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ON THE EQUIVALENCY OF BALANCEDNESS AND STABILITY IN EFFECTIVITY FUNCTION GAMES

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Abstract In this paper we first introduce effectivity functions and some of their properties, especially balancedness. By using a specific characteristic function which enables us to transform a game in the effectivity function form into that of the characteristic function form, we show that balancedness of the effectivity functions is sufficient for the stability, *i.e.*, the existence of the core whatever preference ordering each player has. Our main result states that balancedness is a necessary and sufficient condition for the stability as long as the effectivity functions satisfy anonymity and neutraility.

1. Introduction

This paper deals with the problem of the existence of the core in a social choice theory. Consider a **society** N consisting of n players. Each player has a **preference ordering** over A, the set of m possible alternatives. A **social choice function**(SCF), which is a rule of aggregating the preference orderings of all players' to choose some alternatives socially desirable (*e.g.*, the majority voting rule, the Borda voting rule, *etc.*) is given.

Some players may form a **coalition** S in order to reflect their preferences in the group to the social choice when doing such a behaviour is more beneficial than acting by himself. Corresponding to each SCF, a mapping E from every coalition to a family of subsets of A, called an **effectivity function**, is defined. For every B, subset of alternatives $B \in E(S)$ implies that the coalition S can restrict the result of social choice within B. A triple (A, E, R^N) is called an **n-person cooperative game in the effectivity function form** — a game in which the society chooses some alternatives from A through the preference ordering of each player under the rule prescribed in the effectivity function. An alternative $\tilde{a} \in A$ is said to be **dominated** by the coalition S if there exists $B \in E(S)$ such that every members of S prefers strictly any alternative of B to $\tilde{a}$. The **core** is the set of all alternatives which cannot be dominated by any coalition.

An effectivity function is called **stable** when the existence of the corresponding core is guaranteed whatever preference ordering each player has. Demange [3] showed that any core of game with a **strictly stable** effectivity function, including **convex** one, is nonmanipulable in an optimistic sence. This implies that if the final social choice belongs to the core given a preference profile, then all players are convinced to accept it without complaint. Meanwhile, if there exists preference orderings under which the core is empty, the corresesopnding SCF is considered to be incomplete. Therefore, it is important to show the stability of the effectivity function. Actually, several approaches to this problem has been attempted. The pioneering work was made by Moulin and Peleg [11]. They proved that an **additional** effectivity function is always stable. Peleg [12] showed that a **convex** effectivity function is also stable. Successively, it was shown that **acyclicity** is a necessary and sufficient condition for the stability of the effectivity function by Keiding [7]. Following this work, we shall try to investigate the relation between **balancedness** of the effectivity function and stability.

The concept of balancedness has been known as one of the typical conditions which gurantee the existence of the core in a game with a characteristic function. Indeed, in 1962 Bondareva [2] found that balancedness is necessary and sufficient condition for the existence of the core of a game with side-payments by using the duality theorem of linear programming. Following this work, Shapley [14] derived the relaxed condition of balancedness from Bondareva's theorem. In 1967, Scarf [13] developed an elegant procedure to obtain an element of the core of any balanced game without side-payments by applying the Lemke's complimentarity method. In this paper we shall extend these results to the game with an effectivity function. By considering a special type of the characteristic function used by Peleg in [12], we can easily transform a game in the effectivity function form into that of the characteristic function form. We first show that Scarf's theorem is applicable in general to our scheme, which enables us to obtain an element of the core of social choice problem systematically. Furthermore, we shall prove that balancedness is a necessary and sufficient condition for stability whenever our effectivity function satisfies anonymity and neutrality, of which precise definitions will be given in the succeding section.

2. Notation

Let $N = \{1, \cdots, n\}, n \geqq 2$ be the set of **players** (called a **society**) and $A = \{a_1, \ldots, a_m\}$, $m \geqq 2$ be the set of **alternatives**. Any nonempty subset S of N is called a **coalition**. $P(D)$ denotes a family of all nonempty subsets of D and $P^2(D) = P(P(D))$. Each player i in N is assumed to have a nonnegative real-valued **utility function** $u^i : A \rightarrow \Omega^1_+$ [†1]. U^i denotes the set of all feasible utility functions of player i. A **utility profile** is a combination of all players' utility functions, written by $\boldsymbol{u}^N = (u^1, u^2, \cdots, u^n) \in U^N$, where $U^N = \Pi_{i \in N} U^i$ is the Cartesian product of U^i over the society. A **social choice function**(SCF) $F : U^N \rightarrow P(A)$ is a mapping by which socially desirable alternatives are determined through the preference orderings of all players. Such a function is assumed to be given throughout the paper.

