Journal of the Operations Research
Society of Japan
Vol. 37, No. 3, September 1994



# EULER'S FORMULA VIA POTENTIAL FUNCTIONS

Ryuichi Hirabayashi, *Science University of Tokyo*
Hubertus Th. Jongen, *RWTH-Aachen*
Masayuki Shida, *Kanagawa University*

(Received March 10, 1993; Revised December 10, 1993)

*Abstract* A new proof of Euler's formula for polytopes is presented via an approach using potential functions. In particular, a connection between Euler's formula and the Morse relation from differential topology is established.

## 1 Introduction

Karmarkar's approach in linear programming using potential functions (cf. [6]) has been very seminal. In this paper, we'll use such a potential function approach in order to give a new proof of the famous Euler formula for polytopes. The main point consists in establishing a connection between Euler's formula and the Morse relation from differential topology.

Let $P \subset I\!R^n$ be an $n$-dimensional polytope, and let $f_i(P)$ denote the number of $i$-dimensional faces of $P$, $i = 0, 1, \cdots, n$. Then, Euler's formula is the following (cf. [2]):

$$\sum_{i=0}^{n-1}(-1)^i f_i(P) = 1 + (-1)^{n-1}. \tag{1.1}$$

In Morse theory, relations are established between the various critical points of a real valued function on a manifold $M$ on one hand, and the topology of $M$ on the other hand (cf. [9]). In nonlinear optimization, the concept of Karush-Kuhn-Tucker point (KKT-point, for short) plays the role of that of a critical point. A Morse theory in this context (also for certain types of nondifferentiable functions) was established in [5]. However, the nondegeneracy concept used in [5] can be relaxed considerably in order to obtain Morse relations. On one hand, under the Mangasarian-Fromovitz constraint qualification, Morse theory was developed in [4], where the KKT-points are assumed to be strongly stable in the sense of Kojima (cf. [7]). On the other hand, in feasible sets which allow a certain regular decomposition into manifolds (so-called Whitney stratification, cf. [3]), another approach with wide applicability has been established by M. Goresky and R. MacPherson in [3]. It is not difficult to see that the decomposition of a polytope into the set of relative interiors of its faces is such a regular decomposition. We note that the nondegeneracy concepts in [4] ("strong stability") and [3] ("nondepravedness") are not the same (cf. also the subsequent Example 1.3). In [1], a nondegeneracy concept of a KKT-point is introduced which is subsumed in the corresponding one in [3]. For the purpose of this paper the nondegeneracy concept in [1] turns out to be very appropriate.

Let the $n$-dimensional polytope $P$ be described by means of a system of linear inequalities:

$$P = \{\boldsymbol{x} \in I\!R^n | \boldsymbol{a}_j^\top \boldsymbol{x} - b_j \geq 0, j \in J\}. \tag{1.2}$$

The set $J_0(\boldsymbol{x}) = \{j \in J | \boldsymbol{a}_j^\top \boldsymbol{x} - b_j = 0\}$ denotes the set of active inequality constraints at $\boldsymbol{x}$. The tangent space $T_{\boldsymbol{x}}P$ of $P$ at $\boldsymbol{x} \in P$ is defined as follows:

$$T_{\boldsymbol{x}}P = \{\boldsymbol{\xi} \in \mathbb{R}^n | \boldsymbol{a}_j^\top \boldsymbol{\xi} = 0, j \in J_0(\boldsymbol{x})\}. \tag{1.3}$$

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a twice continuously differentiable function. A point $\overline{\boldsymbol{x}} \in P$ is called a Karush-Kuhn-Tucker point for $f|_P$ if we have:

$$D^\top f(\overline{\boldsymbol{x}}) = \sum_{j \in J_0(\overline{\boldsymbol{x}})} \lambda_j \boldsymbol{a}_j, \text{ and } \lambda_j \geq 0, j \in J_0(\overline{\boldsymbol{x}}), \tag{1.4}$$

where $Df$ stands for the row vector of the first partial derivatives. Note that (1.4) means $D^\top f(\overline{\boldsymbol{x}}) \in C(\boldsymbol{a}_j, j \in J_0(\overline{\boldsymbol{x}}))$, where $C(\boldsymbol{a}_j, j \in J_0(\overline{\boldsymbol{x}}))$ denotes the nonnegative cone generated by the vectors $\boldsymbol{a}_j, j \in J_0(\overline{\boldsymbol{x}})$. Following [1] we define:

