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DYNAMICALLY OPTIMAL REPLACEMENT POLICY FOR A SHOCK MODEL IN A MARKOV RANDOM ENVIRONMENT

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Abstract We consider a system existing in a random environment. The environment is described by a Markov process called *Markov environment process* (MEP). The system is subject to a sequence of randomly occurring shocks, and each shock causes a random amount of damage which accumulates additively. The shock arrival and shock magnitude are influenced by changes of the environment. The damage process is assumed to be a *piecewise semi-Markov process* (PSMP) which is constructed by the shock process and the environment process. The optimal maintenance-replacement problem for the system is examined. A control-limit rule dependent on the MEP is driven.

1. Introduction

The present work deals with an optimal maintenance-replacement problem for a system subject to shocks. The cumulative damage is determined by a process called *piecewise semi-Markov process* (PSMP). In recent years, the replacement models with additive damage have been extensively investigated. An excellent survey of the theory, specifying results up to 1989, of optimal replacement of systems subject to shocks can be found in Valdez-Flores and Feldman (1989). Taylor (1975) studied the shock model where the cumulative damage process is a compound Poisson process. Siedersleben (1981) considered a continuously deteriorating system where the problem can be regarded as a shock model. Other researchers, such as Feldman, Bergman, Gottlieb, Posner and Zuckerman, dealt with various semi-Markov shock models with additive damages. Feldman (1977) and Bergman (1978) allowed replacement to take place only at shock times or at failure times, while Zuckerman (1978) and Gottlieb (1982) considered general stopping rules. Furthermore, Posner and Zuckerman (1986) generalized other restriction conditions of the these earlier results in this area. In these models, the influence of a "randomly varying environment" on systems was not considered. Only Waldmann (1985), we know, has given a shock model in which an "environment process" was introduced to the shock process, for a lattice damage process and discrete time case.

In many applications, the behavior of the cumulative damage processes depends not only on shock processes, but also on "environments" where systems operate. The environmental process may be external factors of an economical or technical nature as well as internal factors of a statistical nature. For example,

(a) Consider a system that receives two types of shocks at random points of time. The corresponding damage processes are related each other, and each type of shocks may cause the system to fail. One of them can be regarded as an "environment" process.

(b) Consider a system with a modulator whose state changes can be described by a Markov jump process. The system is subject to shocks, and the stochastic characteristics of shocks (for instance, the distributions of intershock times and shock magnitudes) depend on the state of the modulator. Hence, the Markov jump process of the modulator can be taken as an "environment" process.

Also there are many other cases where shock processes are influenced by a secondary process which happens to be Markovian should be considered (see Waldmann (1985)). In these cases, the successive replacements of identical systems no longer form a renewal process because the environment state may not return to the initial state, when the system is replaced. Therefore, analysis is difficult by general renewal arguments. The purpose of the present paper is to investigate an optimal replacement problem for such a shock model by means of Dynamic programming method. In this model, we assume that the environment process as well as the damage process be continuous time processes. We consider these policies for which the maintenance or replacement actions can be taken at any time, and permit that the damage level of the system has a randomly decreasing magnitude after a maintenance action is accomplished. We prove that there exists an optimal control-limit policy minimizing the total expected randomly discounted cost. Differing from traditional control-limit policies, here, the control-limit policy is a function dependent on the Markov environment process.

Consider a system existing in a random environment. The environment is described by a Markov jump process. The system is subject to a sequence of randomly occurring shocks, and each shock causes a random amount of damage which accumulates additively over time. The shock arrival and shock magnitude are influenced by changes of the environment. The damage process is assumed to be a piecewise semi-Markov process. The failure of the system can occur only at times of shock arrival or the environment change. The survival probability at these times is determined by a known function of the accumulated damage level of the system, the environment state and the realized shock magnitude. Upon failure, the system must be replaced by a new one having properties that are statistically equivalent to the original, and a cost is incurred. The replacement cycles are repeated indefinitely. The system may be maintained or preventively replaced before failure at a smaller cost. Here, a maintenance task is an action such as cleaning and lubrication of the system, checking and replacing deteriorating units or parts of the system, or making adjustments in the system. The maintenance time and replacement time are assumed to be negligible.

The paper is organized as follows. In Section 2, the piecewise semi-Markov shock model is formulated. In Sections 3 and 4, properties of the total expected randomly discounted cost is discussed, and an optimal maintenance-replacement policy with control-limit is derived. In Section 5, two applications are given. Throughout the paper, the term "increasing" will be used to mean "non-decreasing" and "decreasing" to mean "non-increasing" , and the following will be standard notation:

$$E_{(\xi,z)}[\,\cdot\,] = E[\,\cdot\,|\xi_0 = \xi, Z_0 = z]$$
$$E_{(\xi,z)}[\,\cdot\,|A] = E[\,\cdot\,|\xi_0 = \xi, Z_0 = z, A]$$

where A is an event. Moreover $R_+ = [0,\infty)$ and $\Im$ is Borel-field on R_+.

2. Preliminaries and The Model

Let $\{\xi_t\}_{t\geq 0}$ be a stochastic process specifying the environment of the system. The process $\{\xi_t\}_{t\geq 0}$ is assumed to be a stationary regular Markov jump process with the state space Γ and the initial state ξ_0. Let $\Re$ be a σ-field on Γ such that one point set $\{\xi\} \in \Re$ and $\{\omega_n\}_{n\geq 0}$ ($\omega_0 = 0$) the jump points of $\{\xi_t\}_{t\geq 0}$. The $Q(\xi, A)$ is a Markov kernel on $(\Gamma, \Re)$ with $Q(\xi, \{\xi\}) = 0$, i.e., $Q(\xi, \cdot)$ is a probability measure for every $\xi \in \Gamma$, and $Q(\cdot, A)$ is a

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$\Re$-measurable function for every $A \in \Re$. For any $A \in \Re$ and $t \in R_+$, let

$$(2.1) \qquad P(\xi_{\omega_{n+1}} \in A, \omega_{n+1} - \omega_n \le t | \xi_s, s \le \omega_n) = Q(\xi_{\omega_n}, A)(1 - e^{-\eta(\xi_{\omega_n})t})$$

where $\eta : \Gamma \to R_+$ is a finite function. The process $\{\xi_t\}_{t\ge 0}$ is called *Markov environment process* (MEP).

On $\{\omega_n \le t < \omega_{n+1}, \quad \xi_{\omega_n} = \xi\}$, let $\{Z_{(\xi,z_0)}(t-\omega_n)\}_{t\ge 0}$ be a semi-Markov process representing the cumulative damage of the system. The state space is $E = R_+ \cup \{\infty\}$, and the initial state is $Z_{(\xi,z_0)}(0) = z_0$ which is the damage level just prior to the time ω_n. For any $x, z \in R_+$, and $t \in R_+$, the semi-Markov kernel of $\{Z_{(\xi,z_0)}(t-\omega_n)\}_{t\ge 0}$ is defined by

$$(2.2) \qquad \begin{aligned} &P(Z_{(\xi,z_0)}(\tau^{\xi}_{n+1}) - Z_{(\xi,z_0)}(\tau^{\xi}_n) \le x, \tau^{\xi}_{n+1} - \tau^{\xi}_n \le t | Z_{(\xi,z_0)}(\tau^{\xi}_n) = z) \\ &= G^{\xi}_z(x) H^{\xi}(t) \end{aligned}$$

where $\{\tau^{\xi}_n\}_{n\ge 0}$ ($\tau^{\xi}_0 = 0$) are the jump points of the $\{Z_{(\xi,z_0)}(t)\}_{t\ge 0}$, $H^{\xi}(\cdot)$ is the probability distribution function of the intershock time $\tau^{\xi}_{n+1} - \tau^{\xi}_n$ and $G^{\xi}_z(\cdot)$ is the conditional distribution function of $Z_{(\xi,z_0)}(\tau^{\xi}_{n+1}) - Z_{(\xi,z_0)}(\tau^{\xi}_n)$ given $Z_{(\xi,z_0)}(\tau^{\xi}_n) = z$. We suppose that the $\{Z_{(\xi,z_0)}(t)\}_{t\ge 0}$ be a right-continuous regular process with left-hand limits.

The stochastic process $\{Z(t)\}_{t\ge 0}$ specifying the cumulative damage of the system in one replacement cycle is defined by $Z(0) = 0$, and

$$(2.3) \qquad Z(t) = Z_{(\xi_0,0)}(t) I_{\{0<t<\omega_1\}} + \textstyle\sum_{n=1}^{\infty} Z_{(\xi_{\omega_n}, Z(\omega_n -))}(t - \omega_n) I_{\{\omega_n \le t < \omega_{n+1}\}}.$$

From the definitions of $\{Z_{(\xi,z_0)}(t)\}_{t\ge 0}$, $(\xi, Z_0) \in \Gamma \times R_+$, we know that the process $\{Z(t)\}_{t\ge 0}$ is also a right-continuous regular process with left-hand limits . At the points $\omega_n, n \ge 1$, $Z(\omega_n) = Z(\omega_n -)$, and on the interval $[\omega_n, \omega_{n+1})$, the process $\{Z(t)\}_{t\ge 0}$ is a semi-Markov process dependent on the environment state ξ_{ω_n}. We call $\{Z(t)\}_{t\ge 0}$ *piecewise semi-Markov process*(PSMP). It can be seen that $\{Z(t)\}_{t\ge 0}$ is a general semi-Markov process if there is only one state in Γ. The state space of $\{Z(t)\}_{t\ge 0}$ is $E = [0, \infty]$. Here, $Z = 0$ means that the system is new and $Z = \infty$ indicates a failure state.

