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A GREEDY ALGORITHM FOR MINIMIZING A SEPARABLE CONVEX FUNCTION OVER AN INTEGRAL BISUBMODULAR POLYHEDRON

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Abstract We present a new greedy algorithm for minimizing a separable convex function over an integral bisubmodular polyhedron. The algorithm starts with an arbitrary feasible solution and a current feasible solution incrementally moves toward an optimal one in a greedy way. We also show that there exists at least one optimal solution in the coordinate-wise steepest descent direction from a feasible solution if it is not an optimal one.

1. Introduction

As shown by Edmonds [6], the problem of minimizing a linear function over a polymatroid can be solved by a greedy algorithm. Fujishige [8] solved the problem of minimizing a separable convex quadratic function over the base polyhedron associated with a polymatroid and gave a decomposition algorithm (also see [9]). Federgruen and Groenevelt [7] introduced a concept of weak concavity, generalizing that of separable concavity, and showed that a greedy algorithm finds an optimal solution that maximizes a given weakly concave function if and only if the feasible solutions form a polymatroid. Groenevelt [10] gave a decomposition algorithm and the so-called bottom up algorithm for maximizing a separable concave function over a polymatroid.

Chandrasekaran and Kabadi [3] introduced a concept of polypseudomatroid, on which a greedy algorithm works. A polypseudomatroid is defined by a system of linear inequalities with a $\{0, \pm 1\}$-coefficient matrix and a right-hand side expressed by a bisubmodular function. Nakamura [11] showed that polypseudomatroids are exactly those polyhedra characterized by a greedy algorithm of Dunstan and Welsh [5]. Similar polyhedra and their set-theoretical versions are, independently, studied by Bouchet [1] as Δ-matroids, by Dress and Havel [4] as metroids, by Qi [12] as ditroids, and by Nakamura [11] as universal polymatroids (also see [9]).

Recently, Bouchet and Cunningham [2] have introduced a concept of jump system. A jump system is a set of integral points satisfying a certain exchange axiom. The convex hull of a jump system, if bounded, is a polypseudomatroid ([2]).

We consider a version of the polypseudomatroid, called an *integral bisubmodular polyhedron*, which consists of the set of integral points of a polypseudomatroid. We present a greedy algorithm for minimizing a separable convex function over a bisubmodular polyhedron. It starts with an arbitrary initial feasible point and repeats coordinate-wise augmentations and/or exchanges in a greedy way. We also examine the behavior of the greedy algorithm.

2. Definitions

Let E be a nonempty finite set. Denote by 3^E the set of all the ordered pairs (X, Y) of disjoint subsets X and Y of E. Let $f : 3^E \to \mathbf{Z}$ be a function from 3^E to the set $\mathbf{Z}$ of integers such that $f(\emptyset, \emptyset) = 0$ and for each $(X_i, Y_i) \in 3^E$ (i=1,2)

$$\begin{aligned} &f(X_1, Y_1) + f(X_2, Y_2) \\ &\quad \geq f((X_1 \cup X_2) - (Y_1 \cup Y_2), (Y_1 \cup Y_2) - (X_1 \cup X_2)) + f(X_1 \cap X_2, Y_1 \cap Y_2). \end{aligned} \tag{2.1}$$

We call such an f a *bisubmodular function*, which was first considered by Chandrasekaran and Kabadi [3]. Define a polyhedron

$$\mathrm{P}_*(f) = \{x \mid x \in \mathbf{Z}^E, \forall (X, Y) \in 3^E : x(X) - x(Y) \leq f(X, Y)\} \tag{2.2}$$

associated with f, where $x(X) = \sum_{e \in X} x(e)$ for any $X \subseteq E$ and $x \in \mathbf{Z}^E$. Here, we use the convention that $x(\emptyset) = 0$. We call the polyhedron $\mathrm{P}_*(f)$ an *integral bisubmodular polyhedron.*

Bouchet and Cunningham [2] introduced a concept of *jump system*. A *step* is a (0,±1)-vector with a unique nonzero component. Denote by S the set of all the steps in $\mathbf{Z}^E$. For any $x, y \in \mathbf{Z}^E$ a *step s from x to y* is a step such that $\sum_{e \in E} |x(e) + s(e) - y(e)| = \sum_{e \in E} |x(e) - y(e)| - 1$. We denote by $\mathrm{St}(x, y)$ the set of all the steps from x to y. A *jump system* on a finite set E is a pair $(E, \mathcal{F})$ of E and a nonempty $\mathcal{F} \subseteq \mathbf{Z}^E$ which satisfies the following *2-step axiom*:

