:*

$$y_1 \triangleq \frac{1-r_1}{r_0},$$
$$y_n \triangleq \frac{-r_n}{r_0} \qquad (n\geq 2).$$

□

Definition 4.2 *Suppose that a_1 and a_2 are given as an initial condition. The series a_n is defined recursively as*

$$a_n \triangleq \sum_{k=1}^{n-1} y_{n-k}a_k \qquad (n\geq 3). \tag{4.1}$$

□

In the next lemma, the series P_n^R in (3.4) satisfies the above recursive relation when the initial values P_0 and P_1^R are given.

Lemma 4.1 *Suppose that as initial values P_0 and P_1^R are given. If we put*

$$a_1 := P_0+P_1^R,$$
$$a_2 := P_2^R = y_1\left(P_0+P_1^R\right)-\frac{1}{r_0}P_0,$$

then P_n^R can be represented by a_n such that

$$P_n^R = a_n \qquad (3\leq n\leq m-1),$$

and the boundary condition is

$$a_m = 0. \tag{4.2}$$

Proof : From (3.4), we obtain

$$P_n^R = \frac{1}{r_0}P_{n-1}^R + \sum_{k=1}^{n-1} \frac{-r_{n-k}}{r_0} P_k^R + \frac{-r_{n-1}}{r_0} P_0 \qquad (2 \le n \le m-1). \tag{4.3}$$

It follows from the boundary condition of (4.3) and (3.5) for $n = m-1$ that

$$\begin{aligned} P_{m-1}^R &= r_{m-1}P_0 + \sum_{k=1}^{m-1} r_{m-k}P_k^R, \\ 0 &= \frac{1}{r_0}P_{m-1}^R + \sum_{k=1}^{m-1} \frac{-r_{m-k}}{r_0}P_k^R + \frac{-r_{m-1}}{r_0}P_0. \end{aligned}$$

And from Definition 4.1 and Definition 4.2, this lemma is proved. □

From the above lemma, we have that $\{P_n^R\}_{n=1}^{m-1}$ and P_0 satisfy homogeneous linear equations in which unknown variables are P_0 and P_1^R. In order to get the simple form of their coefficients, we introduce a supplementary series as follows:

Definition 4.3 *A supplementary series* f_n *is defined as*

$$f_0 \triangleq 1, \tag{4.4}$$

$$f_n \triangleq \sum_{k=1}^{n} y_k f_{n-k} \qquad (n \ge 1), \tag{4.5}$$

$$\left(= \sum_{k=0}^{n-1} y_{n-k} f_k = \sum_{k=1}^{n} y_{n+1-k} f_{k-1} \right).$$

□

Remark that f_n is the solution of a discrete type recursive equation of y_n. Even though y_n is not a probability mass function, (4.5) is similar to the renewal equation. A simple relation between a_n and f_n is shown in the next lemma.

Lemma 4.2 *If series* $\{a_n\}$ *and* $\{f_n\}$ *are given by* (4.1), (4.4) *and* (4.5), *then we have*

$$a_n = (f_{n-1} - y_1 f_{n-2})\, a_1 + f_{n-2}a_2 \qquad (n \ge 2). \tag{4.6}$$

Proof : This lemma is proved by induction on n. As it is trivial for the case of $n = 2$, we give a general proof as follows:

$$\begin{aligned} a_{n+1} &= \sum_{k=1}^{n} y_{n+1-k}a_k \\ &= \sum_{k=2}^{n} y_{n+1-k}a_k + y_n a_1 \\ &= \sum_{k=2}^{n} y_{n+1-k}\left[(f_{k-1} - y_1 f_{k-2})\, a_1 + f_{k-2}a_2\right] + y_n a_1 \\ &= a_1 \sum_{k=1}^{n} y_{n+1-k} f_{k-1} + (a_2 - a_1 y_1) \sum_{k=2}^{n} y_{n+1-k} f_{k-2} \\ &= a_1 f_n + (a_2 - a_1 y_1)\, f_{n-1} \\ &= (f_n - y_1 f_{n-1})\, a_1 + f_{n-1}a_2. \end{aligned}$$