In this model, a failure of the system can occur only at times of shock arrival or jump of the MEP. Let T be such a time point, suppose $\xi_{T-} = \xi$ and $Z(T-) = z$. At time T, if a shock of magnitude x occurs, then the system fails with known probability $1 - \gamma(z, \xi, x)$, and if a jump of the MEP into the state ζ occurs, then the system fails with known probability $1 - \gamma(z, \zeta, 0)$. The function $\gamma : R_+ \times \Gamma \times E \to [0, 1]$ is referred to as the survival function. Let $\delta = \inf\{t, \quad Z(t) = \infty\}$, then δ is the first failure time of the system. We assume throughout that $E[\delta] < \infty$.

Let $\mathbf{A}$ be a set of maintenance-replacement decisions defined as follows

$$\begin{aligned} \mathbf{A} \equiv \{A(\cdot,\cdot) = (a(\cdot,\cdot), i(\cdot,\cdot)) \mid a(\cdot,\cdot) : \Gamma \times E \to [0,\infty], i(\cdot,\cdot) : \Gamma \times E \to \{0,1\} \\ \text{are } \Re \times \Im\text{-measurable and } a(\cdot,\infty) = 0, i(\cdot,\infty) = 1, a(\cdot,0) = \infty\}. \end{aligned}$$

Definition 2.1. A decision policy $A = (a(\xi,z), i(\xi,z)) \in \mathbf{A}$ is called a control-limit policy if for every possible state $\xi \in \Gamma$, there is a real-number $f(\xi)$ such that

$$(2.4) \qquad A(\xi, z) = \begin{cases} (a_1(\xi,z), 0) & \text{if } z < f(\xi) \\ (a_2(\xi,z), 1) & \text{otherwise.} \end{cases}$$

The function $f(\cdot)$ is called a control-limit.

An infinite stage maintenance-replacement policy is a sequence $\pi = (A_0, A_1, \cdots)$; $A_i \in$

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A. If $A_i = A$ for all $i = 0, 1, \cdots$, we call π a stationary policy. Let Π be the set of all policies such as π. For every $\pi \in \Pi$, we can obtain a decision process $\{Z^\pi(t)\}_{t\geq 0}$ which describes the accumulated damage level of the system at time t under the policy π.

Suppose at a decision time T, $\xi_T = \xi, Z_T^\pi = z$, $\xi \in \Gamma, z \in (0,\infty)$. For $a \in [0,\infty)$, the decision $A(\xi, z) = (a, 0)$ means that we maintain the system at time $T + a$ and incur a cost $m(\xi, z)$, and the decision $A(\xi, z) = (a, 1)$ means that we replace the system at time $T + a$ and incur a cost $c(\xi, z)$. For $a = \infty$, the decisions $A(\xi, z) = (\infty, i)$ $(i = 0, 1)$ means that we neither maintain nor replace the system at any time, but wait for the next decision time. If $Z_T^\pi = \infty$, in particular, we immediately replace the system and incur a cost $c(\xi, \infty)$. After execution of an action (maintenance or replacement), the behaviors of the damage process and environment process are influenced as follows.

1. For $\xi \in \Gamma, z \in R_+$, let $Y(\xi, z)$ be a [0, 1)-valued random variable with the distribution function $F_z^\xi(y)$. If a maintenance action is taken at state (ξ, z), the damage level z decreases to the level $z \cdot Y(\xi, z)$. Here, we assume that
$F_z^\xi(y)$ is stochastically increasing in z.

2. The environment state ξ does not change after an action is taken, i.e., the damage process evolves still as a PSMP with the initial environment state ξ and damage level $z \cdot Y(\xi, z)$ (maintenance) or 0 (replacement).

Informally, the assumption **1** implies that the system gradually becomes hard of maintaining with increasing of the damage level. The assumption **2** shows that every maintenance action influences not only the damage process but the shock process as well. The set of the decision points is $\{T_n\}_{n\geq 0}$ which are the successive jump points of the two-dimensional process $\{\xi_t, Z^\pi(t)\}_{t\geq 0}$ defined by $T_0 = 0$, and for $n \geq 0$

$$T_{n+1} = \inf\{t \geq T_n; \quad \xi_t \neq \xi_{T_n} \text{ or } Z^\pi(t) \neq Z^\pi(T_n)\}. \tag{2.5}$$

Since any maintenance or replacement action changes the damage level, we see that $\{T_n\}_{n\geq 0}$ contains three-type points (a) shock points, (b) jump points of the MEP, and (c) action points (i.e. at which an action is executed). At point T_n, if we immediately take an action(i.e. $a = 0$), then $T_{n+1} = T_n$. Let

$$\begin{cases} Z_n = Z^\pi(T_n) \\ \xi_n = \xi_{T_n} \end{cases} \qquad \text{for } n \geq 0. \tag{2.6}$$

For the Markov-decision process $\{\xi_n, Z_n, T_n, A_n\}_{n\geq 0}$, we have the following proposition.

Proposition 2.2. At $T_n < \delta$, $\xi_n = \xi, Z_n = z$, if $a(\xi, z) = a$, then
(a) $\Phi_1(\xi, a) \equiv P(T_{n+1} \text{ is a shock point } |\xi_n = \xi, Z_n = z, a(\xi, z) = a)$
$= \int_0^a H^\xi(t)\eta(\xi)e^{-\eta(\xi)t}dt + H^\xi(a)e^{-\eta(\xi)a}$.
(b) $\Phi_2(\xi, a) \equiv P(T_{n+1} \text{ is a jump point } |\xi_n = \xi, Z_n = z, a(\xi, z) = a)$
$= \int_0^a(1 - e^{-\eta(\xi)t})H^\xi(dt) + (1 - e^{-\eta(\xi)a})\bar{H}^\xi(a)$.
(c) $\Phi_3(\xi, a) \equiv P(T_{n+1} \text{ is an action point } |\xi_n = \xi, Z_n = z, a(\xi, z) = a)$
$= \bar{H}^\xi(a)e^{-\eta(\xi)a}$
where $\bar{H}^\xi(a) = 1 - H^\xi(a)$.

Proof: Let S_1 and S_2 respectively represent the first interval length from T_n to the next shock arrival and the first interval length from T_n to the next jump of the MEP. For part (a), we have that

$$\Phi_1(\xi, a) = P(S_1 \leq S_2, S_1 \leq a|\xi_n = \xi, Z_n = z, a(\xi, z) = a)$$

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$= \int_0^\infty P(S_1 \leq t, S_1 \leq a | \xi_n = \xi, Z_n = z, a(\xi, z) = a) dP(S_2 \leq t | \xi_n = \xi, Z_n = z)$
$= \int_0^a H^\xi(t)\eta(\xi)e^{-\eta(\xi)t}dt + H^\xi(a)e^{-\eta(\xi)a}.$

Similarly, we can obtain part (b). For pat (c), we have that

$$\Phi_3(\xi, a) = P(S_1 \geq a, S_2 \geq a | Z_n = z, \xi_n = \xi, a(\xi, z) = a) = \bar{H}^\xi(a)e^{-\eta(\xi)a}. \quad \square$$

Let $B \equiv \{V;\ \Gamma \times E \to R | V$ is bounded and $\Re \times \Im-$measurable$\}$, $B^+ \equiv \{V;\ V \in B | V(\xi, z)$ is increasing in z for any $\xi \in \Gamma\}$, and $\|\cdot\|$ the sup-norm defined on B. Hence B is a Banach space.

In this model, we consider a randomly discounted cost case. The discounted rate is a function of the MEP and is denoted by $\lambda(\xi_t)$. The discounted factor at T_n is $e^{-\Lambda(T_n)}$ where

$$\Lambda(T_n) \equiv \textstyle\sum_{j=1}^{n-1}(\lambda(\xi_{j-1}) - \lambda(\xi_j))T_j + \lambda(\xi_{n-1})T_n, \tag{2.7}$$

and the discounted cost incurred at T_n is

$$K_n = \begin{cases} e^{-\Lambda(T_n)}k(\xi_{n-1}, Z_{n-1}, A_{n-1}) & \text{if } T_n \text{ is an action point} \\ 0 & \text{otherwise} \end{cases} \tag{2.8}$$

where

$$k(\xi, z, A) = \begin{cases} m(\xi, z) & \text{if } i(\xi, z) = 0 \\ c(\xi, z) & \text{otherwise.} \end{cases} \tag{2.9}$$

Note that although the right-hand of (2.7) is the function of $\xi_0, \xi_1, \ldots, \xi_{n-1}; T_0, T_1, \ldots, T_n$, it is denoted by $\Lambda(T_n)$ for the notational simplicity.