(2-SA) For any $x, y \in \mathcal{F}$ and $s \in \mathrm{St}(x, y)$ with $x + s \notin \mathcal{F}$ there exists $t \in \mathrm{St}(x + s, y)$ such that

$$x + s + t \in \mathcal{F}. \tag{2.3}$$

A jump system with a finite $\mathcal{F}$ is called a *finite jump system.* Based on the results of Bouchet and Cunningham [2], we have the following two theorems.

Theorem 2.1 ([2]): *An integral bisubmodular polyhedron is a jump system.* □

Theorem 2.2 ([2]): *The convex hull of a finite jump system in* $\mathbf{R}^E$ *coincides with the convex hull of an integral bisubmodular polyhedron in* $\mathbf{R}^E$. □

Denote by $\mathrm{Co}(\mathcal{F})$ the convex hull of $\mathcal{F}$.

As shown in [2], the converse of Theorem 2.1 is not true. For example, $\mathcal{F} = \{0, 2\} \subseteq \mathbf{Z}$ is a jump system but not an integral bisubmodular polyhedron. However, $\mathrm{Co}(\mathcal{F}) = \{x \mid 0 \leq x \leq 2\}$ is the convex hull of an integral bisubmodular polyhedron.

Proposition 2.3: *Suppose that* $\mathcal{F} \subseteq \mathbf{Z}^E$ *satisfies* $\mathcal{F} = \mathrm{Co}(\mathcal{F}) \cap \mathbf{Z}^E$. *Then, 2-step axiom* (2-SA) *is equivalent to the following variant:*

(2-SA') *For any* $x, y \in \mathcal{F}$ *and* $s \in \mathrm{St}(x, y)$ *with* $x + s \notin \mathcal{F}$ *there exists* $t \in \mathrm{St}(x, y)$ *such that*

$$x + s + t \in \mathcal{F}. \tag{2.4}$$

Proof: In general, $s \in \mathrm{St}(x, y)$ implies

$$\mathrm{St}(x + s, y) \subseteq \mathrm{St}(x, y) \tag{2.5}$$

and

$$\mathrm{St}(x + s, y) \cup \{s\} = \mathrm{St}(x, y). \tag{2.6}$$

We see from (2.5) that if $\mathcal{F}$ satisfies (2-SA), then $\mathcal{F}$ also satisfies (2-SA'). On the other hand, since $\mathcal{F}$ satisfies $\mathcal{F} = \mathrm{Co}(\mathcal{F}) \cap \mathbf{Z}^E$ by the assumption, if $x+s+s \in \mathcal{F}$, then we must have $x+s \in \mathcal{F}$. Therefore, $s \neq t$ for s and t in (2-SA'). It follows from (2.6) that if $\mathcal{F}$ satisfies (2-SA'), then $\mathcal{F}$ also satisfies (2-SA). □

An integral bisubmodular polyhedron $\mathrm{P}_*(f)$ satisfies (2-SA') because of its convexity. We shall use this axiom, (2-SA'), repeatedly in the following argument.

3. A Greedy Algorithm

Let $w : \mathbf{R}^E \to \mathbf{R}$ be a separable convex function given by

$$w(x) = \sum_{e \in E} w_e(x(e)), \tag{3.1}$$

where for each $e \in E$ w_e is a convex function on $\mathbf{R}$. Consider a discrete optimization problem described as

$$\begin{aligned} \mathbf{IP}: \quad \text{Minimize} \quad & w(x) = \sum_{e \in E} w_e(x(e)) \\ \text{subject to} \quad & x \in \mathrm{P}_*(f). \end{aligned} \tag{3.2}$$

When w is a linear function, Problem **IP** can be solved by the (non-incremental) greedy algorithm of Chandrasekaran and Kabadi [3], Dunstan and Welsh [5], et al.

An integral polymatroid is a special case of an integral bisubmodular polyhedron (see [2]). When $\mathrm{P}_*(f)$ is an integral polymatroid, Problem **IP** can be solved by the incremental greedy algorithm of Federgruen and Groenevelt [7] which starts from the origin. By their algorithm the current solution is monotonically increased (component-wise), starting from the origin, the zero vector. For a general integral bisubmodular polyhedron we are not given a special starting point such as the origin in a polymatroid. Consequently, our algorithm starts from an arbitrary feasible solution and requires component-wise augmentations and exchanges.

