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$G/M^{a,b}/1$ QUEUES WITH SERVER VACATIONS

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Abstract In this paper we analyze a $G/M^{a,b}/1$ queue with multiple vacation discipline. Customers are served in batches according to the following bulk service rule in which at least $'a'$ customers are needed to start a service and maximum capacity of the server at a time is $'b'$. When the server either finishes a service or returns from a vacation, if he finds less than $'a'$ customers in the system, he takes a vacation with exponential distribution. When the server either finishes a service or returns from a vacation, if he hinds more than $'a'$ customers in the system, he serves a bulk of maximum of $'b'$ customers at a time. With the supplementary variable method, we explicitly obtain the queue length probabilities at arrival time points and arbitrary time points simultaneously. The shift operator method is used to solve simultaneous non-homogeneous difference equations. The results for our model in the special case of $a = b = 1$ coincide with known results for $G/M/1$ queue with multiple vacation obtained by imbedded Markov chain method.

1. Introduction.

In recent years there have been significant contributions to the theory of queue with server vacations. For complete references on vacation models, see Doshi[4,5] and Takagi[11]. Vacation models have been widely used to model many problems in computer, communication and production system. In the literature, very few results are available for batch service models with server vacations. For complete reference on batch service models without vacations, see Chaudhry and Templeton[1].

Jacob and Madhusoodanan[8] investigated the finite capacity $M/G^{a,b}/1$ queueing system with multiple vacation. Using the theory of regenerative process, they derived expression for the time dependent system size probability.

In this paper we analyze the $G/M^{a,b}/1$ queue with multiple vacation discipline. Customers are served in batches according to the following bulk service rule in which at least $'a'$ customers are needed to start a service and maximum capacity of the server at a time is $'b'$. When the server finishes a service, if he finds less than $'a'$ customers in the system, he goes away for a random length of time called vacation. If the server returns from a vacation to find less than $'a'$ customers waiting, he immediately takes another vacation, and continues in the manner until he finds at least $'a'$ customers waiting when he returns from a vacation. If the server finds more than $'a'$ customers waiting after his return from vacation or service completion, he serves a bulk of maximum of $'b'$ customers from the queue simultaneously.

With the supplementary variable method, we explicitly obtain the queue length probabilities at arrival time points and arbitrary time points simultaneously. It is well known that shift operator $\mathcal{D}$ and its polynomial $f(\mathcal{D})$ are very useful tools for solving simple

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difference equations (see, for example, Gross and Harris[6]). The shift operator method is used for solving complicated simultaneous non-homogeneous difference equations.

This paper is organized as follows. In section 2, we discuss operational calculus about the shift operator. In section 3, queue length probabilities at arrival points and arbitrary points are derived explicitly by solving simultaneous non-homogeneous difference equations. In section 4, special cases are treated. The results for our model in the special case of $a = b = 1$ coincide with known results for $G/M/1$ queue with exponential vacation obtained by imbedded Markov chain method (Choi and Park[2], Tien et al.[12]). As the rate of multiple vacation time tends to infinite, queue size distribution for queueing system with vacation will approach to that for queueing system without vacation. We show that the above facts hold for $G/M^{1,b}/1$ queue (Chaudhry and Templeton[1]).

2. Operational calculus.

In this section we will discuss operational calculus for later use. For a sequence $\{x_n\}$ of complex numbers, the right shift operator $\mathcal{D}$ is defined by $\mathcal{D}x_n = x_{n+1}$ for all n. If $f(z) = \alpha_0 + \alpha_1 z + \cdots + \alpha_k z^k$ is a polynomial with complex coefficients α_i, then the symbol $f(\mathcal{D}) = \alpha_0 + \alpha_1\mathcal{D} + \cdots + \alpha_k\mathcal{D}^k$ is naturally defined by

$$f(\mathcal{D}) \cdot x_n = \alpha_0 x_n + \alpha_1 x_{n+1} + \cdots + \alpha_k x_{n+k}.$$

It is well-known (see, for example, Gross and Harris[6], Spiegel[10]) that this type of polynomial $f(\mathcal{D})$ in $\mathcal{D}$ is used to find the general solution of difference equations.