The total expected randomly discounted cost incurred under π, starting at time 0 in state (ξ, z) is given by

$$V_\pi(\xi, z) \equiv E\{\textstyle\sum_{n=1}^\infty K_n | \xi_0 = \xi, Z_0 = z\}. \tag{2.10}$$

Let

$$V^* \equiv \inf_{\pi\in\Pi} V_\pi. \tag{2.11}$$

Definition 2.3. If $\pi \in \Pi$ and $V_\pi = V^*$, then π is called optimal.

Assumption 2.4.
(a) $\gamma(z, \xi, x)$ is decreasing in z and x for any $\xi \in \Gamma$.
(b) The cost function m and c are in B^+, $0 < m(\xi, z) < m(\xi, \infty), 0 < c(\xi, z) < c(\xi, \infty)$ and $m(\xi, \infty) = c(\xi, \infty)$ for any $\xi \in \Gamma$.
(c) $H^\xi(t)$ has a continuous density function $h^\xi(t)$ for any $\xi \in \Gamma$.
(d) $\lambda \equiv \inf_{\xi\in\Gamma} \lambda(\xi) > 0$.

3. The total expected randomly discounted cost

In this section, we discuss the total expected randomly discounted cost over infinite horizon. First by the Proposition 2.2, we get the following lemma.

Lemma 3.1. For any $\xi \in \Gamma, z \in R_+, a \in [0, \infty]$ and $V \in B$

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$$E_{(\xi,z)}[e^{-\lambda(\xi)T_1}V(\xi_1,Z_1)|A_0(\xi,z)]$$

$$=\begin{cases}[V(\xi,\infty)-L_1V(\xi,z)]\Psi_1(\xi,a) \\ \quad +[E_\xi[V(\xi_1,\infty)]-L_2V(\xi,z)]\Psi_2(\xi,a) \\ \quad\quad +\int_0^1 V(\xi,zy)F_z^\xi(dy)e^{-\lambda(\xi)a}\Phi_3(\xi,a) & \text{if } A_0(\xi,z)=(a,0) \\ [V(\xi,\infty)-L_1V(\xi,z)]\Psi_1(\xi,a) \\ \quad +[E_\xi[V(\xi_1,\infty)]-L_2V(\xi,z)]\Psi_2(\xi,a) \\ \quad\quad +V(\xi,0)e^{-\lambda(\xi)a}\Phi_3(\xi,a) & \text{otherwise}\end{cases}$$

where
$$L_1V(\xi,z)\equiv\int_{R_+}[V(\xi,\infty)-V(\xi,z+x)]\gamma(z,\xi,x)G_z^\xi(dx)$$
$$L_2V(\xi,z)\equiv\int_\Gamma[V(\zeta,\infty)-V(\zeta,z)]\gamma(z,\zeta,0)Q(\xi,d\zeta)$$
$$E_\xi[V(\xi_1,\infty)]\equiv\int_\Gamma V(\zeta,\infty)Q(\xi,d\zeta)$$
$$\Psi_1(\xi,a)\equiv\int_0^a\int_0^t e^{-\lambda(\xi)u}H^\xi(du)\eta(\xi)e^{-\eta(\xi)t}dt+\int_0^a e^{-\lambda(\xi)u}H^\xi(du)e^{-\eta(\xi)a}$$
$$\Psi_2(\xi,a)\equiv\int_0^a\int_0^t \eta(\xi)e^{-(\lambda(\xi)+\eta(\xi))u}du H^\xi(dt)+\int_0^a \eta(\xi)e^{-(\lambda(\xi)+\eta(\xi))u}du\bar{H}^\xi(a).$$

Proof: For the case that $A_0(\xi,z)=(a,0)$, considering whether or not the sojourn time in the state (ξ,z) exceeds a and using S_1,S_2 defined in Proposition 2.2.1, we have that

$$\begin{aligned}&E_{(\xi,z)}[e^{-\lambda(\xi)T_1}V(\xi_1,Z_1)|A_0(\xi,z)=(a,0)]\\ &=E_{(\xi,z)}[e^{-\lambda(\xi)T_1}V(\xi_1,Z_1);S_1\le S_2;S_1\le s|A_0(\xi,z)=(a,0)]\\ &\quad +E_{(\xi,z)}[e^{-\lambda(\xi)T_1}V(\xi_1,Z_1);S_2\le S_1;S_2\le s|A_0(\xi,z)=(a,0)]\\ &\quad\quad +E_{(\xi,z)}[e^{-\lambda(\xi)T_1}V(\xi_1,Z_1);\min\{S_1,S_2\}\ge a|A_0(\xi,z)=(a,0)]\\ &=\int_0^a\int_0^t e^{-\lambda(\xi)u}\int_{R_+}[V(\xi,z+x)\gamma(z,\xi,x)+V(\xi,\infty)(1-\gamma(z,\xi,x))]\\ &\qquad\qquad \times G_z^\xi(dx)H^\xi(du)\eta(\xi)e^{-\eta t}dt\\ &\quad +\int_0^a\int_0^t e^{-\lambda(\xi)u}\int_\Gamma[V(\zeta,z)\gamma(z,\zeta,0)+V(\zeta,\infty)(1-\gamma(z,\zeta,0))]\\ &\qquad\qquad \times Q(\xi,d\zeta)\eta(\xi)e^{-\eta(\xi)u}duH^\xi(dt)\\ &\quad +\int_0^1 V(\xi,zy)F_z^\xi(dy)e^{-\lambda(\xi)a}\Phi_3(\xi,a).\end{aligned}$$

By rearranging the right-hand of the above equality, we can obtain lemma 3.1 when $A_0(\xi,z)=(a,0)$. The case that $A_0(\xi,z)=(a,1)$ can be proved by similar manner. □

Now, we define the following operators U_1,U_2, and U.

Definition 3.2. For any $V\in B,\xi\in\Gamma$, $z\in[0,\infty)$ and $a\in[0,\infty]$, let

(3.1) $$U_1(a)V(\xi,z)\equiv U_1(\xi,z,a,V)$$
$$\equiv(m(\xi,z)+\int_0^1 V(\xi,zy)F_z^\xi(dy))e^{-\lambda(\xi)a}\Phi_3(\xi,a)$$
$$+[V(\xi,\infty)-L_1V(\xi,z)]\Psi_1(\xi,a)+[E_\xi[V(\xi_1,\infty)]-L_2V(\xi,z)]\Psi_2(\xi,a)$$

(3.2) $$U_2(a)V(\xi,z)\equiv U_2(\xi,z,a,V)$$
$$\equiv(c(\xi,z)+V(\xi,0))e^{-\lambda(\xi)a}\Phi_3(\xi,a)$$
$$+[V(\xi,\infty)-L_1V(\xi,z)]\Psi_1(\xi,a)+[E_\xi[V(\xi_1,\infty)]-L_2V(\xi,z)]\Psi_2(\xi,a)$$

(3.3) $$UV(\xi,z)=\min\{\inf_{a\in[0,\infty]}U_1(a)V(\xi,z),\inf_{a\in[0,\infty]}U_2(a)V(\xi,z)\}$$

(3.4) $$UV(\xi,\infty)=U_2(0)V(\xi,\infty).$$

In the following, we first consider an operator U^ϵ on the restricted action space $R_\epsilon=[\epsilon,\infty]$ for any $\epsilon>0$, i.e. for $V\in B$

(3.5) $$U^\epsilon V(\xi,z)=\min\{\inf_{a\in R_\epsilon}U_1(a)V(\xi,z),\inf_{a\in R_\epsilon}U_2(a)V(\xi,z)\}.$$

Lemma 3.3. For fixed $\epsilon>0$, U^ϵ is a monotone contraction operator .