We describe a greedy algorithm for solving the above problem **IP**.

A greedy algorithm

Input: an arbitrary vector $x \in \mathrm{P}_*(f)$.

Output: an optimal solution x of Problem **IP**.

Step 1: If neither of the following two conditions (1) and (2) is satisfied, then stop (x is an optimal solution):

(1) there exists a step $s \in S$ such that $x+s \in \mathrm{P}_*(f)$ and $w(x+s) < w(x)$,

(2) there exist steps $s, s' \in S$ such that $x+s \notin \mathrm{P}_*(f), x+s+s' \in \mathrm{P}_*(f)$ and $w(x+s+s') < w(x)$.

Step 2: Compute

$$\begin{aligned} w_1 &= \min\{w(x+s) \mid \text{step } s \text{ satisfying Condition (1) in Step 1}\}, && (3.3) \\ w_2 &= \min\{w(x+s) \mid \text{steps } s, s' \text{ satisfying Condition (2) in Step 1}\}, && (3.4) \end{aligned}$$

where the minimum over the empty set is defined to be $+\infty$.

Put $\hat{w} := \min\{w_1, w_2\}$.

If we have $\hat{w} = w_1$, let $\hat{s}$ be the step s that attains the minimum of (3.4), put $x := x + \hat{s}$ and go to Step 1.

If $\hat{w} \neq w_1$, let $\hat{s}$ and $\hat{s}'$ be the steps s and s' that attain the minimum of (3.4), put $x := x + \hat{s} + \hat{s}'$ and go to Step 1.

(End)

It should be noted that in (3.4) $w(x+s)$ but not $w(x+s+s')$ is minimized and that each step s in the above algorithm is chosen in a greedy way.

4. Validity of the Greedy Algorithm

In this section we prove the following

Theorem 4.1: *The greedy algorithm described in Section 3 terminates in finitely many steps and finds an optimal solution of Problem* **IP**.

Proof: Since $\mathrm{P}_*(f)$ is bounded, the algorithm terminates in finitely many steps. Let x be the solution found by the greedy algorithm when it terminates. Suppose that x is not an optimal solution of **IP**. Then, we can choose an optimal solution $x^*(\neq x)$ that satisfies the following conditions (i) and (ii):

(i) If $s \in \mathrm{St}(x^*, x)$ and $x^* + s \in \mathrm{P}_*(f)$, then $w(x^*) < w(x^* + s)$.

(ii) If $s, t \in \mathrm{St}(x^*, x), x^* + s \notin \mathrm{P}_*(f)$ and $x^* + s + t \in \mathrm{P}_*(f)$, then $w(x^*) < w(x^* + s + t)$.

(If an optimal solution x^* does not satisfy (i), i.e., if for a step $s \in \mathrm{St}(x^*, x)$ we have $x^*+s \in \mathrm{P}_*(f)$ and $w(x^*) = w(x^*+s)$, then replace x^* by a new optimal solution x^*+s. Note that $\sum_{e\in E} |x^*(e)+s(e)-x(e)| < \sum_{e\in E} |x^*(e)-x(e)|$. Similarly, if x^* does not satisfy (ii), i.e., if for steps $s, t \in \mathrm{St}(x^*, x)$ we have $x^*+s \notin \mathrm{P}_*(f)$, $x^*+s+t \in \mathrm{P}_*(f)$ and $w(x^*) = w(x^*+s+t)$, replace it by a new optimal solution $x^* + s + t$. Note that $\sum_{e\in E} |x^*(e) + s(e) + t(e) - x(e)| < \sum_{e\in E} |x^*(e)-x(e)|$. By such repetitions we eventually reach an optimal solution that satisfies (i) and (ii).)

Claim. There exists some $s \in \mathrm{St}(x^*, x)$ such that $w(x^*) < w(x^* + s)$.