3. Definitions and Basic Properties

First, we shall introduce the concept of effectivity functions with some of their properties and next, define the core of an n-person cooperative game in the effectivity function form.

Definition 3.1 *An* **effectivity function** *is a mapping* $E : P(N) \rightarrow P^2(A)$ *which satisfies the following conditions:*

(1) $A \in E(S)$ *for every* $S \in P(N)$,

(2) $B \in E(N)$ *for every* $B \in P(A)$.

An effectivity function which assignes to every coalition a family of subsets of alternatives must satisfy that (1) any coalition can enforce A, the set of all alternatives (which is same to say that it is always possible for every coalition to make no use of its power) and (2) the society as a collectivity can always enforce any subset of alternatives (namely, it has mighty power to exclude any undesirable alternatives).

We shall call a triple $(A, E, \boldsymbol{u}^N)$ an **n-person cooperative game in the effectivity function form**, where A is the set of alternatives, E is an effectivity function, and $\boldsymbol{u}$ is a given utility profile. $(A, E, \boldsymbol{u}^N)$ is a game in which a society chooses some alternatives from A through the utility profile under the rule prescribed in the effectivity function. And every player can try to form coalitions if he thinks that it is more beneficial to him than acts alone. Such a game makes us possible to treat the topics about the core in a scheme of social choice theory. Next, let us define the core of the game $(A, E, \boldsymbol{u}^N)$.

[†1] $\Omega^1_+ \equiv [0, \infty)$

Definition 3.2 *Let $E : P(N) \to P^2(A)$ be an effectivity function and let a utility profile $\boldsymbol{u}^N$ be given. Futhermore, let $B \in P(A)$ and $a \in A \backslash B$*[†2]*. B dominates an alternative a via a coalition $S \in P(N)$, written by $B dom(\boldsymbol{u}^N, S)a$, if $B \in E(S), u^i(B) > u^i(a)$*[†3] *for every $i \in S$.*

The core is defined as follows:

$$C(A, E, \boldsymbol{u}^N) = \{a \in A \mid \text{there exists no } S \in P(N) \text{ and no } B \in P(A) \text{ such that } B dom(\boldsymbol{u}^N, S)a\}.$$

Definition 3.3 *A function $E : P(N) \to P^2(A)$ is* **stable**, *if and only if*

$$\text{for every } \boldsymbol{u}^N \in U^N,\ C(A, E, \boldsymbol{u}^N) \neq \emptyset.$$

An effectivity function is called stable when the existence of the core is guaranteed whatever utility function each player has. That is to say, the core corresponding to each logically feasible preference ordering over A should be nonempty[†4].

Definition 3.4 *A function $E : P(N) \to P^2(A)$ is called* **monotone**, *if and only if for every $S, S' \in P(N)$ and every $B, B' \in P(A)$,*

$$B \in E(S), B' \supset B \text{ and } S' \supset S \Rightarrow B' \in E(S').$$

Monotonicity means that when some players join a coalition, the coalition maintains its previous power, and that every coalition is allowed to be not fully influencing the social choice.

Definition 3.5 *Let $E : P(N) \to P^2(A)$ be an effectivity function. Let $B \in P(A), S \in P(N)$ and $B \in E(S)$. The* **monotonic cover** *E^m of E is defined as follows:*

$$B \in E^m(S) \Leftrightarrow \text{there exists } B' \text{ and } S' \text{ such that } B' \subset B, S' \subset S \text{ and } B' \in E(S').$$

It immediately follows that E^m is a monotone effectivitity function.

Theorem 3.6 (Peleg[12, Lemma 6.3.4.], 1983) *Let E^m be the monotonic cover of an effectivity function E. Then, for every $\boldsymbol{u}^N \in U^N$, $C(A, E^m, \boldsymbol{u}^N) = C(A, E, \boldsymbol{u}^N)$.*

This theorem permits us to assume, without loss of generality, that E is monotone. Hereafter, instead of the effectivity function itself, we shall use its monotonic cover implicitly as long as the core of the game in the effectivity function form is concerned.