Proof: The monotonicity of U^ϵ is obvious from Lemma 3.1 and Definition 3.2. We prove the contraction property of U^ϵ. For any ξ,z and V, $U_i(a)V(\xi,z)(i=1,2)$ are bounded

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continuous function in a on $[\epsilon, \infty]$. Hence for $V, W \in B$, there exist $a_1^*(\xi, z), a_2^*(\xi, z) \in R_\epsilon$ satisfying the following equalities

$$\inf_{a \in R_\epsilon} U_1(a) V(\xi, z) = U_1(a_1^*(\xi, z)) V(\xi, z),$$

$$\inf_{a \in R_\epsilon} U_1(a) W(\xi, z) = U_1(a_2^*(\xi, z)) W(\xi, z).$$

Thus $\mid \inf_{a\in R_\epsilon} U_1(a)V(\xi,z) - \inf_{a\in R_\epsilon} U_1 W(\xi,z) \mid$
$=\mid U_1(a_1^*(\xi,z))V(\xi,z) - U_1(a_2^*(\xi,z))W(\xi,z) \mid$
$\leq \mid U_1(a_2^*(\xi,z))V(\xi,z) - U_1(a_2^*(\xi,z))W(\xi,z) \mid$
$\leq (\Psi_1(\xi, a_2^*(\xi,z)) + \Psi_2(\xi, a_2^*(\xi,z)) + e^{-\lambda(\xi)a_2^*(\xi,z)}\Phi_3(\xi, a_2^*(\xi,z))) \parallel V - W \parallel$
$\leq (\Phi_1(\xi, a_2^*(\xi,z)) + \Phi_2(\xi, a_2^*(\xi,z)) + e^{-\lambda(\xi)a_2^*(\xi,z)}\Phi_3(\xi, a_2^*(\xi,z))) \parallel V - W \parallel$
$\equiv \beta^\xi(a_2^*(\xi,z)) \parallel V - W \parallel$
where
$\beta^\xi(a_2^*(\xi,z)) = \Phi_1(\xi, a_2^*(\xi,z)) + \Phi_2(\xi, a_2^*(\xi,z)) + e^{-\lambda(\xi)a_2^*(\xi,z)}\Phi_3(\xi, a_2^*(\xi,z))$.
Since for any $\xi \in \Gamma$, $z \in R_+, \Phi_1(\xi, a_2^*(\xi,z)) + \Phi_2(\xi, a_2^*(\xi,z)) + \Phi_3(\xi, a_2^*(\xi,z)) = 1$, and for $a_2^*(\xi,z) \geq \epsilon$, $\sup_{\xi,z} \Phi_i(\xi, a_2^*(\xi,z)) \neq 0 (i = 1,2)$, $\sup_{\xi,z} \Phi_3(\xi, a_2^*(\xi,z)) \neq 1$ and $\sup_{\xi,z} e^{-\lambda(\xi)a_2^*(\xi,z)} < 1$, we have that $\beta_\epsilon \equiv \sup_{\xi,z} \beta^\xi(a_2^*(\xi,z)) < 1$. Therefore,

$$\parallel \inf_{a\in R_\epsilon} U_1(a)V - \inf_{a\in R_\epsilon} U_1(a)W \parallel \leq \beta_\epsilon \parallel V - W \parallel .$$

Similarly , there exists a $\acute{\beta}_\epsilon < 1$ such that

$$\parallel \inf_{a\in R_\epsilon} U_2(a)V - \inf_{a\in R_\epsilon} U_2(a)W \parallel \leq \acute{\beta}_\epsilon \parallel V - W \parallel .$$

Since

$$\mid U^\epsilon V(\xi,z) - U^\epsilon W(\xi,z) \mid \leq \max_{i=1,2}\{\mid \inf_{a\in R_\epsilon} U_i(a)V(\xi,z) - \inf_{a\in R_\epsilon} U_i(a)W(\xi,z) \mid\},$$

it follows that $\parallel UV - UW \parallel \leq \max\{\beta_\epsilon, \acute{\beta}_\epsilon\} \parallel V - W \parallel$. □

As U^ϵ is a monotone contraction operator, it has a unique fixed point $V_\epsilon^{**} \in B$. Now we discuss the properties of this fixed point. Using the operator U^ϵ, we define a mapping sequence $\{V_n\}_{n\geq 0}$ by

(3.6) $$V_0 = 0$$
$$V_n = U^\epsilon V_{n-1} \quad n \geq 1.$$

We have that $V_n \in B$ and V_n is non-negative function for $n \geq 0$.

Lemma 3.4. Assume that for any $\xi \in \Gamma, t \in R_+, G_z^\xi(\cdot)$ is stochastically increasing in z, then (i) $V_n \in B^+$,
(ii) $L_1 V_n(\xi,z)$ and $L_2 V_n(\xi,z)$ are decreasing in z.

Proof: By induction, we prove the assertions (i) and (ii). Since $V_0 = 0$ and

$$V_1(\xi,z) = \min\{\inf_{a\in R_\epsilon} m(\xi,z)e^{-\lambda(\xi)a}\Phi_3(\xi,a), \inf_{a\in R_\epsilon} c(\xi,z)e^{-\lambda(\xi)a}\Phi_3(\xi,a)\},$$

(i) and (ii) hold certainly when $n = 0, 1$. Suppose that (i) and (ii) are true for an integer n. Consider two cases for $z_1 \leq z_2$:
CASE 1: if $U^\epsilon V_n(\xi, z_2) = \inf_{a\in R_\epsilon} U_1(a)V_n(\xi, z_2)$, then
$V_{n+1}(\xi, z_2) - V_{n+1}(\xi, z_1) = \inf_{a\in R_\epsilon} U_1^\epsilon(a)V_n(\xi, z_2) - U^\epsilon V_n(\xi, z_1)$
$\geq \inf_{a\in R_\epsilon} U_1(a)V_n(\xi, z_2) - \inf_{a\in R_\epsilon} U_1(a)V_n(\xi, z_1)$

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$\geq \inf_{a\in R_\epsilon}\{U_1(a)V_n(\xi,z_2) - U_1(a)V_n(\xi,z_2)\}$
$= \inf_{a\in R_\epsilon}\{[m(\xi,z_2) - m(\xi,z_1) + \int_0^1 V_n(\xi,z_2y)F_{z_2}^{\xi}(dy)) - \int_0^1 V_n(\xi,z_1y)F_{z_1}^{\xi}(dy))]$
$\times e^{-\lambda(\xi)a}\Phi_3(\xi,a) + [L_1V_n(\xi,z_1) - L_1V_n(\xi,z_2)]\Psi_1(\xi,a)$
$+[L_2V_n(\xi,z_1) - L_2V_n(\xi,z_2)]\Psi_2(\xi,a)\}$
$\geq \inf_{a\in R_\epsilon}\{\ [m(\xi,z_2) - m(\xi,z_1)]e^{-\lambda(\xi)a}\Phi_3(\xi,a)\ \} \geq 0,$
where the third inequality follows from that $L_iV_n(\xi,z_1) - L_iV_n(\xi,z_2) \geq 0$ $(i=1,2)$, and
$\int_0^1 V_n(\xi,z_2y)F_{z_2}^{\xi}(dy)) - \int_0^1 V_n(\xi,z_1y)F_{z_1}^{\xi}(dy))$
$\geq \int_0^1 V_n(\xi,z_2y)F_{z_2}^{\xi}(dy)) - \int_0^1 V_n(\xi,z_2y)F_{z_1}^{\xi}(dy)) \geq 0$
since $V_n(\xi,z_2y) \geq V_n(\xi,z_1y)$ and $F_z^{\xi}(\cdot)$ is stochastically increasing in z.

CASE 2: if $U^\epsilon V_n(\xi,z_2) = \inf_{a\in R_\epsilon} U_2(a)V_n(\xi,z_2)$, similarly we have that
$V_{n+1}(\xi,z_2) - V_{n+1}(\xi,z_1)$
$\geq \inf_{a\in R_\epsilon}\{\ [c(\xi,z_2) - c(\xi,z_1)]e^{-\lambda(\xi)a}\Phi_3(\xi,a)\ \} \geq 0.$
Since $(V_{n+1}(\xi,\infty) - V_{n+1}(\xi,z+x))\gamma(z,\xi,x)$ is decreasing in z, and $G_z^{\xi}(\cdot)$ is stochastically increasing in z, $L_1V_{n+1}(\xi,z)$ and $L_2V_{n+1}(\xi,z)$ are decreasing in z. (i) and (ii) hold for $n+1$. These complete the proof. □

Corollary 3.5. For any fixed $\epsilon > 0$, we have that
(a) $V_\epsilon^{**} = \lim_{n\to\infty} V_n \in B^+$.
(b) $L_1V_\epsilon^{**}(\xi,z)$ and $L_2V_\epsilon^{**}(\xi,z)$ are decreasing in z for any $\xi \in \Gamma$.

Lemma 3.6. There exists a unique fixed point $V^{**} \in B^+$ for operator U.