Now we will introduce symbol $f(\mathcal{D})$ for other functions f in such a way that $f(\mathcal{D})$ has a natural meaning for particular geometric sequence $\{\omega^n\}$. For $f(z) = \sum_{k=0}^{\infty} \alpha_k z^k$, it is natural to define $f(\mathcal{D}) = \sum_{k=0}^{\infty} \alpha_k \mathcal{D}^k$ by

$$f(\mathcal{D}) \cdot \omega^n = (\sum_{k=0}^{\infty} \alpha_k \mathcal{D}^k) \cdot \omega^n = f(\omega) \cdot \omega^n.$$

For instance, since $\exp(z) = \sum_{n=0}^{\infty} \frac{1}{k!} z^k$, it follows that

$$\exp(\mathcal{D}) \cdot \omega^n = \exp(\omega) \cdot \omega^n.$$

Let $a^*(\theta) = \int_0^\infty e^{-\theta x} a(x)\, dx$ be the Laplace transform of function $a(x)$. By the same reason as above, we have $a^*(\mathcal{D}) \cdot \omega^n = a^*(\omega) \cdot \omega^n$.

When $f(\mathcal{D}) \cdot x_n = \omega^n$, the inverse operator $\frac{1}{f(\mathcal{D})}$ of $f(\mathcal{D})$ is defined by $\frac{1}{f(\mathcal{D})} \cdot \omega^n = x_n$. For instance, since $(\mathcal{D} + \alpha) \cdot \omega^n = (\omega + \alpha) \cdot \omega^n$, it follows that

$$(\frac{1}{\mathcal{D} + \alpha}) \cdot \omega^n = (\frac{1}{\omega + \alpha}) \cdot \omega^n.$$

For a geometric sequence $\{\omega^n\}$, we summarize operator formula which are very useful in obtaining a particular solution of difference equations.

Proposition 2.1. For $\alpha_1, \alpha_2, \omega \in \mathbf{C}$ and $m \in \mathbf{N}$,

(i) $(\alpha_1\mathcal{D}^m + \alpha_2) \cdot \omega^n = (\alpha_1\omega^m + \alpha_2) \cdot \omega^n$.

(ii) $\dfrac{1}{\alpha_1\mathcal{D}^m + \alpha_2} \cdot \omega^n = \dfrac{1}{\alpha_1\omega^m + \alpha_2} \cdot \omega^n$, if $\alpha_1\omega^m + \alpha_2 \neq 0$.

(iii) $a^*(\alpha_1 + \alpha_2\mathcal{D}^m) \cdot \omega^n = a^*(\alpha_1 + \alpha_2\omega^m) \cdot \omega^n$.

(iv) $\dfrac{1}{\mathcal{D} - a^*(\alpha_1 + \alpha_2\mathcal{D}^m)} \cdot \omega^n = \dfrac{1}{\omega - a^*(\alpha_1 + \alpha_2\omega^m)} \cdot \omega^n$, if $\omega - a^*(\alpha_1 + \alpha_2\omega^m) \neq 0$.

3. Queue length probabilities.

We consider a $G/M^{a,b}/1$ queueing system with server vacations. The interarrival times of customers are independent and identically distributed with *p.d.f.* $a(x)$, with mean $1/\lambda$ and Laplace transform $a^*(\theta)$. The service times and vacation times are independent and exponentially distributed with mean $1/\mu$ and $1/\nu$, respectively. Since the statistical equilibrium conditions for both the queue with vacation and the queue without vacation are same [4], we always assume that $\rho = \frac{\lambda}{b\mu} < 1$ in the remaining of this paper.

Now we will investigate the distribution of the queue length in the system at arrival time points and at arbitrary time points simultaneously by the supplementary variable method. Here we take supplementary variable as the remaining interarrival time.