4. Balancedness

In this section, first, we shall briefly enumerate the definition of a cooperative game in the characteristic function form and discuss some related issues about its balancedness. Next, with these backgrounds, we shall define balancedness of an effectivity function and introduce a specific characteristic function $v^{E,\boldsymbol{u}^N}$ which can be obtained from the effectivity function given and enables us to transform a game in the effectivity function form into that of characteristic function form. It will be shown that the balancedness of the effectivity function is equivalent to that of a game in the characteristic function form with $v^{E,\boldsymbol{u}^N}$. Finally, we shall prove that if an effectivity function is balanced, then it is stable, *i.e.*, the core corresponding to any feasible preference profile always exists.

Definition 4.1 *An* **n-person cooperative game without side-payment** *is a pair (N, v), where N is a society and $v : P(N) \to \Omega_+^N$*[†5] *(called a* **characteristic function***) which satisfies the following conditions:*

[†2] $a \in A \backslash B \Leftrightarrow a \in A$ and $a \notin B$

[†3] $u^i(B) > u^i(a) \Leftrightarrow u^i(b) > u^i(a)$ for $^\forall b \in B$

[†4] Let $|A| = m$ and $|N| = n$, where $|X|$ denotes the cardinarity of X. Assume that every player has a strict utility over A which admits no tie between any two different alternatives. Then, the total number of every possible utility profile is $(m!)^n$ and thus, the stability of the effectivity function requires that the same number of the cores be guranteed to exist.

[†5] Ω_+^N denotes the nonnegative orthant of dimension n.

(1) $v(S) \neq \emptyset$ *for every* $S \in P(N)$,
(2) $v(S)$ *is closed for every* $S \in P(N)$,
(3) $\boldsymbol{x} \in v(S), \boldsymbol{y} \in \Omega_+^N$ *and* $\boldsymbol{x}^S \geqq \boldsymbol{y}^S \Rightarrow \boldsymbol{y} \in v(S)$,
(4) for every $S \in P(N)$, $\overline{v}(S) = \{\boldsymbol{x}^S \mid \boldsymbol{x} \in v(S)\}$ *is bounded.*

Note that $\boldsymbol{x}^S$ is a vector obtained by projecting $\boldsymbol{x} \in \Omega_+^N$ onto the nonnegative orthant of dimension equal to the number of players in S, *i.e.*, $S = \{i_1, \ldots, i_s\} \Rightarrow \boldsymbol{x}^S = (x_{i_1}, \ldots, x_{i_s}) \in \Omega_+^S$ and $v(S) = \overline{v}(S) \times \Omega_+^{N \setminus S}$. For $\boldsymbol{x},\boldsymbol{y} \in \Omega_+^S$, $\boldsymbol{x} \geqq (>) \boldsymbol{y}$ means that $x_i \geqq (>) y_i$ for every $i \in S$. Next, we shall define the core of (N, v).

Definition 4.2 *Let* $\boldsymbol{y} \in v(N)$. $\boldsymbol{x}$ *dominates* $\boldsymbol{y}$ *via* S, *written by* $\boldsymbol{x}\,dom(S)\boldsymbol{y}$, *if there exists* $\boldsymbol{x} \in v(S)$ *such that* $\boldsymbol{x}^S > \boldsymbol{y}^S$.

The core is defined as follows:

$C(N, v) = \{\boldsymbol{x} \in v(N) \mid$ *there exists no* $S \in P(N)$ *and no* $\boldsymbol{z} \in v(S)$ *such that* $\boldsymbol{z}\,dom(S)\boldsymbol{x}\}$.

Note that $v(N)$ is the set of all feasible vectors for the grand coalition N. Thus, the core of (N, v) is the set of all undominated feasible vectors, *i.e.*, $\boldsymbol{x} \in C(N, v)$ means that it is impossible for any coalition to block $\boldsymbol{x}$.

Definition 4.3 $\mathcal{S} = \{S_j\}_{j \in K}, K = \{1, 2, \cdots, k\}$ *is called a* **balanced collection**, *if it satisfies the following condition:*

$$\text{there exists } \boldsymbol{\delta} \in \Omega_+^K \text{ such that } \sum_{j \in K(i)} \delta_j = 1 \text{ for every } i \in N,$$

where $\boldsymbol{\delta} = (\delta_1, \ldots, \delta_k)^T$ [†6] *and* $K(i) = \{j \in K \mid i \in S_j$ *and* $S_j \in \mathcal{S}\}$.