Proof: For $\xi \in \Gamma, z \in R_+$, $V_\epsilon^{**}(\xi,z)$ is a non-negative decreasing function in ϵ. Let

(3.7) $V^{**}(\xi,z) = \lim_{\epsilon\to 0} V_\epsilon^{**} = \lim_{\epsilon\to 0}\min\{\inf_{a\in R_\epsilon} U_1(\xi,z,a,V), \inf_{a\in R_\epsilon} U_1(\xi,z,a,V)\}.$

Then V^{**} is a uniquely determined non-negative function, and $V^{**} \in B^+$ by Corollary 3.5. Moreover, since $U_i(\xi,z,a,V^{**})(i=1,2)$ are right-continuous function at $a=0$, i.e., $\lim_{a\downarrow 0} U_i(\xi,z,a,V^{**}) = U_i(\xi,z,0,V^{**})$, it follows that $\lim_{\epsilon\to 0}\inf_{a\in R_\epsilon} U_i(\xi,z,a,V^{**}) = \inf_{a\in[0,\infty]} U_i(\xi,z,a,V^{**})(i=1,2)$. From the monotonicities of U_1, U_2, we have that

$$\lim_{\epsilon\to 0}\min\{\inf_{a\in R_\epsilon} U_1(\xi,z,a,V_\epsilon^{**}), \inf_{a\in R_\epsilon} U_1(\xi,z,a,V_\epsilon^{**})\}$$

$$\geq \lim_{\epsilon\to 0}\min\{\inf_{a\in R_\epsilon} U_1(\xi,z,a,V^{**}), \inf_{a\in R_\epsilon} U_1(\xi,z,a,V^{**})\}$$

$$\geq \min\{\lim_{\epsilon\to 0}\inf_{a\in R_\epsilon} U_1(\xi,z,a,V^{**}), \lim_{\epsilon\to 0}\inf_{a\in R_\epsilon} U_1(\xi,z,a,V^{**})\}$$

$$= \min\{\inf_{a\in[0,\infty]} U_1(\xi,z,a,V^{**}), \inf_{a\in[0,\infty]} U_2(\xi,z,a,V^{**})\}. \tag{3.8}$$

First we consider the case that $UV^{**}(\xi,z) = \inf_{a\in(0,\infty]} U_1(\xi,z,a,V^{**})$. For any $\sigma > 0$, there exists an a_0 satisfying

$$\inf_{a\in[0,\infty]} U_1(\xi,z,a,V) > U_1(\xi,z,a_0,V) - \sigma. \tag{3.9}$$

Also by the monotone convergence theorem, we have that

$$\lim_{\epsilon\to 0} U_1(\xi,z,a_0,V_\epsilon^{**}) = U_1(\xi,z,a_0,V^{**}), \tag{3.10}$$
and
$$\lim_{\epsilon\to 0}\inf_{a\in R_\epsilon} U_1(\xi,z,a,V_\epsilon^{**}) \leq U_1(\xi,z,a_0,V^{**}) \tag{3.11}$$

since $a_0 \in R_\epsilon$ for $\epsilon \leq a_0$. Furthermore, by (3.10), we have that

$$\inf_{a\in[0,\infty]} U_1(\xi,z,a,V^{**}) > \lim_{\epsilon\to 0}\inf_{a\in R_\epsilon} U_1(\xi,z,a,V_\epsilon^{**}) - \sigma.$$

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As $\sigma \to 0$, it holds that

$$\inf_{a\in[0,\infty]} U_1(\xi, z, a, V^{**}) \geq \lim_{\epsilon\to 0} \inf_{a\in R_\epsilon} U_1(\xi, z, a, V_\epsilon^{**})$$

$$\geq \lim_{\epsilon\to 0} \min\{\inf_{a\in R_\epsilon} U_1(\xi, z, a, V_\epsilon^{**}), \inf_{a\in R_\epsilon} U_2(\xi, z, a, V_\epsilon^{**})\}. \tag{3.12}$$

From (3.8) and (3.12), we have that

$$\lim_{\epsilon\to 0} \min\{\inf_{a\in R_\epsilon} U_1(\xi, z, a, V_\epsilon^{**}), \inf_{a\in R_\epsilon} U_2(\xi, z, a, V_\epsilon^{**})\} = \inf_{a\in[0,\infty]} U_1(\xi, z, a, V^{**}).$$

That is

$$V^{**}(\xi, z) = \lim_{\epsilon\to 0} V_\epsilon^{**}(\xi, z) = \lim_{\epsilon\to 0} U^\epsilon V_\epsilon^{**}(\xi, z) = UV^{**}(\xi, z).$$

When $UV^{**}(\xi, z) = \inf_{a\in[0,\infty]} U_2(\xi, z, a, V^{**})$, the proof is similar. □

For any $\xi \in \Gamma$, let

$$\alpha(\xi) = \inf\{z, \quad m(\xi, z) - c(\xi, z) \geq 0\} \tag{3.13}$$

and $\inf\{\emptyset\} = \infty$.

Theorem 3.7. Assume that $m(\xi, z) - c(\xi, z)$ is increasing in z for $z \in [0, \alpha(\xi))$. Then there exists a function $f(\xi)$ satisfying

(i) $f(\xi) \leq \alpha(\xi)$ for $\xi \in \Gamma$.

(ii)

$$UV^{**}(\xi, z) = \begin{cases} \inf_{a\in[0,\infty]} U_1(a)V^{**}(\xi, z) & \text{if } z < f(\xi) \\ \inf_{a\in[0,\infty]} U_2(a)V^{**}(\xi, z) & \text{otherwise.} \end{cases}$$

Proof: For any fixed $\xi \in \Gamma$, let

$$f(\xi) = \inf\{z, \quad m(\xi, z) - c(\xi, z) + \textstyle\int_0^1 V^{**}(\xi, zy)F_z^\xi(dy) - V^{**}(\xi, 0) \geq 0\} \tag{3.14}$$

and $\inf\{\emptyset\} = \infty$.

(i) Since $V^{**}(\xi, z)$ is increasing in z, and $F_z^\xi(\cdot)$ is stochastically increasing in z, we have that $\int_0^1 V^{**}(\xi, zy)F_z^\xi(dy)$ is increasing in z, and

$$\textstyle\int_0^1 V^{**}(\xi, zy)F_z^\xi(dy) \geq \int_0^1 V^{**}(\xi, 0)F_z^\xi(dy) = V^{**}(\xi, 0).$$

Hence $m(\xi, z) - c(\xi, z) + \int_0^1 V^{**}(\xi, zy)F_z^\xi(dy) - V(\xi, 0) \geq m(\xi, z) - c(\xi, z)$.
We obtain the result (i).

(ii) For $z < f(\xi)$, we have $c(\xi, z) - m(\xi, z) + V^{**}(\xi, 0) - \int_0^1 V^{**}(\xi, zy)F_z^\xi(dy) \leq 0$.

$$\begin{aligned} &\inf_{a\in[0,\infty]} U_2(a)V^{**}(\xi, z) - \inf_{a\in[0,\infty]} U_1(a)V^{**}(\xi, z) \\ &\geq \inf_{a\in[0,\infty]}\{U_2(a)V^{**}(\xi, z) - U_1(a)V^{**}(\xi, z)\} \\ &= \inf_{a\in[0,\infty]}\{(c(\xi, z) - m(\xi, z) + V^{**}(\xi, 0) - \textstyle\int_0^1 V^{**}(\xi, zy)F_z^\xi(dy))e^{-\lambda(\xi)a}\Phi_3(\xi, a)\} \geq 0. \end{aligned}$$

Thus $\inf_{a\in[0,\infty]} U_2(a)V^{**}(\xi, z) \geq \inf_{a\in[0,\infty]} U_1(a)V^{**}(\xi, z)$. For $z \geq f(\xi)$, we have that

$$\begin{aligned} &\inf_{a\in[0,\infty]} U_1(a)V^{**}(\xi, z) - \inf_{a\in[0,\infty]} U_2(a)V^{**}(\xi, z) \\ &\geq \inf_{a\in[0,\infty]}\{(m(\xi, z) - c(\xi, z) + \textstyle\int_0^1 V^{**}(\xi, zy)F_z^\xi(dy) - V^{**}(\xi, 0))e^{-\lambda(\xi)a}\Phi_3(\xi, a)\} \geq 0. \end{aligned}$$

Thus $\inf_{a\in[0,\infty]} U_1(a)V^{**}(\xi, z) \geq \inf_{a\in[0,\infty]} U_2(a)V^{**}(\xi, z)$. These complete the proof of result (ii). □

Theorem 3.8. For $\xi \in \Gamma$, let $r^\xi(t)$ is the hazard rate associated with the distribution

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function $H^{\xi}(t)$. If $r^{\xi}(t)$ is monotonic function in t, then there exists a unique function $a^* = a^*(\xi, z)$ satisfying the following equality

$$UV^{**}(\xi, z) = \min\{\inf_{a\in[0,\infty]} U_1(a)V^{**}(\xi, z), \inf_{a\in[0,\infty]} U_2(a)V^{**}(\xi, z)\}, \tag{3.15}$$

and

$$a^*(\xi, z) = \begin{cases} a_1^*(\xi, z) & \text{if } z < f(\xi) \\ a_2^*(\xi, z) & \text{otherwise,} \end{cases} \tag{3.16}$$

where $a_1^*(\xi, z), a_2^*(\xi, z)$ are unique minimal solutions of the following equations respectively.