□

Lemma 4.3 (The boundary condition) *By using* P_0,

$$P_0 + P_1^R = \frac{f_{m-2}}{r_0 f_{m-1}} P_0 \tag{4.7}$$

or

$$P_1^R = \left(\frac{f_{m-2}}{r_0 f_{m-1}} - 1\right) P_0$$

is obtained. Furthermore, we have

$$P_n^R = \left(\frac{f_{m-2}}{f_{m-1}} f_{n-1} - f_{n-2}\right) \frac{P_0}{r_0} \qquad (2 \leq n \leq m-1).$$

Proof: From the boundary condition (4.2) and (4.6) in Lemma 4.2, we have

$$0 = (f_{m-1} - y_1 f_{m-2}) a_1 + f_{m-2} a_2.$$

Then the above three equations can be obtained. □

From Lemma 4.3, the unknown Z-transform $\phi(z)$ can be represented using f_n with given P_0 as follows:

$$\begin{aligned}
\phi(z) &= \sum_{n=1}^{m-1} z^n P_n^R \\
&= z P_1^R + \sum_{n=2}^{m-1} z^n P_n^R \\
&= z\left(\frac{f_{m-2}}{r_0 f_{m-1}} - 1\right) P_0 + \sum_{n=2}^{m-1} z^n \left(\frac{f_{m-2}}{f_{m-1}} f_{n-1} - f_{n-2}\right) \frac{P_0}{r_0} \\
&= \frac{z P_0}{r_0}\left[\left(\frac{f_{m-2}}{f_{m-1}} - z\right) \sum_{n=0}^{m-2} z^n f_n + z^{m-1} f_{m-2} - r_0\right].
\end{aligned}$$

Then we get

$$\phi(1) = \frac{P_0}{r_0}\left[\left(\frac{f_{m-2}}{f_{m-1}} - 1\right) \sum_{n=0}^{m-2} f_n + f_{m-2} - r_0\right], \tag{4.8}$$

$$\phi'(1) = \phi(1) + \frac{P_0}{r_0}\left[\left(\frac{f_{m-2}}{f_{m-1}} - 1\right) \sum_{n=0}^{m-2} n f_n - \sum_{n=0}^{m-2} f_n + (m-1) f_{m-2}\right]. \tag{4.9}$$

Now, an unknown factor in Theorem 3.2 is only P_0. In the next theorem, P_0 is obtained by Lemma 3.4.

Theorem 4.1 *The idle probability* P_0 *is given by*

$$P_0 = \frac{r_0 f_{m-1}(1 - \rho_H)}{\left[\begin{array}{l} r_0 f_{m-1}(1 + \rho_V) - r_0 f_{m-2} \rho_V \\ \quad + (\rho_R - \rho_H)\left(f_{m-2} f_{m-1} + (f_{m-2} - f_{m-1}) \sum_{n=0}^{m-2} f_n\right) \end{array}\right]}.$$

Proof: Substituting (4.7) and (4.8) into (3.12), this theorem is proved. □

4.2 An Algorithm

In Section 3, we first get $E[L]$ in the cases of $m = 1$ and $m = 2$ in (3.1) and (3.2), respectively. For $m \geq 3$, the Z-transform $\tilde{P}(z)$ of the stationary probability mass function P_n is obtained in Theorem 3.1 under the condition that $\{P_n^R\}_{n=1}^{m-1}$ and P_0 are given. In Section 4, it follows from the boundary condition, (3.4) and (3.5) that $\{P_n^R\}_{n=1}^{m-1}$ and P_0 satisfy homogeneous linear equations and their coefficients are given by f_n in Lemma 4.3. From the normalizing condition, P_0 is obtained in Theorem 4.1. We are now in position to summarize our algorithm of the computation $E[L]$ for $m \geq 3$.