At an arbitrary time, the steady state of the system can be characterized by the following variables;

$$N = \text{the number of customers in the queue;}$$
$$\tilde{A} = \text{the remaining interarrival time;}$$
$$\xi = \begin{cases} 0, & \text{if server is on vacation;} \\ j, & \text{if server is busy with } j \text{ customers in a batch, } a \le j \le b. \end{cases}$$

Define

$$p_{i0}(x)dx = P(N = i, \xi = 0, \tilde{A} \in (x, x + dx]), \quad i \ge 0,$$
$$p_{ij}(x)dx = P(N = i, \xi = j, \tilde{A} \in (x, x + dx]), \quad i \ge 0, \ a \le j \le b,$$

and their Laplace transforms

$$p^*_{ij}(\theta) = \int_0^\infty e^{-\theta x} p_{ij}(x)dx.$$

Note that $p^*_{i0}(0)$ ($\frac{p_{i0}(0)}{\lambda} \equiv p^{(a)}_{i0}$) are steady state probabilities that there are i customers in the queue and server is on vacation at arbitrary time points (arrival time points, respectively), and $p^*_{ij}(0)$ ($\frac{p_{ij}(0)}{\lambda} \equiv p^{(a)}_{ij}$) are steady state probabilities that there are i customers in the queue and server is busy with j customers in a batch at arbitrary time points (arrival time points, respectively) for $a \le j \le b$.

Using a typical argument of the supplementary variable method (Choi and Park[3], Hokstad[7]), we have the following system of differential difference equations;

$$(3.1a)\qquad -p'_{00}(x)=\sum_{n=a}^{b}\mu p_{0n}(x),$$

$$(3.1b)\qquad -p'_{i0}(x)=\sum_{n=a}^{b}\mu p_{in}(x)+a(x)p_{i-1,0}(0),\quad i<a,$$

$$(3.1c)\qquad -p'_{i0}(x)=-\nu p_{i0}(x)+a(x)p_{i-1,0}(0),\quad i\geq a,$$

$$(3.1d)\qquad -p'_{0j}(x)=-\mu p_{0j}(x)+\nu p_{j0}(x)+\sum_{n=a}^{b}\mu p_{jn}(x),\ a\leq j\leq b,$$

$$(3.1e)\qquad -p'_{ij}(x)=-\mu p_{ij}(x)+a(x)p_{i-1,j}(0),\ i\geq 1,\ a\leq j\leq b-1,$$

$$(3.1f)\qquad \begin{aligned}-p'_{ib}(x)=&-\mu p_{ib}(x)+\nu p_{i+b,0}(x)\\&+\sum_{n=a}^{b}\mu p_{i+b,n}(x)+a(x)p_{i-1,b},\ i\geq 1.\end{aligned}$$

By taking Laplace transform to the above equations, it follows that

$$(3.2a)\qquad \theta p^*_{00}(\theta)=p_{00}(0)-\sum_{n=a}^{b}\mu p^*_{0n}(\theta),$$

$$(3.2b)\qquad \theta p^*_{i0}(\theta)=p_{i0}(0)-a^*(\theta)p_{i-1,0}(0)-\sum_{n=a}^{b}\mu p^*_{in}(\theta),\quad i<a,$$

$$(3.2c)\qquad (\theta-\nu)p^*_{i0}(\theta)=p_{i0}(0)-a^*(\theta)p_{i-1,0}(0),\quad i\geq a,$$

$$(3.2d)\qquad (\theta-\mu)p^*_{0j}(\theta)+\sum_{n=a}^{b}\mu p^*_{jn}(\theta)+\nu p^*_{j0}(\theta)=p_{0j}(0),\quad a\leq j\leq b,$$

$$(3.2e)\qquad (\theta-\mu)p^*_{ij}(\theta)=p_{ij}(0)-a^*(\theta)p_{i-1,j}(0),\quad i\geq 1,\ a\leq j\leq b-1,$$

$$(3.2f)\qquad \begin{aligned}&(\theta-\mu)p^*_{ib}(\theta)+\sum_{n=a}^{b}\mu p^*_{i+b,n}(\theta)+\nu p^*_{i+b,0}(\theta)\\&\qquad=p_{ib}(0)-a^*(\theta)p_{i-1,b}(0),\ i\geq 1.\end{aligned}$$

The rest of this section is devoted to find the queue length probabilities $p^{(a)}_{ij}$ at arrival time points, and queue length probabilities $p^*_{ij}(0)$ at arbitrary time points.