$$M(\xi, z, a, V^{**}) = (m(\xi, z) + \textstyle\int_0^1 V^{**}(\xi, zy)F_z^{\xi}(dy))(r^{\xi}(a) + \lambda(\xi) + \eta(\xi)), \tag{3.17}$$

$$M(\xi, z, a, V^{**}) = (c(\xi, z) + V^{**}(\xi, 0))(r^{\xi}(a) + \lambda(\xi) + \eta(\xi)) \tag{3.18}$$

where $M(\xi, z, a, V^{**})$

$$= [V^{**}(\xi, \infty) - L_1V^{**}(\xi, z)]r^{\xi}(a) + [E_{\xi}[V^{**}(\xi_1, \infty)] - L_2V^{**}(\xi, z)]\eta(\xi). \tag{3.19}$$

Proof: We have $V^{**}(\xi, z) = \inf_{a\in(0,\infty]} U_1(a)V^{**}(\xi, z)$ for $z < f(\xi)$. Differentiating with respect to a and rearranging, we obtain (3.17). By the monotonicity of $r^{\xi}(t)$, this minimal solution is unique. The proof of the case for $z \geq f(\xi)$ is similar. □

We can determinate concretely the minimal solutions $a_1^*(\xi, z)$ and $a_2^*(\xi, z)$ according to the monotonicity of $r^{\xi}(t)$. In the following, we give $a_1^*(\xi, z)$ and $a_2^*(\xi, z)$ when $r^{\xi}(t)$ is increasing function in t. The case that $r^{\xi}(t)$ is decreasing function in t can be discussed similarly. Let

$$h_1(\xi) = \inf\{z;\ \ V^{**}(\xi, \infty) - L_1V^{**}(\xi, z) - m(\xi, z) - \textstyle\int_0^1 V^{**}(\xi, zy)F_z^{\xi}(dy) \geq 0\},$$
$$h_2(\xi) = \inf\{z;\ \ (m(\xi, z) + \textstyle\int_0^1 V^{**}(\xi, zy)F_z^{\xi}(dy))(\eta(\xi) + \lambda(\xi)) - (E_{\xi}[V^{**}(\xi, \infty)] - L_2V^{**}(\xi, z))\eta(\xi) \geq 0\},$$

and

$$g_1(\xi) = \inf\{z;\ \ V^{**}(\xi, \infty) - L_1V^{**}(\xi, z) - c(\xi, z) - V^{**}(\xi, 0) \geq 0\},$$
$$g_2(\xi) = \inf\{z;\ \ (c(\xi, z) + V^{**}(\xi, 0))(\eta(\xi) + \lambda(\xi)) - (E_{\xi}[V^{**}(\xi, \infty)] - L_2V^{**}(\xi, z))\eta(\xi) \geq 0\}.$$

Corollary 3.9 If $r^{\xi}(t)$ is increasing function in t, we have that

$$a_1^*(\xi, z) =$$

$$\begin{cases} \inf_{a\in[0,\infty]}\{a;\ \ r^{\xi}(a) \leq \frac{(m(\xi,z)+\int_0^1 V^{**}(\xi,zy)F_z^{\xi}(dy))(\eta(\xi)+\lambda(\xi))-(E_{\xi}[V^{**}(\xi,\infty)]-L_2V^{**}(\xi,z))\eta(\xi)}{V^{**}(\xi,\infty)-L_1V^{**}(\xi,z)-m(\xi,z)-\int_0^1 V^{**}(\xi,zy)F_z^{\xi}(dy)}\} & \text{if } z < \min\{h_1(\xi), h_2(\xi)\} \\ 0 & \text{if } h_1(\xi) \leq z < h_2(\xi) \\ \infty & \text{if } h_2(\xi) \leq z < h_1(\xi) \\ \inf_{a\in[0,\infty]}\{a;\ \ r^{\xi}(a) \geq \frac{(m(\xi,z)+\int_0^1 V^{**}(\xi,zy)F_z^{\xi}(dy))(\eta(\xi)+\lambda(\xi))-(E_{\xi}[V^{**}(\xi,\infty)]-L_2V^{**}(\xi,z))\eta(\xi)}{V^{**}(\xi,\infty)-L_1V^{**}(\xi,z)-m(\xi,z)-\int_0^1 V^{**}(\xi,zy)F_z^{\xi}(dy)}\} & \text{if } \max\{h_1(\xi), h_2(\xi)\} \leq z \end{cases}$$

and $a_2^*(\xi, z) =$

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$$\begin{cases} \inf_{a\in[0,\infty]}\{a;\ \ r^{\xi}(a) \le \frac{(c(\xi,z)+V^{**}(\xi,0))(\eta(\xi)+\lambda(\xi))-(E_{\xi}[V^{**}(\xi,\infty)]-L_2V^{**}(\xi,z))\eta(\xi)}{V^{**}(\xi,\infty)-L_1V^{**}(\xi,z)-c(\xi,z)-V^{**}(\xi,0)}\} & \\ & \text{if } z < \min\{g_1(\xi), g_2(\xi)\} \\ 0 & \text{if } g_1(\xi) \le z < g_2(\xi) \\ \infty & \text{if } g_2(\xi) \le z < g_1(\xi) \\ \inf_{a\in[0,\infty]}\{a;\ \ r^{\xi}(a) \ge \frac{(c(\xi,z)+V^{**}(\xi,0))(\eta(\xi)+\lambda(\xi))-(E_{\xi}[V^{**}(\xi,\infty)]-L_2V^{**}(\xi,z))\eta(\xi)}{V^{**}(\xi,\infty)-L_1V^{**}(\xi,z)-c(\xi,z)-V^{**}(\xi,0)}\} & \\ & \text{if } \max\{g_1(\xi), g_2(\xi)\} \le z. \end{cases}$$

Corollary 3.10. If $c(\xi, z) = c(\xi)$ for $z \in R_+$, then $a_2^*(\xi, z)$ is decreasing in z.

Proof: From Corollary 3.5 (b), $V^{**}(\xi,\infty) - L_1V^{**}(\xi, z)$ and $E_{\xi}[V^{**}(\xi_1,\infty)] - L_2V^{**}(\xi, z)$ are increasing in z. We get $M(\xi, z, a, V^{**})$ is increasing in z. On the other hand, if $c(\xi, z) = c(\xi)$, the right-hand of (3.18) becomes $(c(\xi)+V^{**}(\xi,0))(r^{\xi}(a)+\lambda(\xi)+\eta(\xi))$ which is not dependent on z. Hence, the minimal solution $a_2^*(\xi, z)$ satisfying the equation (3.18) is decreasing in z. □

In the following, we examine the influences of the maintenance action and the environment state on the control-limit $f(\xi)$ defined in (3.14). $F_z^{\xi}(\cdot)$ is the distribution function of the discount rate $Y(\xi, z)$ if a maintenance action is executed at state (ξ, z). An extreme case is that $F_z^{\xi}(0) = P(Y(\xi, z) = 0) = 1$ for $z \in R_+$. This case means that every maintenance action restores the system to a new one. We have that $\int_0^1 V^{**}(\xi, zy)F_z^{\xi}(dy) = V^{**}(\xi, 0)$ and $f(\xi) = \alpha(\xi)$. Hence, $f(\xi) = \infty$ if $m(\xi, z) < c(\xi, z)$ for all z, i.e. it is always optimal to maintain the system. $f(\xi) = 0$ if $m(\xi, z) \ge c(\xi, z)$ for all z, i.e. it is always optimal to replace the system. For any $\xi \in \Gamma$, let

$$\beta(\xi) = \inf\{z, \quad m(\xi, z) - c(\xi, z) + V^{**}(\xi, z) - V^{**}(\xi, 0) \ge 0\}.$$

For a general distribution function $F_z^{\xi}(\cdot)$, we have the following theorem.

Theorem 3.11. (i) Let $f(\xi)$ be a control-limit associated with $F_z^{\xi}(\cdot)$, then
$\beta(\xi) \le f(\xi) \le \alpha(\xi)$ for $\xi \in \Gamma$.
(ii) Let $f_i(\xi)$ be control-limits associated with $F_{i,z}^{\xi}(\cdot)$ for $i = 1, 2$.
If $F_{1,z}^{\xi}(y) \le F_{2,z}^{\xi}(y)$ for $0 \le y < 1$, then $f_1(\xi) \ge f_2(\xi)$ for $\xi \in \Gamma$.

Proof: For part (i), we have that $0 \le \int_0^1 V^{**}(\xi, zy)F_z^{\xi}(dy) - V^{**}(\xi, 0) \le V^{**}(\xi, z) - V^{**}(\xi, 0)$, and for part (ii), $\int_0^1 V^{**}(\xi, zy)F_{1,z}^{\xi}(dy) \le \int_0^1 V^{**}(\xi, zy)F_{2,z}^{\xi}(dy)$. From definition (3.14) of $f(\cdot)$, these lead to the desired results. □

In general, influences of the environment are complex because changes of the environment influence simultaneously the shock arrival, shock magnitude and the failure rate. In some cases, it is difficult to compare influencing affects of two environment. Let $\xi_1, \xi_2 \in \Gamma$, for instance, $H^{\xi_1}(\cdot) \ge H^{\xi_2}(\cdot)$, and $G_z^{\xi_1}(\cdot) \ge G_2^{\xi_2}(\cdot)$ for $z \in [0, \infty)$. Roughly speaking, these imply that at state ξ_1, the shock arrival is faster than that at state ξ_2, while shock magnitude is smaller than that at state ξ_2. So that, we can not appreciate simply which of the states ξ_1 and ξ_2 is a better environment to the system. Here, we consider a particular case as follows.