The concept of the balanced collection can be rewritten in the matrix version. Let $\boldsymbol{e} = (1, 1, \cdots, 1)^T \in \Omega_+^N$ and $\Gamma := (\gamma_{ij}), i = 1, \cdots, n, j = 1, \cdots, k$ such that $\gamma_{ij} = 1$ if $i \in S_j$ and $\gamma_{ij} = 0$ if $i \notin S_j$. Then, the above definition can be restated as the existence of an nonnegative vector $\boldsymbol{\delta}$ which satisfies $\Gamma \cdot \boldsymbol{\delta} = \boldsymbol{e}$. That is to say, $\boldsymbol{e}$ is spanned by $\boldsymbol{\gamma}^1, \cdots, \boldsymbol{\gamma}^k$, the column vectors of the matrix Γ and can be written as the nonnegative combination of such vectors.

Definition 4.4 *An n-person cooperative game without side-payment* (N, v) *is* **balanced**, *if the following is satisfied: for any balanced collection* $\mathcal{S} = \{S_j\}_{j \in K}$,

$$\boldsymbol{x}^{S_j} \in \overline{v}(S_j) \text{ for every } S_j \in \mathcal{S} \Rightarrow \boldsymbol{x}^N \in v(N).$$

Scarf proved that a sufficient condition for the existence of the core of an n-person cooperative game without side-payment (N, v) is balancedness.

Theorem 4.5 (Scarf[13], 1967) *If* (N, v) *is balanced, then the core* $C(N, v)$ *is nonempty.*

Lemma 4.6 *Given* $E : P(N) \rightarrow P^2(A)$ *and* $\boldsymbol{u}^N$, *define* $v^{E,\boldsymbol{u}^N}$ *as follows:*

$v^{E,\boldsymbol{u}^N}(S) \equiv \{\boldsymbol{x}^N \in \Omega_+^N \mid$ *there exists* $B \in E(S)$ *such that* $x^i \leqq \min_{b \in B} u^i(b)$ *for every* $i \in S\}$.

Then, $(N, v^{E,\boldsymbol{u}^N})$ *is an n-person cooperative game without side-payment.*

Proof: Relations (1) to (4) of **Definition 4.1** directly follow from our definition of $v^{E,\boldsymbol{u}^N}$. Q.E.D.

Lemma 4.7 (Peleg[12, Theorem 6.A.7.a.], 1983) *Given an n-person cooperative game in the characteristic function form without side-payments* $(N, v^{E,\boldsymbol{u}^N})$,

$$C(N, v^{E,\boldsymbol{u}^N}) \neq \emptyset \Rightarrow C(A, E, \boldsymbol{u}^N) \neq \emptyset.$$

[†6] $\boldsymbol{x}^T$ denotes the transpose of the vector $\boldsymbol{x}$.

Definition 4.8 *A function $E : P(N) \to P^2(A)$ is* **balanced** *if the following condition is satisfied: for any balanced collection $\mathcal{S} = (S_j)_{j \in K}$ and $B_j \in E(S_j), j \in K$,*

$$\bigcap_{i \in N} \Big(\bigcup_{j \in K(i)} B_j \Big) \neq \emptyset,$$

where $K = \{1, 2, \cdots, k\}$, $K(i) = \{j \in K \mid i \in S_j \text{ and } S_j \in \mathcal{S}\}$.

An effectivity function is balanced when there exists at least one alternative which any player in the society can commonly enforce through some coalition to which he belongs.

Lemma 4.9 *A function $E : P(N) \to P^2(A)$ is* **balanced**, *if and only if for any utility profile $\boldsymbol{u}^N$, a game $(N, v^{E,\boldsymbol{u}^N})$ is balanced.*

Proof: (*Necessity.*) Take an arbitrary balanced collection $\mathcal{S} = \{S_j\}_{j \in K}$ and let $\boldsymbol{x}^N \in v^{E,\boldsymbol{u}^N}(S_j)$, for every $S_j \in \mathcal{S}$. Then, from **Lemma 4.6**, for every $j \in K$, there exists B_j such that $\min_{b \in B_j} u^i(b) \geqq x^i$ for every $i \in S_j$. Hence,

$$\min_{b \in \bigcup_{j \in K(i)} B_j} u^i(b) \geqq x^i \text{ for every } i \in N. \tag{1}$$