(Proof of Claim) Suppose on the contrary that for any $s \in \mathrm{St}(x^*, x)$

$$w(x^* + s) \leq w(x^*). \tag{4.1}$$

Since x^* satisfies (i) and $\mathrm{P}_*(f)$ satisfies (2-SA'), for any $s \in \mathrm{St}(x^*, x)$ we have $x^* + s \notin \mathrm{P}_*(f)$ and there exists $s' \in \mathrm{St}(x^*, x)$ such that $x^* + s + s' \in \mathrm{P}_*(f)$. It follows from (4.1) that

$$\begin{aligned} w(x^* + s + s') - w(x^*) &= w(x^* + s) - w(x^*) + w(x^* + s') - w(x^*) \\ &\leq 0, \end{aligned} \tag{4.2}$$

where note that the nonzero components of s and s' are distinct and that w is separable. This contradicts the fact that x^* satisfies (ii). Therefore, the present claim holds. //

Let s be an element of $\mathrm{St}(x^*, x)$ that satisfies

$$w(x^* + s) - w(x^*) = \max_{t\in \mathrm{St}(x^*,x)} (w(x^* + t) - w(x^*)). \tag{4.3}$$

The above claim implies

$$w(x^*) < w(x^* + s). \tag{4.4}$$

Because of the separable convexity of w and the fact that $s \in \mathrm{St}(x^*, x)$, we have from (4.4)

$$w(x) - w(x - s) \geq w(x^* + s) - w(x^*) > 0. \tag{4.5}$$

Since $-s \in \mathrm{St}(x, x^*)$ and the greedy algorithm terminated with x, it follows from (4.5) that $x - s \notin \mathrm{P}_*(f)$. Therefore, there exists $-t \in \mathrm{St}(x, x^*)$ such that

$$x - s - t \in \mathrm{P}_*(f). \tag{4.6}$$

Since we have $x - s \notin \mathrm{P}_*(f)$ and the algorithm terminated with x, we must have from (4.6)

$$w(x - s - t) \geq w(x). \tag{4.7}$$

Consequently, from the separability of w,

$$w(x - t) - w(x) \geq w(x) - w(x - s). \tag{4.8}$$

It follows from (4.5), (4.8) and the separable convexity of w that

$$w(x^*) - w(x^* + t) \geq w(x - t) - w(x) \geq w(x) - w(x - s) \geq w(x^* + s) - w(x^*) > 0, \tag{4.9}$$

where note that $-t \in \mathrm{St}(x, x^*)$. Since x^* is an optimal solution, we see from (4.9) that $x^* + t \notin \mathrm{P}_*(f)$. Since $t \in \mathrm{St}(x^*, x)$, there exists $t' \in \mathrm{St}(x^*, x)$ such that $x^* + t + t' \in \mathrm{P}_*(f)$. Recall that x^* is an optimal solution which satisfies (ii). Therefore,

$$w(x^* + t + t') > w(x^*), \tag{4.10}$$

i.e.,

$$w(x^* + t') - w(x^*) > w(x^*) - w(x^* + t). \tag{4.11}$$

From (4.9) and (4.11) we have

$$w(x^* + t') - w(x^*) > w(x^* + s) - w(x^*), \tag{4.12}$$

which contradicts (4.3). Hence, x is an optimal solution of Problem **IP**. □

A vector $x \in \mathrm{P}_*(f)$ is called a *local optimal solution* of Problem **IP** if the following two hold:

(L1) For any $s \in S$ such that $x + s \in \mathrm{P}_*(f)$ we have $w(x) \leq w(x + s)$.

(L2) For any $s, s' \in S$ such that $x + s \notin \mathrm{P}_*(f)$ and $x + s + s' \in \mathrm{P}_*(f)$ we have $w(x) \leq w(x + s + s')$.

Corollary 4.2: *Every local optimal solution of Problem* **IP** *is also an optimal solution.*

Proof: Let x be a local optimal solution of **IP**. If we start the greedy algorithm with x, then the algorithm terminates with x because of **(L1)** and **(L2)**, and x is an optimal solution due to Theorem 4.1. □

One may notice that Conditions **(L1)** and **(L2)** are equivalent to Conditions **(L1)** and the following **(L3)**.

(L3) For any $s, s' \in S$ such that $x + s + s' \in \mathrm{P}_*(f)$ we have $w(x) \leq w(x + s + s')$.

5. Properties of the Greedy Algorithm

By the greedy algorithm, if the current x is not an optimal solution, then x is changed into $x + s$ or $x + s + s'$. Concerning such a step s we have the following theorems, Theorems 5.1 and 5.2.