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For $\xi \in \Gamma$, let $H^{\xi}(\cdot) = H(\cdot)$, $\lambda(\xi) = \lambda, \eta(\xi) = \eta$. We introduce an order $\prec$ on the state space Γ. For $\xi_1 \prec \xi_2$, we refer to as the following

(i) $1 - \gamma(z, \xi_1, x) \leq 1 - \gamma(z, \xi_2, x)$,
(ii) $G_z^{\xi_1}(\cdot) \geq G_z^{\xi_2}(\cdot)$, and $Q(\xi_1, \cdot) \geq Q(\xi_2, \cdot)$

for $z, x \in R_+$.

The meaning of (i) is obvious. The (ii) means that distribution functions $G_z^{\xi}(\cdot)$ and $Q(\xi, \cdot)$ are stochastically increasing in order $\prec$. In this case, we call ξ_1 is a better environment than ξ_2 to the system. If $m(\xi, z), c(\xi, z)$ are increasing in order $\prec$, similar to the proof of Theorem 3.4, we also have $V^{**}(\xi, z)$ is increasing in order $\prec$, and $L_1V_n(\xi, z)$, $L_2V_n(\xi, z)$ are decreasing in order $\prec$. Furthermore, suppose the environment state restores the initial state ξ_0 when the system is replaced (this corresponds to the case where the environment is an internal factor of the system). We have that

$$f(\xi) = \inf\{z, \quad m(\xi, z) - c(\xi, z) + \textstyle\int_0^1 V^{**}(\xi, zy)F_z^{\xi}(dy) - V^{**}(\xi_0, 0) \geq 0\}.$$

Corollary 3.12. (i) If $m(\xi, z) - c(\xi, z)$ is increasing in order $\prec$, then for $\xi_1 \prec \xi_2$, $f(\xi_1) \geq f(\xi_2)$.
(ii) If $c(\xi, z) = c(z)$, then for $\xi_1 \prec \xi_2$, $a_2^*(\xi_1, z) \geq a_2^*(\xi_2, z)$.

Remark 1. Note that we do not require the environment process $\{\xi_t\}_{t\geq 0}$ be an increasing process in order $\prec$. This Corollary shows that the control-limit corresponding to a worse environment is lower. In this case, the system may be replaced early. For a general state space Γ without order, the control-limit $f(\cdot)$ can be taken as a criterion function. That is, if $f(\xi_1) \geq f(\xi_2)$, we can think that ξ_1 is a better environment then ξ_2.

4. Optimal Maintenance-Replacement Policy

Let $A^* \in \mathbf{A}$ be a control-limit policy defined by

$$A^*(\xi, z) = \begin{cases} (a_1^*(\xi, z), 0) & \text{if } z < f(\xi) \\ (a_2^*(\xi, z), 1) & \text{otherwise,} \end{cases} \tag{4.1}$$

where $f(\xi)$ is defined in (3.14), and $a_1^*(\xi, z), a_2^*(\xi, z)$ are respectively the minimal solutions of the equations (3.17) and (3.18), then $A^*(\xi, z)$ exists and is uniquely determined. Let $\pi^* = (A^*, A^*, \ldots)$, then π^* presents such a maintenance-replacement policy: at decision point T_n, the decision is $A_n(\xi_n, Z_n) = A^*(\xi_n, Z_n)$; if $Z_n < f(\xi_n)$, and the sojoun time at state (ξ_n, Z_n) exceeds $a_1^*(\xi_n, Z_n)$, we maintain the system at time $T_n + a_1^*(\xi_n, Z_n)$; if $Z_n \geq f(\xi_n)$, and the sojoun time at state (ξ_n, Z_n) exceeds $a_2^*(\xi_n, Z_n)$, we replace the system at time $T_n + a_2^*(\xi_n, Z_n)$. We will prove that π^* is an optimal replacement policy . For any $\pi \in \Pi$, let

$$N^{\pi} \equiv \inf\{n \mid i_n(\xi_n, Z_n) = 1\} \tag{4.2}$$
$$N(t) \equiv \textstyle\sum_{n\geq 0} I_{\{T_n \leq t\}}. \tag{4.3}$$

Then, $T_{N^{\pi}}$ is the first replacement time of the system under π, and $N(t)$ is a point process corresponding to the stationary Markov renewal process $\{\xi_n, Z_n, T_n\}_{n\geq 0}$. Using $T_{N^{\pi}}$ and $N(t)$, we define the operator $H_{N^{\pi}}$ on B by

(4.4) $\quad H_{N^{\pi}}V(\xi, z)$

$$\equiv E_{(\xi,z)}[\int_{0+}^{T_{N^{\pi}}-} e^{-\Lambda(t)}m(\xi_t, Z^{\pi}(t))dN(t) + e^{-\Lambda(T_{N^{\pi}})}(c(\xi_{N^{\pi}}, Z_{N^{\pi}}) + V(\xi_{N^{\pi}}, 0))]$$

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where $\Lambda(t) = \Lambda(T_n)$ if $T_n \leq t < T_{n+1}$, $n \geq 0$.

Remark 2.

(1) $H_{N^\pi}V(\xi, z)$ can be interpreted as follows: let V be the 'remaining cost' , that is, we have to pay the discounted amount $e^{-\Lambda(t)}V(\xi, z)$ if the process is stopped at time t in state (ξ, z). After the execution of a replacement action the system moves immediately into the state $z = 0$ and the environment state does not change. Employing the policy π, we have that the replacement causes the first cost $\int_{0+}^{T_{N^\pi}-} e^{-\Lambda(t)}m(\xi_t, Z^\pi(t))dN(t) + e^{-\Lambda(T_{N^\pi})}c(\xi_{N^\pi}, Z_{N^\pi})$ which is equal to $\sum_{i=1}^{N^\pi-1} e^{-\Lambda(T_i)}m(\xi_i, Z_i) + e^{-\Lambda(T_{N^\pi})}\, c(\xi_{N^\pi}, Z_{N^\pi})$, and after that there remain cost $e^{-\Lambda(T_{N^\pi})}V(\xi_{N^\pi}, 0)$. So that $H_{N^\pi}V(\xi, z)$ means the expected randomly discounted cost of the first replacement under π, starting at time 0 in state (ξ, z).

(2) By Proposition 2.1, the process $\{\xi_n, Z_n, T_n\}_{n\geq 0}$ is a stationary Markov renewal process under a stationary policy π. Since $ET_{N^\pi} \leq E\delta < \infty$, H_{N^π} is well-defined.

The expected randomly discounted costs incurred under π until n-th replacement can be given by

(4.5) $$V_\pi^n \equiv H_{N_1^\pi} \dots H_{N_n^\pi} V_0,$$

where the terminal cost function V_0 is set to be 0. Let

(4.6) $$u_n \equiv \inf_\pi V_\pi^n$$

(4.7) $$u_\infty \equiv \lim_{n\to\infty} u_n.$$

Lemma 4.1. (a) $\lim_{n\to\infty} V_\pi^n = V_\pi$.
(b) $V^* \geq u_\infty$.
(c) $V_{\pi^*} = V^{**}$.

Proof: (a) For every $n \geq 0$, there is an integer $m \geq n$ such that

$$E_{(\xi,z)}[\sum_{i=1}^{m} K_n] \leq H_{N_1^\pi} \dots H_{N_n^\pi} V_0 \leq E_{(\xi,z)}[\sum_{i=1}^{m+1} K_n].$$

Since $V_\pi \in B$ for any $\pi \in \Pi$, we get that $V_\pi \leq \lim_{n\to\infty} V_\pi^n \leq V_\pi$.
(b) By $V_\pi^n \geq u_n$ and (a), we have $\lim_{n\to\infty} V_\pi^n \geq \lim_{n\to\infty} u_n$, which yields $\inf_\pi V_\pi \geq u_\infty$.
(c) Under the policy π^*, we have that

$$\begin{aligned} H_{N^{\pi^*}}V_0(\xi, z) &= \sum_{n=1}^{\infty} E_{(\xi,z)}[\int_{0+}^{T_{N^\pi}-} e^{-\Lambda(t)}m(\xi_t, Z^\pi(t))dN(t) + e^{-\Lambda(T_{N^\pi})}(c(\xi_{N^\pi}, Z_{N^\pi}) \\ &\qquad + V(\xi_{N^\pi}, 0))|N^{\pi^*} = n]P_{(\xi,z)}(N^{\pi^*} = n) \\ &= \sum_{n=1}^{\infty} U^n V_0(\xi, z) P_{(\xi,z)}(N^{\pi^*} = n) \\ &= E_{(\xi,z)}[U^{N^{\pi^*}} V_0], \end{aligned}$$

and

$$\begin{aligned} H_{N_1^{\pi^*}} H_{N_2^{\pi^*}} V_0(\xi, z) &= E_{(\xi,z)}[U^{N_1^{\pi^*}} E_{(\xi,z)}[U^{N_2^{\pi^*}} V_0]] \\ &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} U^{n+m} V_0(\xi, z) P_{(\xi,z)}(N_1^{\pi^*} = n) P_{(\xi,z)}(N_2^{\pi^*} = m) \end{aligned}$$