From the definition of balancedness of the effectivity function, we can take an alternative (say $\tilde{a}$) which belongs to the term of the right-hand side of **Definition 4.8**. Then, for every player i, $\tilde{a} \in \bigcup_{j \in K(i)} B_j$. Hence,

$$\min_{b \in \bigcup_{j \in K(i)} B_j} u^i(b) \leqq u^i(\tilde{a}) \text{ for every } i \in N. \tag{2}$$

From (1),(2) and **Definition 3.1**, N is effective over any subset of alternatives, and thus, we can get

$$\text{there exists } \{\tilde{a}\} \in E(N) \text{ such that } u^i(\tilde{a}) \geqq x^i \text{ for every } i \in N.$$

We obtain the desired result, $\boldsymbol{x}^N \in v^{E,\boldsymbol{u}^N}(N)$.

(*Sufficiency.*) Assume, on the contrary, that there exists a balanced collection $\mathcal{S}$ such that $\bigcap_{i \in N}(\bigcup_{j \in K(i)} B_j) = \emptyset$. Then, the following is satisfied:

$$\text{For every } a \in A, \text{ there exists } i_a \in N \text{ such that } a \notin \bigcup_{j \in K(i_a)} B_j.$$

Set I_A to $\{i_a \mid a \in A\}$. And let u^{i_a} satisfy $u^{i_a}(\bigcup_{j \in K(i_a)} B_j) > u^{i_a}(a)$ for every $i_a \in I_A$. Then,

$$\min_{b \in \bigcup_{j \in K(i_a)} B_j} u^{i_a}(b) > u^{i_a}(a), \text{ for every } i_a \in I_A.$$

Define $\boldsymbol{x}^N$ as follows:

$$\boldsymbol{x}^N \equiv \begin{cases} \min_{b \in \bigcup_{j \in K(i)} B_j} u^i(b) & \text{if } i \in I_A \\ 0 & \text{if } i \notin I_A. \end{cases}$$

This vecter satisfies the condition $\boldsymbol{x}^{S_j} \in \overline{v}^{E,\boldsymbol{u}^N}(S_j)$ for every $S_j \in \mathcal{S}$. But for every $a \in A$ there exists i_a such that $x^{i_a} > u^{i_a}(a)$, and therefore $\boldsymbol{x}^N \notin v^{E,\boldsymbol{u}^N}(N)$. Thus, the game $(N, v^{E,\boldsymbol{u}^N})$ is not balanced. Q.E.D.

Theorem 4.10 *A function $E : P(N) \to P^2(A)$ is stable if E is balanced.*

Proof: E is balanced if and only if for every $\boldsymbol{u}^N$, $(N, v^{E,\boldsymbol{u}^N})$ is balanced. Then, for every $\boldsymbol{u}^N$, $C(N, v^{E,\boldsymbol{u}^N})$ is nonempty from **Theorem 4.5** and so does $C(A, E, \boldsymbol{u}^N)$ from **Lemma 4.7**. From **Definition 3.3**, E is stable. Q.E.D.

Theorem 4.10 makes us possible to use Scarf's algorithm for obtaining an element of the core by considering the specific characteristic function obtained from a balanced effectivity function[†7]. We showed in **Theorem 4.10** that if an effectivity function is balanced, then it

[†7] In [9] we were given a simple example how to utilize Scarf's algorithm to obtain the elements of the core of the game in the effectivity function form.

is stable. But, in general, the reverse does not always hold true. The example given below treats the case where the effectivity function is stable, but not balanced.