Theorem 5.1: *If x is changed into $x + s$ by the greedy algorithm, then there exists an optimal solution x^* such that $s \in \mathrm{St}(x, x^*)$.*

Proof: Suppose that x is changed into $x + s$. Choose an optimal solution x^* that satisfies conditions (i) and (ii) in the proof of Theorem 4.1. Let us denote by $e(s)$ the nonzero component of a step s.

[Case 1]: $x^*(e(s)) = x(e(s))$.

In this case, since $w(x^*+s)-w(x^*)=w(x+s)-w(x)$ and $w(x+s)-w(x)<0$ because of the choice of step s, we have

$$w(x^*+s)-w(x^*)<0. \tag{5.1}$$

Hence,

$$x^*+s\notin \mathrm{P}_*(f). \tag{5.2}$$

Since $x^*, x+s\in \mathrm{P}_*(f)$ and $s\in \mathrm{St}(x^*, x+s)$, it follows from (5.2) that there exists a step

$$t_1\in \mathrm{St}(x^*, x+s) \tag{5.3}$$

such that

$$x^*+s+t_1\in \mathrm{P}_*(f). \tag{5.4}$$

Since $e(t_1)\neq e(s)$, we have from (5.3) $t_1\in \mathrm{St}(x^*, x)$, i.e.,

$$-t_1\in \mathrm{St}(x, x^*). \tag{5.5}$$

Also, by the optimality of x^* we have from (5.4)

$$w(x^*)\leq w(x^*+s+t_1), \tag{5.6}$$

i.e.,

$$w(x^*)-w(x^*+s)\leq w(x^*+t_1)-w(x^*). \tag{5.7}$$

Here, recall that $e(s)\neq e(t_1)$. If (5.6) and (5.7) hold with equality, then x^*+s+t_1 is also an optimal solution such that $s\in \mathrm{St}(x, x^*+s+t_1)$ and the present theorem with x^* replaced by x^*+s+t_1 holds. Therefore, suppose that (5.6) and (5.7) hold with strict inequality:

$$w(x^*)-w(x^*+s)<w(x^*+t_1)-w(x^*). \tag{5.8}$$

It follows from (5.5), (5.8) and the separable convexity of w that

$$\begin{aligned} 0 &< w(x)-w(x+s) \\ &= w(x^*)-w(x^*+s) \\ &< w(x^*+t_1)-w(x^*) \\ &\leq w(x)-w(x-t_1). \end{aligned} \tag{5.9}$$

We have from (5.9) $x-t_1\notin \mathrm{P}_*(f)$ due to the description of the greedy algorithm. Hence, from (5.5) there exists

$$-t_2\in \mathrm{St}(x, x^*) \tag{5.10}$$

such that

$$x-t_1-t_2\in \mathrm{P}_*(f). \tag{5.11}$$

Since $e(t_1)\neq e(t_2)$, it follows from (5.9), (5.11) and the greedy algorithm that

$$w(x)\leq w(x-t_1-t_2), \tag{5.12}$$

i.e.,

$$w(x)-w(x-t_1)\leq w(x-t_2)-w(x). \tag{5.13}$$

From (5.9) and (5.13),

$$0 < w(x - t_2) - w(x). \tag{5.14}$$

Since w is separable convex, we have from (5.10)

$$w(x - t_2) - w(x) \le w(x^*) - w(x^* + t_2). \tag{5.15}$$

We have from (5.14) and (5.15) that

$$w(x^* + t_2) < w(x^*). \tag{5.16}$$

From (5.5), (5.13) and (5.15) we have

$$\begin{aligned} w(x^* + t_1) - w(x^*) &\le w(x) - w(x - t_1) \\ &\le w(x - t_2) - w(x) \\ &\le w(x^*) - w(x^* + t_2). \end{aligned} \tag{5.17}$$

Therefore, from (5.8) and (5.17),

$$w(x^*) - w(x^* + s) < w(x^*) - w(x^* + t_2). \tag{5.18}$$

Since from (5.16) we have $x^* + t_2 \notin \mathrm{P}_*(f)$, it follows from (5.10) that there exists a step $t_3 \in \mathrm{St}(x^*, x)$ such that

$$x^* + t_2 + t_3 \in \mathrm{P}_*(f). \tag{5.19}$$

Since x^* satisfies (ii) in the proof of Theorem 4.1, we have

$$w(x^*) < w(x^* + t_2 + t_3), \tag{5.20}$$

i.e.,

$$w(x^*) - w(x^* + t_2) < w(x^* + t_3) - w(x^*). \tag{5.21}$$

Now, we can apply the same argument in (5.8)$\sim$(5.20) for the pair (s, t_1) to the pair (t_2, t_3) and obtain a pair (t_4, t_5) of steps such that