Step 1 Compute $\{y_n\}_{n=1}^{m-1}$ from $\{r_n\}_{n=0}^{m-1}$ by Definition 4.1.
Step 2 Compute $\{f_n\}_{n=0}^{m-1}$ from $\{y_n\}_{n=1}^{m-1}$ by Definition 4.3.
Step 3 Calculate $\sum_{n=0}^{m-2} f_n$ and $\sum_{n=0}^{m-2} nf_n$.
Step 4 Calculate P_0 by Theorem 4.1.
Step 5 Calculate $P_0 + P_1^R$ by Lemma 4.3.
Step 6 Calculate $\phi(1)$ by (4.8).
Step 7 Calculate $\phi'(1)$ by (4.9).
Step 8 Calculate $E[L]$ from $P_0, P_0 + P_1^R, \phi(1)$ and $\phi'(1)$ by Theorem 3.2.

4.3 The Analytical Results for Some Regular Service Distributions

Using previous results, an explicit expression of $E[L]$ for some distributions of the regular service time X_R can be obtained. At first, the Z-transform of f_n is represented by $\tilde{r}(z)$.

Lemma 4.4 *The Z-transform of* f_n

$$\tilde{f}(z) \triangleq \sum_{n=0}^{\infty} z^n f_n$$

is

$$\tilde{f}(z) = \frac{r_0}{\tilde{r}(z) - z}.$$

Proof: From the definition of f_n, we obtain

$$\begin{aligned}
\tilde{f}(z) &= 1 + \sum_{n=1}^{\infty} z^n f_n \\
&= 1 + \sum_{n=1}^{\infty} z^n \sum_{k=1}^{n} y_k f_{n-k} \\
&= 1 + \tilde{f}(z) \sum_{k=1}^{\infty} z^k y_k \\
&= 1 + \tilde{f}(z) \left(z\frac{1-r_1}{r_0} + \sum_{k=2}^{\infty} z^k \frac{-r_k}{r_0} \right) \\
&= 1 + \frac{\tilde{f}(z)}{r_0} \left(z(1-r_1) - (\tilde{r}(z) - zr_1 - r_0) \right), \\
r_0 \tilde{f}(z) &= r_0 + \tilde{f}(z)(z + r_0 - \tilde{r}(z)).
\end{aligned}$$

Therefore, this lemma is proved. □

Since $\tilde{r}(z) = R^*(\lambda - \lambda z)$, $\tilde{f}(z)$ is similar to the *Pollaczek-Khinchin transform equation* of the number of customers in $M/G/1$ queues. When the service distribution is exponential or constant, the distribution of the number of customers in an $M/G/1$ queue is given (*Gross and Harris* [3]). We can apply these methods and obtain the explicit formula of f_n.

Proposition 4.1 *If X_R is exponential, that is*

$$R(x) \triangleq \Pr(X_R \le x) = 1 - e^{-\frac{\lambda}{\rho_R}x},$$

then

$$\begin{aligned} r_n &= \frac{1}{1+\rho_R}\left(\frac{\rho_R}{1+\rho_R}\right)^n \qquad (n \ge 0), \\ \tilde{r}(z) &= \frac{1}{1+\rho_R-\rho_R z}, \\ \sum_{n=0}^{m-2} f_n &= \frac{1}{(1-\rho_R)(1+\rho_R)}\left(m-1-\frac{\rho_R^2}{1-\rho_R}\left(1-\rho_R^{m-1}\right)\right), \\ \sum_{n=0}^{m-2} nf_n &= \frac{1}{(1-\rho_R)(1+\rho_R)}\left(\frac{(m-2)(m-1)}{2}\right. \\ &\qquad \left. -\frac{\rho_R^3}{(1-\rho_R)^2}\left(1-(m-1)\rho_R^{m-2}+(m-2)\rho_R^{m-1}\right)\right). \end{aligned}$$

Proof: From the direct calculation, r_n and $\tilde{r}(z)$ are obtained. And it follows from Lemma 4.4 that

$$\begin{aligned} \tilde{f}(z) &= \frac{1+\rho_R-\rho_R z}{(1+\rho_R)(1-z)(1-\rho_R z)} \\ &= \frac{1}{(1-\rho_R)(1+\rho_R)}\left(\frac{1}{1-z}-\frac{\rho_R^2}{1-\rho_R z}\right), \\ f_n &= \frac{1-\rho_R^{n+2}}{(1-\rho_R)(1+\rho_R)} \qquad (n \ge 0). \end{aligned}$$