Example 4.11 *Let* $N = \{1,2,3,4\}$ *and* $A = \{a_1, a_2, a_3, a_4\}$*. Define an effectivity function as follows:* $E(N) = P(A)$, $E(\{1,2,3\}) = \{a_1\}^+$, $E(\{1,4\}) = \{a_2, a_3\}^+$, $E(\{2,4\}) = \{a_2, a_4\}^+$, $E(\{3,4\}) = \{a_3, a_4\}^+$, $E(\{1,2,4\}) = E(\{1,4\}) \cup E(\{2,4\})$, $E(\{1,3,4\}) = E(\{1,4\}) \cup E(\{3,4\})$, $E(\{2,3,4\}) = E(\{2,4\}) \cup E(\{3,4\})$ *and otherwise,* $E(S) = \{A\}$*, where* $B^+ = \{B' \mid B' \in P(A), B' \supset B\}$*. Peleg[12, Example 6.3.16.] showed that it is stable. Take* $S_1 = \{1,2,3\}, S_2 = \{1,4\}, S_3 = \{2,4\}$ *and* $S_4 = \{3,4\}$*. Then,* $\mathcal{S} = \{S_1, S_2, S_3, S_4\}$ *is a balanced collection with nonnegative weights* $d_1 = 2/3, d_2 = d_3 = d_4 = 1/3$*. Let* $B_1 = \{a_1\}, B_2 = \{a_2, a_3\}, B_3 = \{a_2, a_4\}$ *and* $B_4 = \{a_3, a_4\}$ *such that* $B_j \in E(S_j), j = 1,2,3,4$*. Then,* $\bigcup_{j \in K(1)} B_j = B_1 \cup B_2 = \{a_1, a_2, a_3\}$, $\bigcup_{j \in K(2)} B_j = B_1 \cup B_3 = \{a_1, a_2, a_4\}$, $\bigcup_{j \in K(3)} B_j = B_1 \cup B_4 = \{a_1, a_3, a_4\}$ *and* $\bigcup_{j \in K(4)} B_j = B_2 \cup B_3 \cup B_4 = \{a_2, a_3, a_4\}$*. Then,* $\bigcap_{i \in N}(\bigcup_{j \in K(i)} B_j) = \emptyset$*. Thus, the effectivity function is not balanced.*

5. A Necessary and Sufficient Condition for Stability

The example given in the previous section motivates us to search for conditions guaranteeing that an effectivity function is balanced whenever it is stable. In this section we shall introduce some concepts and related properties of an effectivity function which is indispensable for our proof.

Definition 5.1 *A function* $E : P(N) \to P^2(A)$ *is* **anonymous**, *if and only if for every* $S \in P(N)$ *and every* $B \in P(A)$,

$$B \in E(S), S' \in P(N) \text{ and } |S'| = |S| \Rightarrow B \in E(S').$$

Definition 5.2 *A function* $E : P(N) \to P^2(A)$ *is* **neutral**, *if and only if for every* $S \in P(N)$ *and every* $B \in P(A)$,

$$B \in E(S), B' \in P(A) \text{ and } |B'| = |B| \Rightarrow B' \in E(S).$$

In the former definition, not identity but number of the players in a coalition matters only and in the latter, number of a subset of alternatives.

Definition 5.3 *Let* $E : P(N) \to P^2(A)$ *be a neutral effectivity function. The* **veto function** *is defined in the following fashion:*

$$v_E(S) = m - e_E(S), \text{ where } e_E(S) = \min_{B \in E(S)} |B|.$$

Let $e_E(\emptyset) = m + 1$ and, then $v(\emptyset) = -1$. Given E and S, $e_E(S)$ indicates the minimum number of alternatives which S can enforce. Thus, the veto function reflects the power of the coalition S, *i.e.*, the maximum number of alternatives that it can block.

Definition 5.4 *For every* $S \in P(N)$ *the* **proportional veto function** *is to be defined as follows:*

$$\check{v}(S) = \left\lceil \frac{m|S|}{n} \right\rceil - 1,$$

where $[x]$ *is a upper Gaussian number of* x*, i.e., the smallest integer* z *such that* $z \geqq x$*.*

If the given effectivity function E satisfies the following a proportional condition concerning the power of each coalition: $|B| = m + 1 - [m|S|/n]$ for every $S \in P(N)$, then we directly have $v_E = \check{v}$. It is easy to extend this relation between v_E and $\check{v}$. Indeed, $v_E(S) \leqq \check{v}(S)$ holds for every $S \in P(N)$, if and only if the SCF corresponding E does not permit proportional power to each S in the sense that $|B| \geqq m+1-[m|S|/n]$, or equivalently $|B| > m(n-|S|)/n$.

Theorem 5.5 (Moulin[10], 1981) *Let* $E : P(N) \to P^2(A)$ *be an effectivity function with anonymity and neutrality. Then* E *is stable if and only if* $v_E(S) \leqq \check{v}(S)$ *for every* $S \in P(N)$*, where* v_E *is the veto function and* $\check{v}$ *is the proportional veto function.*

Finally, we reach the last thresh-hold of our main theorem through the complicated chains of definitions, lemmata and theorems. And our last work is to prove the proposition that an effectivity function is balanced whenever it is stable.