$$w(x^*) - w(x^* + t_2) < w(x^*) - w(x^* + t_4), \tag{5.22}$$

$$t_4,\ t_5 \in \mathrm{St}(x^*, x), \tag{5.23}$$

$$x^* + t_4 \notin \mathrm{P}_*(f),\ \ x^* + t_4 + t_5 \in \mathrm{P}_*(f), \tag{5.24}$$

$$w(x^*) - w(x^* + t_4) < w(x^* + t_5) - w(x^*). \tag{5.25}$$

Since this process can be repeated indefinitely, this contradicts the finiteness of $\mathrm{St}(x^*, x)$. Therefore, (5.6) must hold with equality.

[Case 2]: $x^*(e(s)) \neq x(e(s))$.
In this case, if $s \in \mathrm{St}(x, x^*)$, then the present theorem holds. So, suppose that $s \notin \mathrm{St}(x, x^*)$. Then, we have $s \in \mathrm{St}(x^*, x)$. It follows from the separable convexity of w that

$$0 < w(x) - w(x + s) \le w(x^*) - w(x^* + s). \tag{5.26}$$

Hence, $x^* + s \notin \mathrm{P}_*(f)$. Since $s \in \mathrm{St}(x^*, x)$, there exists $t_1 \in \mathrm{St}(x^*, x)$ such that

$$x^* + s + t_1 \in \mathrm{P}_*(f), \tag{5.27}$$

where we have

$$w(x^*) < w(x^* + s + t_1). \tag{5.28}$$

By the same argument in (5.8)~(5.25) of **[Case 1]** we reach a contradiction, and hence, we must have $s \in \mathrm{St}(x, x^*)$. □

We also have the following.

Theorem 5.2: *If x is changed into $x + s + s'$ such that $x + s \notin \mathrm{P}_*(f)$ and $x + s + s' \in \mathrm{P}_*(f)$ by the greedy algorithm, then there exists an optimal solution x^* such that $s \in \mathrm{St}(x, x^*)$.*

Proof: Choose an optimal solution x^* that satisfies conditions (i) and (ii) in the proof of Theorem 4.1.

[Case 1]: $x^*(e(s)) = x(e(s))$ and $x^*(e(s')) = x(e(s'))$.
Since $x^*(e(s)) = x(e(s))$, we have

$$w(x^* + s) < w(x^*) \tag{5.29}$$

by the choice of s in the greedy algorithm, where note that $w(x^* + s) - w(x^*) = w(x + s) - w(x)$. Hence, $x^* + s \notin \mathrm{P}_*(f)$ and there exists a step

$$t_1 \in \mathrm{St}(x^*, x + s + s') \tag{5.30}$$

such that $x^* + s + t_1 \in \mathrm{P}_*(f)$. If $t_1 = s'$, we would have

$$w(x^* + s + s') - w(x^*) = w(x + s + s') - w(x) < 0, \tag{5.31}$$

which contradicts the optimality of x^*. Therefore,

$$t_1 \neq s'. \tag{5.32}$$

Since $t_1 \neq s$, we have from (5.30) and (5.32)

$$t_1 \in \mathrm{St}(x^*, x). \tag{5.33}$$

Moreover, by the optimality of x^* we have

$$w(x^*) \leq w(x^* + s + t_1), \tag{5.34}$$

i.e.,

$$w(x^*) - w(x^* + s) \leq w(x^* + t_1) - w(x^*). \tag{5.35}$$

By the same argument in the proof of Theorem 5.1 we can show that (5.35) holds with equality, i.e., $x^* + s + t_1$ is also an optimal solution.

[Case 2]: $x^*(e(s)) = x(e(s))$ and $x^*(e(s')) \neq x(e(s'))$.
We have (5.29) and (5.30), where $t_1 \neq s$. Since $x^*(e(s')) \neq x(e(s'))$, we thus have (5.33). Therefore, we can show that $x^* + s + t_1$ is an optimal solution by the same argument as in **[Case 1]**.

[Case 3]: $x^*(e(s)) \neq x(e(s))$.
By the same argument as in **[Case 2]** of the proof of Theorem 5.1 we can show the present theorem. □

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