□

Proposition 4.2 *If X_R is a constant, that is*

$$\Pr\left(X_R = \frac{\rho_R}{\lambda}\right) = 1,$$

then

$$\begin{aligned} R^*(s) &= e^{-\frac{\rho_R}{\lambda}s}, \\ r_0 &= e^{-\rho_R}, \\ \tilde{r}(z) &= e^{-\rho_R(1-z)}, \\ \tilde{f}(z) &= \frac{e^{-\rho_R}}{e^{-\rho_R(1-z)}-z}, \end{aligned}$$

are obtained and

$$f_n = \sum_{k=0}^{n} e^{\rho_R k}\frac{(-(k+1)\rho_R)^{n-k}}{(n-k)!} \qquad (n \ge 0). \tag{4.10}$$

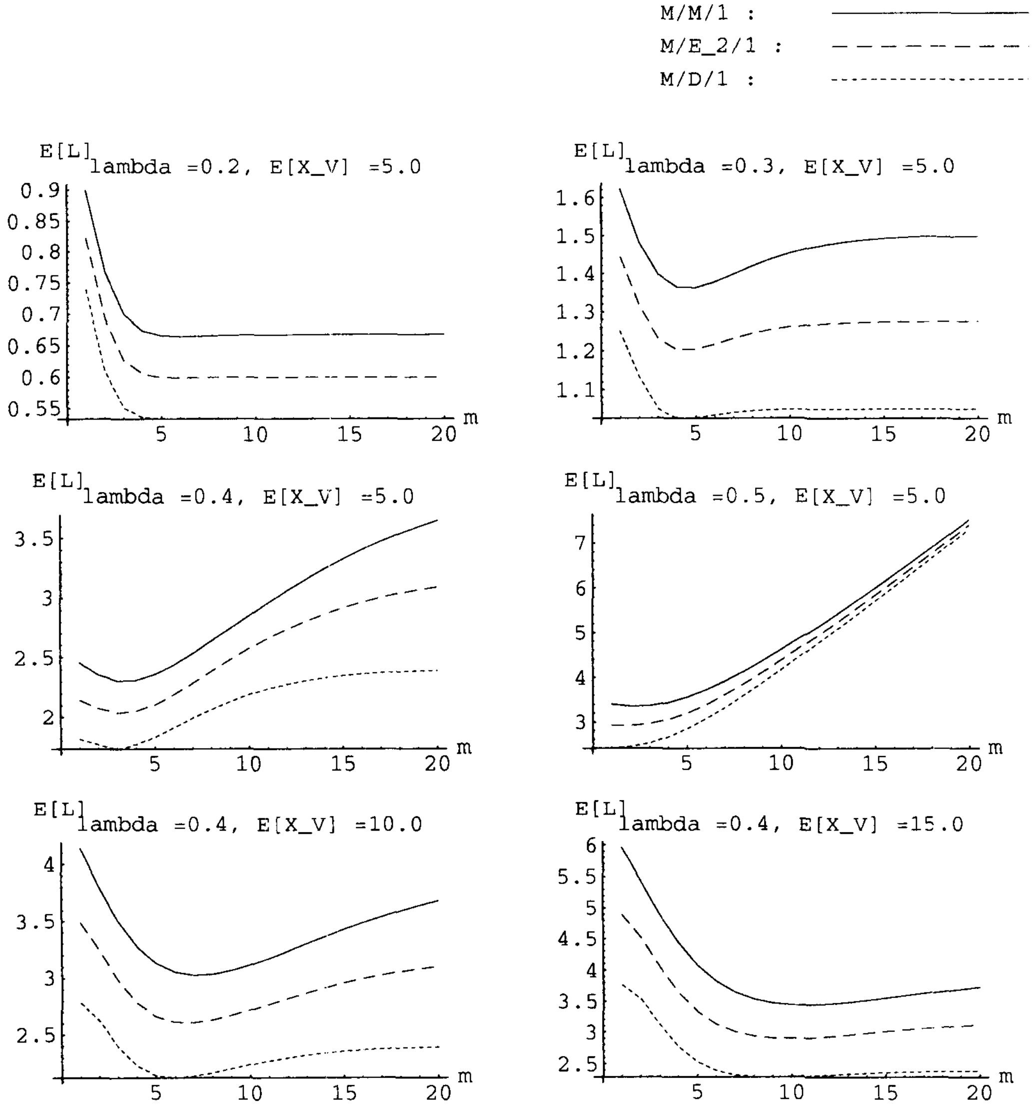

Figure 1: The mean number of customers $E[L]$ in $M/M/1$, $M/E_2/1$ and $M/D/1$ type models: $E[X_R] = 2.0$ and $E[X_H] = 1.0$.

Proof: By the expansion of $1/\left(1 - ze^{\rho_R(1-z)}\right)$ (see pp.270 in *Gross and Harris* [3]), we have

$$\begin{aligned}\tilde{f}(z) &= \frac{e^{-\rho_R}}{e^{-\rho_R(1-z)} - z} = \frac{e^{-\rho_R}e^{\rho_R(1-z)}}{1 - ze^{\rho_R(1-z)}} \\ &= e^{-\rho_R}e^{\rho_R(1-z)}\sum_{k=0}^{\infty} e^{k\rho_R(1-z)}z^k\end{aligned}$$

$$
\begin{aligned}
&= e^{-\rho_R} \sum_{k=0}^{\infty} e^{(k+1)\rho_R(1-z)} z^k \\
&= e^{-\rho_R} \sum_{k=0}^{\infty} e^{(k+1)\rho_R} \sum_{j=0}^{\infty} \frac{(-(k+1)\rho_R z)^j}{j!} z^k \\
&= e^{-\rho_R} \sum_{k=0}^{\infty} e^{(k+1)\rho_R} \sum_{n=k}^{\infty} \frac{(-(k+1)\rho_R)^{(n-k)}}{(n-k)!} z^n \\
&= \sum_{n=0}^{\infty} z^n \sum_{k=0}^{n} e^{\rho_R k} \frac{(-(k+1)\rho_R)^{n-k}}{(n-k)!}.
\end{aligned}
$$

Therefore, (4.10) can be obtained. □

It should be noted that $E[L]$ depends on the form of the regular service distribution through r_n, but it depends only on first and second moments of the high speed service distribution and the vacation time distribution. If the service time is exponential and constant, then we can abbreviate steps 1–3 and 1–2, respectively, of our algorithm in Section 4.2.