Lemma 5.6 (Ichiishi[6], 1988) *If $\mathcal{S} = \{S_j\}_{j\in K}$ is a balanced collection, then $\overline{\mathcal{S}} = \{\overline{S}_j\}_{j\in K}$ is also a balanced collection, where $\overline{S}_j = N\backslash S_j$.*

Theorem 5.7 *Let $E : P(N) \to P^2(A)$ be an effectivity function with anonymity and neutrality. Then E is balanced if and only if it is stable.*

Proof: Necessity comes from **Theorem 4.10**. It remains to prove sufficiency. From **Theorem 5.5**, it is enough to show that an effectivity function is balanced if $|B| > m(n-|S|)/n$ for every $B \in E(S)$ and every $S \in P(N)$. Suppose that E is not balanced. Then, there exists a balanced collection $\{S_j\}_{j\in K}$ which satisfies the following condition:

$$\text{there exists } B_j \in E(S_j), j \in K;\ \bigcap_{i\in N}\Big(\bigcup_{j\in K(i)} B_j\Big) = \emptyset.$$

Thus,

$$\text{for every } a \in A, \text{ there exists } i \in N; a \in B_j \Rightarrow i \in \overline{S}_j.$$

Let us define an $m \times k$ matrix $\Xi = (\xi_{ij})$ such that $\xi_{ij} = 1$ if $a_i \in B_j$ and $\xi_{ij} = 0$ if $a_i \notin B_j$. From **Lemma 5.6**, we know that there exists a nonnegative weight vector $\overline{\boldsymbol{\delta}}$ such that $\overline{\Gamma}\cdot\overline{\boldsymbol{\delta}} = \boldsymbol{e}$[†8]. Then, for every $a_l \in A$, there exists $i \in N$ such that $\overline{\boldsymbol{\gamma}}_i \geqq \boldsymbol{\xi}_l$ for every $l = 1,\cdots,m$, where $\overline{\boldsymbol{\gamma}}_i = (\overline{\gamma}_{i1},\dots,\overline{\gamma}_{ik})$ and $\boldsymbol{\xi}_l = (\xi_{l1},\dots,\xi_{lk})$. As $1 = \overline{\boldsymbol{\gamma}}_i\cdot\overline{\boldsymbol{\delta}} \geqq \boldsymbol{\xi}_l\cdot\overline{\boldsymbol{\delta}}$, for every $l = 1,\dots,m$, $\Xi\cdot\overline{\boldsymbol{\delta}} \leqq \boldsymbol{e}$. Then, by multiplying both sides of the latter inequality relation by $\boldsymbol{e}^T$, we can get $\sum_{j\in K}|B_j|\,\overline{\delta}_j \leqq m$. From the assumption

$$\sum_{j\in k}|B_j|\,\overline{\delta}_j > \sum_{j\in K}\frac{m(n-|S_j|)}{n}\overline{\delta}_j = \frac{m}{n}\sum_{j\in K}|\overline{S}_j|\,\overline{\delta}_j,$$

we can get $\sum_{j\in K}|\overline{S}_j|\,\overline{\delta}_j < n$. On the other hand, from the fact that $\boldsymbol{e}^T\cdot\overline{\Gamma}\cdot\overline{\boldsymbol{\delta}} = \boldsymbol{e}^T\cdot\boldsymbol{e}$, $\sum_{j\in K}|\overline{S}_j|\,\overline{\delta}_j = n$ and a contradiction occurs. Thus, E is balanced. Q.E.D.

6. Concluding Remarks

We showed that if an effectivity function is balanced, then it is stable and if it is anonymous and neutral, then the balancedness of the effectivity function is a necessary and sufficient condition for it to be stable.

Including the majority voting rule and the Borda voting rule, most of actual voting rules have anonymous and neutral effectivity functions. This implies that balancedness is equivalent to stability in a usual social choice scheme. Though we have been discovered no algorithm to check whether or not the given effectivity function is balanced yet, if we obtain an algorithm, we can find whether or not it is stable at once.

Related to the stability of the effectivity function, we can refer to the recent development by Mizutani, Hiraide and Nishino [8]. They showed that the problem to check the unstability of the effectivity function belongs to NPC with respect to the computational complexity. This suggests that the problem to check the balancedness of the given effectivity function is intractably hard to solve.

[†8] $\overline{\Gamma} = (\overline{\gamma}_{ij}), i = 1,\cdots,n, j = 1,\cdots,k$ such that $\overline{\gamma}_{ij} = 0$ if $i \in S_j$ and $\overline{\gamma}_{ij} = 1$ if $i \notin S_j$.

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