Letting $\theta=\nu$ in (3.2c), we have

$$(3.3)\qquad p_{i0}(0)=p_0\alpha^i,\quad i\geq a-1,$$

where $\alpha=a^*(\nu)$ and $p_0=p_{a-1,0}(0)\alpha^{1-a}$. Substitution of the equation (3.3) into (3.2c) yields

$$(3.4)\qquad p^*_{i0}(\theta)=\frac{p_0(\alpha-a^*(\theta))}{\theta-\nu}\alpha^{i-1},\quad i\geq a.$$

Letting $\theta=\mu$ in (3.2e), we have

$$(3.5)\qquad p_{ij}(0)=p_j\omega^i,\quad i\geq 0,\ a\leq j\leq b-1,$$

where $\omega = a^*(\mu)$ and $p_j = p_{0j}(0)$. By inserting (3.5) into (3.2e), we obtain

$$p^*_{ij}(\theta) = \frac{p_j(\omega - a^*(\theta))}{\theta - \mu}\omega^{i-1}, \quad i \geq 1,\ a \leq j \leq b-1. \tag{3.6}$$

We need the following lemmas for solving non-homogeneous difference equations.

Lemma 3.1. If $\frac{\lambda}{b\mu} < 1$, then $z - a^*(\mu - \mu z^b) = 0$ has the unique root γ between 0 and 1.

Proof. See Gross and Harris[6]. □

Lemma 3.2. Let $\{x_n\}_{n=0}^{\infty}$ be an unknown sequence with $\sum_{n=0}^{\infty} |x_n| \leq 1$.

(i) A particular solution of difference equation $(\mathcal{D} - \delta) \cdot x_n = \xi^n$ with $\xi \neq \delta$ is given by

$$x_n = \frac{1}{\xi - \delta} \cdot \xi^n.$$

(ii) The general solution of homogeneous difference equation $(\mathcal{D} - \delta) \cdot x_n = 0$ with $|\delta| < 1$ is given by

$$x_n = c\delta^n,$$

where c is arbitrary constant.

(iii) A particular solution of difference equation $(\mathcal{D} - a^*(\mu - \mu\mathcal{D}^b)) \cdot x_n = \delta^n$ $(\delta \neq \gamma)$ is given by

$$x_n = \frac{\delta^n}{\delta - a^*(\mu - \mu\delta^b)},$$

where γ is the unique root of $z - a^*(\mu - \mu z^b) = 0$ and $0 < \gamma < 1$.

(iv) If $\frac{\lambda}{b\mu} < 1$, then the general solution of homogeneous difference equation $(\mathcal{D} - a^*(\mu - \mu\mathcal{D}^b)) \cdot x_n = 0$ is given by

$$x_n = c\gamma^n,$$

where c is arbitrary constant.

Proof. (i). This is obtained by applying Proposition 2.1.(ii).

(ii). Clear.

(iii). This is obtained by applying Proposition 2.1.(iv).

(iv). In general, when γ_i is a root of $z - a^*(\mu - \mu z^b) = 0$, a solution of $(\mathcal{D} - a^*(\mu - \mu\mathcal{D}^b)) \cdot x_n = 0$ is given by $x_n = c_i\gamma_i^n$. So the general solution is a linear combination of such solutions. But we require that a solution $\{x_n\}$ must satisfies $\sum_{n=0}^{\infty} |x_n| \leq 1$. To satisfy this condition, a root γ of $z - a^*(\mu - \mu z^b) = 0$ must be inside the unit circle. Since there is only one root inside the unit circle of $z - a^*(\mu - \mu z^b) = 0$ under the assumption $\frac{\lambda}{b\mu} < 1$, the general solution of $(\mathcal{D} - a^*(\mu - \mu\mathcal{D}^b)) \cdot x_n = 0$ is given by $x_n = c\gamma^n$. □