5 Numerical Illustrations

In this section, we investigate numerical calculations of the mean number $E[L]$ of customers in the system. In Figure 1, graphs of $E[L]$ are illustrated as a function of m when both service and setup times are exponential, Erlang type 2 distribution and constant, respectively. If $\rho_R < 1$, $E[L]$ converges to the mean number of customers in the $M/G/1$ queueing system with the service time X_R, and if $\rho_R \geq 1$, $E[L]$ diverges as $m \to \infty$. In numerical standpoints, an optimal switching scheduling where the average sojourn time is to be minimized will be discussed. Since the arrival process is assumed to be a Poisson process with rate λ independent of states, from Little's formula, the average sojourn time (the waiting time + the service time) is equal to $E[L]/\lambda$. The optimal switching point m^* which minimizes the average sojourn time is the same as m^* which minimizes $E[L]$. It can be observed that $-E[L]$ is unimodal in m, that is, $E[L]$ is monotone decreasing in $[1, m^*]$ and is monotone increasing in $[m^*, \infty)$. Moreover, m^* is monotone decreasing for increasing arrival rate λ. As was shown in the optimality of N-Policy, it seems that we obtain the optimal switching point m^*.

Acknowledgements

The authors wish to thank the anonymous referees for careful reading of this paper and invaluable comments.

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Shoichi Nishimura
Department of Applied Mathematics
Science University of Tokyo
1-3 Kagurazaka, Shinjuku, Tokyo 162, Japan