**Definition 1.1.** *A KKT-point $\overline{\boldsymbol{x}} \in P$ for $f|_P$ is said to be nondegenerate if*

**(1)** $D^\top f(\overline{\boldsymbol{x}}) \in rel.int.C(\boldsymbol{a}_j, j \in J_0(\overline{\boldsymbol{x}}))$, *where (rel.)int.C denotes the (relative) interior of $C$.*

**(2)** $V^\top D^2 f(\overline{\boldsymbol{x}}) V$ *is nonsingular, where $V$ is a matrix whose columns form a basis for the tangent space $T_{\overline{\boldsymbol{x}}}P$, where $D^2 f$ stands for the Hessian matrix of $f$.*

*The number of negative eigenvalues of $V^\top D^2 f(\overline{\boldsymbol{x}}) V$ is called the index of the KKT-point $\overline{\boldsymbol{x}}$.*

**Remark 1.2.** *Note that the numbers of negative eigenvalues of $V^\top D^2 f(\overline{\boldsymbol{x}}) V$ is independent of the choice of the matrix $V$. The linearity of the constraints describing the polytope $P$ implies that the Hessian of $f$ coincides with the Hessian of any associated Lagrange function.*

**Example 1.3.** *Consider the polytope $P = \{\boldsymbol{x} \in \mathbb{R}^2 | -1 \leq x_1 \leq 1, 0 \leq x_2 \leq 1\}$ and the functions $f_1(\boldsymbol{x}) = -x_1^2 + x_2, f_2(\boldsymbol{x}) = -x_1^2 - x_2^2$. The origin $(0,0)$ is a KKT-point for $f_j|_P, j = 1,2$, and condition (2) in Definition 1.1 is fulfilled. However, condition (1) is only fulfilled in case of $f_1$ (therefore, for $f_1|_P$, the origin is a nondegenerate KKT-point). Consider the lower level sets $P^\alpha(f_j) = \{\boldsymbol{x} \in P | f_j(\boldsymbol{x}) \leq \alpha\}$ as $\alpha$ increases and passes zero (the value of $f_j$ at the KKT-point $0 \in P$). In case of $f_1$, the lower level set changes from being disconnected to connected, whereas in case $f_2$, the connectedness does not change. We remark that the origin is a strongly stable KKT-point for $f_1|_P$, but not for $f_2|_P$. But the concept of the strong stability depends on the defining system of the polytope $P$, whereas the condition (1) in Definition 1.1 does not. In fact, if we use a system $\{\boldsymbol{x} \in \mathbb{R}^2 | -1 \leq x_1 \leq 1, 0 \leq x_2 \leq 1, 0 \leq x_2\}$ as a defining system for $P$ (remark that the constraint $0 \leq x_2$ is redundant), then the origin is not strongly stable. These two facts underline the importance of the condition (1) in Definition 1.1.*

Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is twice continuously differentiable and that all KKT-points of $f|_P$ are nondegenerate. Let $c_i(f|_P)$ denote the number of KKT-points of index $i$. Let $b_j(P)$ denote the Betti-number of the polytope $P$ (roughly speaking, $b_j(P)$ counts the number of $(j+1)$-dimensional "holes"). Then, the Morse relation is valid (cf. [1]):

$$\sum_{i=0}^{n} (-1)^i c_i(f|_P) = \sum_{i=0}^{n} (-1)^i b_i(P) \tag{1.5}$$

which relates the various KKT-points of $f|P$ with the topology of $P$. Since a polytope is obviously contractible, its Betti-numbers coincide with those of a one-point space, i.e., $b_0(P) = 1, b_i(P) = 0$ for $i > 0$. Hence, (1.5) reduces to:

$$\sum_{i=0}^{n} (-1)^i c_i(f|_P) = 1. \tag{1.6}$$

In the next section, we prove Formula (1.1) by means of (1.6) by using a potential function. In addition, it will be shown that an approach via a quadratic function is in general not appropriate. Another way of proving Euler's formula via potential functions might be done with the aid of a dynamical system (this was communicated to us by M. Shub [10]). In fact, one might use as underlying vector fields those which are discussed in [8].