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$$= E_{(\xi,z)}[U^{N_1^{\pi^*}+N_2^{\pi^*}}V_0].$$

By induction, we have $H_{N_1^{\pi^*}} \ldots H_{N_n^{\pi^*}} V_0 = E[U^{N_1^{\pi^*}+\ldots+N_n^{\pi^*}} V_0]$, and $n \leq N_1^{\pi^*} + \ldots + N_n^{\pi^*}$, a.s.. Thus, $P(\lim_{n\to\infty} N_1^{\pi^*} + \ldots + N_n^{\pi^*} = \infty) = 1$ and $U^{N_1^{\pi^*}+\ldots+N_n^{\pi^*}} V_0 \to V^{**}$, a.s. $(n \to \infty)$. Noting $U^n V_0$ is increasing in n, we get that $\lim_{n\to\infty} E[U^{N_1^{\pi^*}+\ldots+N_n^{\pi^*}} V_0] = V^{**}$, i.e., $V_{\pi^*} = V^*$. □

Lemma 4.2. $u_\infty \geq V^*$

Proof: For any $n \geq 1$ and $\pi \in \Pi$, let $m = N_1^\pi +, \ldots, +N_n^\pi$. Then, $V_\pi^n \geq E[U^{m-1}V_0]$ and $u_n = \inf_\pi V_\pi^n \geq E[U^{m-1}V_0]$. Letting $n \to \infty$, we have $u_\infty \geq V^*$. □

Theorem 4.3. π^* is an optimal stationary maintenance-replacement policy.

Proof: From Lemma 4.1 and 4.2, $u_\infty \geq V^{**} = V_{\pi^*} \geq V^* \geq u_\infty$, we get that $V_{\pi^*} = V^*$. Therefore, π^* is an optimal stationary policy with the control-limit type. □

5. Application

In this section, we give two applications for the optimal maintenance-replacement problem of systems.

(1) Consider a network system composed of a main-system and N sub-systems. Such systems constitute the vast majority of most industry's capital. For example, communication network systems, computer network systems, etc. The behavior of the main-system may be influenced by environment changes such as temperature, season or sub-system's state, etc. So that it is necessary to consider these influences when we decide an optimal maintenance-replacement policy for the main-system. Here, assume that the network system be new at time $t = 0$, and the lifetime distributions of the sub-system be independent identical exponential distribution $F_1(t) = 1 - e^{-\mu t}$ for $t \geq 0$. Every failed sub-system is repaired and the repair time is a random variable with the distribution $F_2(t) = 1 - e^{-\lambda t}$ for $t \geq 0$. There is only one repairman and the sub-system repaired is as good as new. We take the process $\{\xi(t)\}_{t\geq 0}$, the number of the functioning sub-system at time t, as the environment of the main-system. $\xi(t)$ is a Markov process with the state space $\Gamma = \{0, 1, \ldots, N\}$ and the initial state $\xi(0) = N$. Let $\{\omega_n\}_{n\geq 0}$ be the transition times of $\{\xi(t)\}_{t\geq 0}$, i.e., ω_n is a time at which a sub-system fails or a failed sub-system is restored to functioning. The Markov transition kernel of the process $\{\xi(\omega_n), \omega_n\}_{n\geq 0}$ is

$$P(\xi(\omega_{n+1}) = j, \omega_{n+1} - \omega_n \leq t \ \ |\xi(\omega_n) = i)$$

$$= \begin{cases} \frac{i\mu}{i\mu+\lambda}(1 - e^{-(i\mu+\lambda)t}) & i = 1, 2, \ldots, N-1; j = i-1 \\ \frac{\lambda}{i\mu+\lambda}(1 - e^{-(i\mu+\lambda)t}) & i = 1, 2, \ldots, N-1; j = i+1 \\ 1 - e^{-N\mu t} & i = N; j = N-1 \\ 1 - e^{-\lambda t} & i = 0; j = 1 \\ 0 & \text{otherwise.} \end{cases}$$

$$\equiv p_{ij}(1 - e^{-\eta(i)t})$$

where $\eta(i) = i\mu + \delta_{iN}\lambda$, and $\delta_{iN} = 0$ if $i = N$ and 1 otherwise.

The main-system is subject to a sequence of randomly occurring shocks. Shock arrivals and magnitudes depend on the accumulated damage level of the main system, and the number of the functioning sub-system. The process $Z(t)$ defined by (2.3) represents the damage level of the main-system. Upon failure of the main-system, it has to be replaced

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and a cost $C + C_0$ is incurred. It may be maintained by a cost $m(i,z)$ or preventively replaced by a cost C before failure. For such a network system, using Theorem 3.7 and 3.8, we can derive an optimal maintenance-replacement policy for the main-system. For example, the control-limit $f(i)$ can be obtained by

$$f(i) = \inf\{z, \quad m(i,z) - C + \int_0^1 V^{**}(i, zy) F_z^i(dy) - V^{**}(i,0) \geq 0\}.$$

(2) Consider an aircraft system subject to shocks. These shocks greatly depend on the aircraft flight state such as flight speed and altitude, and weather changes. Take these as the environment of the aircraft and model their changes as a Markov chain $\{\xi(t)\}_{t\geq 0}$ with a state space $\Gamma = \{s_0, s_1, s_2, \ldots, s_N\}$, where s_0 is a no flight state, s_1 upraising state, s_2 downfall state, and $s_i(i > 2)$ upstairs state at speed s_i. Corresponding to every state s_i, we have a maintenance action set M_i. For example, we can replace deteriorating units in the state s_0, check and adjust the aircraft in the state s_i, etc. Since the aircraft can be replaced only in the state s_0, we have an optimal maintenance-replacement policy as follows.

$$A^*(s_i, z) = (a_1^*(s_i, z), 0_{M_i}) \qquad \text{for} \quad i \geq 1$$

$$A^*(s_0, z) = \begin{cases} (a_1^*(s_0, z), 0_{M_0}) & \text{if } z < f(s_0) \\ (a_2^*(s_0, z), 1) & \text{otherwise,} \end{cases}$$

where 0_{M_i} represents maintenance actions taken in set M_i.

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References

[1] Bergman, B.: Optimal replacement under a general failure model. *Adv. Appl. Prob.*, Vol.10 (1978), 431-451.
[2] Feldman, R. M.: Optimal replacement with semi-Markov shock models. *J. Appl. Prob.*, Vol.13 (1976), 108-117.
[3] Feldman, R. M.: Optimal replacement with semi-Markov shock models using discounted cost. *Math. Opns. Res.*, Vol.2 (1977), 78-90.
[4] Feldman, R. M. and Joo. N. Y.: A state-age dependent policy for shock process. *Stochastic Model*, Vol.1 (1985), 53-76.
[5] Gottlieb, G.: Optimal replacement for shock models with general failure rate. *Opns. Res.*, Vol.30 (1982), 82-92.
[6] Posner, M. J. M. and Zuckerman, D.: A replacement model for an additive damage model with restoration. *Opns. Res. Letters*, Vol.3 (1984), 141-148.
[7] Posner, M. J. M. and Zuckerman, D.: Semi-Markov shock models with additive damage. *Adv. Appl. Prob.*, Vol.18 (1986), 772-790.
[8] Richard, F. S. R.: Conditional Poisson processes. *J. Appl. Prob.*, Vol.9 (1972), 288-302.
[9] Schäl, M.: Condition for optimality in dynamic programming and for the limit of n-stage optimal policies to be optimal. *Zeitschrift fuer Wahrscheinlichkeitstheorie und verwandte Gebiete*, Vol.32 (1975), 179-196.
[10] Siedersleben, J.: Dynamically optimized replacement with a Markovian renewal process. *J. Appl. Prob.*, Vol.18 (1981), 641-651.
[11] Taylor, H. M.: Optimal replacement under additive damage and other failure

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models. *Naval Res. Logist. Quart.*, Vol.22 (1975), 1-18.

[12] Valdez-Flores, C. and Feldman, R. M.: A survey of preventive maintenance models for stochastically deteriorating single-unit systems. *Naval Res. Logist. Quart.*, Vol.36 (1989), 419-446.

[13] Waldmann, K.-H.: Optimal replacement under additive damage in randomly varying environment. *Naval Res. Logist. Quart.*, Vol.30 (1983), 377-386.

[14] Zuckerman, D.: Optimal stopping in a semi-Markov shock model. *J. Appl. Prob.*, Vol.15 (1978), 629-634.

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