By using the shift operator $\mathcal{D}$, the equation (3.2f) can be written as

$$(\theta - \mu + \mu\mathcal{D}^b)p^*_{ib}(\theta) = (\mathcal{D} - a^*(\theta))p_{i-1,b}(0) - \nu p^*_{i+b,0}(\theta) - \sum_{n=a}^{b-1} \mu p^*_{i+b,n}(\theta). \tag{3.7}$$

By considering the imbedded Markov chain at arrival time points, we obtain the following relation

$$p_{ib}(0) = \sum_{j=a}^{b-1}\sum_{k=1}^{\infty} p_{i-1+kb,j}(0)\int_0^\infty \frac{e^{-\mu t}(\mu t)^k}{k!}\,a(t)\,dt$$
$$(3.8)\qquad + \sum_{k=0}^{\infty} p_{i-1+kb,b}(0)\int_0^\infty \frac{e^{-\mu t}(\mu t)^k}{k!} a(t)\,dt$$
$$+ \sum_{k=1}^{\infty} p_{i-1+kb,0}(0)\int_0^\infty\int_0^t \nu e^{-\nu x}\frac{e^{-\mu(t-x)}(\mu(t-x))^{k-1}}{(k-1)!} a(t)\,dx dt.$$

By substituting (3.3) and (3.5) into (3.8), we obtain

$$p_{ib}(0) = \left(\int_0^\infty e^{-\mu(1-\mathcal{D}^b)t}\,a(t)\,dt\right)\cdot p_{i-1,b}(0)$$
$$(3.9)\qquad + p_0\nu\alpha^{i+b-1}\int_0^\infty\int_0^t e^{-\nu x}e^{-\mu(1-\alpha^b)(t-x)}\,a(t)\,dx\,dt$$
$$+ \sum_{j=a}^{b-1} p_j\omega^{i-1}\int_0^\infty \left(e^{-\mu(1-\omega^b)t} - e^{-\mu t}\right)\,a(t)\,dt.$$

This can be rewritten as

$$(3.10)\qquad (\mathcal{D} - a^*(\mu - \mu\mathcal{D}^b))p_{ib}(0) = \frac{p_0\nu(\alpha - a^*(\mu - \mu\alpha^b))}{\mu(1-\alpha^b)-\nu}\alpha^{i+b} - \sum_{j=a}^{b-1} p_j(\omega - a^*(\mu - \mu\omega^b))\omega^i,\ i \ge 0.$$

Note that the another formal way to obtain (3.10) is to substitute $\theta = \mu - \mu\mathcal{D}^b$ into (3.7). By Lemma 3.2(iii), a particular solution of (3.10) is given by

$$(3.11)\qquad p_{ib}^{(p)}(0) = \frac{p_0\nu}{\mu(1-\alpha^b)-\nu}\alpha^{i+b} - \sum_{j=a}^{b-1} p_j\omega^i,\ i \ge 0,$$

where $\alpha \neq \gamma$ and $\omega \neq \gamma$. For the brevity of paper, we treat only the case $\alpha \neq \gamma$ and $\omega \neq \gamma$. By Lemma 3.1 and Lemma 3.2(iv), the general solution of homogeneous difference equation $(\mathcal{D} - a^*(\mu - \mu\mathcal{D}^b))p_{ib}(0) = 0$ of (3.10) is given by

$$(3.12)\qquad p_{ib}^{(h)}(0) = C\gamma^i,$$

where C is arbitrary constant. Since the general solution of non-homogeneous difference equation (3.10) is the sum of the solution of homogeneous equation and a particular solution, the general solution of (3.10) is given by

$$(3.13)\qquad p_{ib}(0) = C\gamma^i + \frac{p_0\nu}{\mu(1-\alpha^b)-\nu}\alpha^{i+b} - \sum_{j=a}^{b-1} p_j\omega^i,\ i \ge 0.$$