## 2 The Proof of Euler's Formula

Let our $n$-dimensional polytope $P \subset \mathbb{R}^n$ again be described as in (1.2). Without loss of generality we assume that $0 \in Int(P)$. Define the potential function $f$:

$$f(\boldsymbol{x}) = \sum_{j \in J} ln(\boldsymbol{a}_j^\top \boldsymbol{x} - b_j). \tag{2.1}$$

It is well-known that $f$ is strictly concave in $Int(P)$. In fact, the Hessian $D^2 f(\boldsymbol{x})(= \sum_{j\in J} -\boldsymbol{a}_j \boldsymbol{a}_j^\top/(\boldsymbol{a}_j^\top \boldsymbol{x} - b_j)^2)$ is negative definite, since the compactness of $P$ implies that the vectors $\boldsymbol{a}_j, j \in J$ span $\mathbb{R}^n$. Unfortunately, the function $f$ becomes singular at the boundary of $P$. Therefore, we make a "desingularization step" by slightly shrinking the polytope $P$.

For $0 < \varepsilon < 1$ we define $P(\varepsilon) = (1-\varepsilon)P$. Note that there is one-to-one correspondence $\rho$ between the $i$-dimensional faces of $P$ and $P(\varepsilon)$. In particular, if $\sigma$ is an $i$-dimensional face of $P$, then $\rho(\sigma) := (1-\varepsilon)\sigma$ is an $i$-dimensional face of $P(\varepsilon)$. For some $I \subset J$, the face $\sigma$ can be described as follows:

$$\sigma = \{\boldsymbol{x} \in P | \boldsymbol{a}_j^\top \boldsymbol{x} = b_j, j \in I\}. \tag{2.2}$$

The potential function $f$ in $\rho(\sigma)$ then takes the form:

$$f(\boldsymbol{x})|_{\rho(\sigma)} = \sum_{j \in J \setminus I} ln(\boldsymbol{a}_j^\top \boldsymbol{x} - b_j) + c(\varepsilon), \tag{2.3}$$

where $c(\varepsilon)$ is some constant (only depending on $\varepsilon$ and $\sigma$). A moment of reflection now shows that, for $\varepsilon$ sufficiently small, each face of $P(\varepsilon)$ contains precisely one KKT-point for $f|_{P(\varepsilon)}$ in its relative interior. Moreover, it is easy to check that every KKT-point is nondegenerate in the sense of Definition 1.1, and the index equals the dimension of the corresponding face. So, we can apply formula (1.6) for $f|_{P(\varepsilon)}$. Noting that $c_i(f|_{P(\varepsilon)}) = f_i(P(\varepsilon)) = f_i(P)$, and the fact that the analytic center (cf. [11]) of $P$ is the only KKT-point for $f|_{P(\varepsilon)}$ of index $n$, formula (1.1) follows immediately.

**Remark 2.1.** *In the above proof we used the fact that every face of $P(\varepsilon)$ contains precisely one (nondegenerated) KKT-point for $f|_{P(\varepsilon)}$ in its relative interior. This can in general not be accomplished by means of a quadratic function $g(\boldsymbol{x}) = -(\boldsymbol{x}-\boldsymbol{x}^*)^\top \boldsymbol{A}(\boldsymbol{x}-\boldsymbol{x}^*)$, $\boldsymbol{A}$ symmetric and positive definite. To see this, consider the following example. Put $P = C(\boldsymbol{p}_i, i = 1, \cdots, 9)$, where $\boldsymbol{p}_1 = (0, 10/3), \boldsymbol{p}_2 = (-\sqrt{3}/3, 3), \boldsymbol{p}_3 = (-5\sqrt{3}/3, -1)$, $\boldsymbol{p}_4 = (-5\sqrt{3}/3, -5/3), \boldsymbol{p}_5 = (-4\sqrt{3}/3, -2), \boldsymbol{p}_6 = (4\sqrt{3}/3, -2), \boldsymbol{p}_7 = (5\sqrt{3}/3, -5/3)$, $\boldsymbol{p}_8 = (5\sqrt{3}/3, -1), \boldsymbol{p}_9 = (\sqrt{3}/3, 3)$, cf. Figure 1. The very construction implies that $\boldsymbol{p}_1\boldsymbol{p}_2 // \boldsymbol{p}_7\boldsymbol{p}_6$, $\boldsymbol{p}_3\boldsymbol{p}_4 // \boldsymbol{p}_8\boldsymbol{p}_7$, $\boldsymbol{p}_4\boldsymbol{p}_5 // \boldsymbol{p}_1\boldsymbol{p}_9$, where the symbol // stands for "parallel". Now, suppose that a quadratic function $g$ as above would work, say $\boldsymbol{x}^* \in int.P$. Then, from the relation of points of tangency between the pararell lines and the family of similar ellipsoids with the same center and the same axis, the center $\boldsymbol{x}^*$ must be in the intersection of the trapezia $\boldsymbol{p}_1\boldsymbol{p}_2\boldsymbol{p}_6\boldsymbol{p}_7$, $\boldsymbol{p}_1\boldsymbol{p}_4\boldsymbol{p}_5\boldsymbol{p}_9$, $\boldsymbol{p}_3\boldsymbol{p}_4\boldsymbol{p}_7\boldsymbol{p}_8$. However, the latter intersection is empty!*

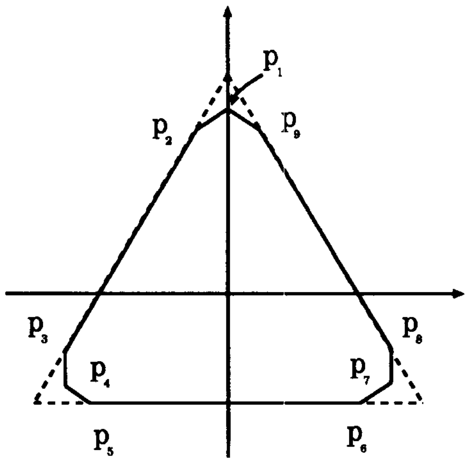


Figure 1:

**Acknowledgment:**
The authors thank two anonymous referees for useful comments and suggestions.

## References


[1] A.A. Agrachev and S.A. Vakhrameev. Morse theory and optimal control problems. *Nonlinear Synthesis (Sopron, 1989), 1-11. Progr. Systems Control Theory 9, Birkhäuser Boston, Boston, MA*, 1991.

[2] A. Brøndsted. *An Introduction to Convex Polytopes.* Springer-Verlag, New York, Heidelberg, Berlin, 1983.

[3] M. Goresky and R. MacPherson. *Stratified Morse Theory.* Springer Verlag, 1988.

[4] J. Guddat, H.Th. Jongen, and J. Rückmann. On stability and stationary points in nonlinear optimization. *Journal of Austral. Math. Soc., Ser.B, 28*, pages 36–56, 1986.

[5] H.Th. Jongen, P. Jonker, and F. Twilt. *Nonlinear optimization in $R^n$, I. Morse Theory, Chebychev Approximation.* Peter Lang Verlag, Frankfurt a.M. - Bern - New York, 1983.

[6] N. Karmarkar. A new polynomial-time algorithm for linear programming. *Combinatorica 4*, pages 373-395, 1984.

[7] M. Kojima. Strongly stable stationary solutions in nonlinear programs. In *Analysis and Computation of Fixed Points*, S.M. Robinson, ed., pages 93–138. Academic Press, New York, 1980.

[8] N. Megiddo and M. Shub. Bounding behavior of interior point algorithms in linear programming. *Mathematics of Operations Research 14*, pages 97–146, 1989.

[9] J. Milnor. *Morse Theory.* Annals of Mathematics Studies, No. 51, Princeton University Press, 1963.

[10] M. Shub. Private communication. 1992(June).

[11] G. Sonnevend. An "analytical centre" for polyhedrons and new classes of global algorithms for linear (smooth, convex) programming. In *Lecture Notes in Control and Information Sciences 84*, pages 866–876, New York, 1985. Springer.


Masayuki SHIDA: Department of Mathematics,
Kanagawa University, Rokkakubashi,
Kanagawa-ku, Yokohama 221, Japan.