Next we find out $p^*_{ib}(\theta)$. Let $z_j(\theta)$, $1 \le j \le b$, be the zeros of $\theta - \mu + \mu z^b = 0$ for fixed θ with $Re(\theta) \ge 0$. Then the general solution of homogeneous difference equation $(\theta - \mu + \mu \mathcal{D}^b)p^*_{ib}(\theta) = 0$ of (3.7) is $\sum_{j=1}^{b} d_j z_j^i(\theta)$ where d_j is arbitrary constant. By inserting (3.4), (3.6) and (3.13) into (3.7) and applying Lemma 3.2(i), a particular solution of (3.7) is given by

$$\text{(3.14)}\quad \begin{aligned} p_{ib}^{*(p)}(\theta) =& \frac{C(\gamma - a^*(\theta))}{\theta - \mu(1-\gamma^b)}\gamma^{i-1} + \frac{p_0\nu(\alpha - a^*(\theta))}{(\mu(1-\alpha^b) - \nu)(\theta - \nu)}\alpha^{i+b-1} \\ & - \sum_{j=a}^{b-1} p_j \frac{(\omega - a^*(\theta))}{\theta - \mu}\omega^{i-1}. \end{aligned}$$

Thus the general solution of (3.7) is

$$\text{(3.15)}\quad \begin{aligned} p^*_{ib}(\theta) =& \sum_{j=1}^{b} d_j z_j^i(\theta) + \frac{C(\gamma - a^*(\theta))}{\theta - \mu(1-\gamma^b)}\gamma^{i-1} \\ & + \frac{p_0\nu(\alpha - a^*(\theta))}{(\mu(1-\alpha^b) - \nu)(\theta - \nu)}\alpha^{i+b-1} - \sum_{j=a}^{b-1} p_j \frac{(\omega - a^*(\theta))}{\theta - \mu}\omega^{i-1}. \end{aligned}$$

Since $\sum_{i=0}^{\infty} p^*_{ib}(0) \le 1$, we have $\sum_{i=0}^{\infty} p^*_{bi+k,b}(0) \le 1$, $k = 1, 2, \cdots, b$. Note that $z_j^{bi+k}(0) = z_j^k(0)$, since $z_j^k(0)\,(1 \le j \le b)$ are b-th roots of 1. By letting $\theta = 0$ in (3.15) and summing over i, we must have the convergence of $\sum_{i=0}^{\infty}\sum_{j=1}^{b} d_j z_j^k(0)$, since $\sum_{i=0}^{\infty} p^*_{bi+k,b}(0)$ converges and γ, α and ω are less than 1. This fact occurs only when $\sum_{j=1}^{b} d_j z_j^k(0) = 0$, for $k = 1, 2 \cdots, b$. The determinant of $b \times b$-matrix $(z_j^k(0))$ is known as Vandermonde determinant of order b [9, page 93] and is not equal zero. Therefore all $d_j\,(1 \le j \le b)$ must be zero. Hence we have

$$\text{(3.16)}\quad \begin{aligned} p^*_{ib}(\theta) =& \frac{C(\gamma - a^*(\theta))}{\theta - \mu(1-\gamma^b)}\gamma^{i-1} + \frac{p_0\nu(\alpha - a^*(\theta))}{(\mu(1-\alpha^b) - \nu)(\theta - \nu)}\alpha^{i+b-1} \\ & - \sum_{j=a}^{b-1} p_j \frac{(\omega - a^*(\theta))}{\theta - \mu}\omega^{i-1}, \quad i \ge 1. \end{aligned}$$

Now we find $p_{i0}(0)$, $0 \le i \le a-2$. By inserting $\theta = 0$ into (3.2b) and using (3.6) and (3.16), we obtain the recursive relation

$$\text{(3.17)}\quad \begin{aligned} p_{i0}(0) - p_{i-1,0}(0) &= \sum_{j=a}^{b-1} \mu p^*_{ij}(0) + \mu p^*_{ib}(0) \\ &= \frac{C(1-\gamma)}{1-\gamma^b}\gamma^{i-1} + \frac{p_0\mu(1-\alpha)}{\mu(1-\alpha^b) - \nu}\alpha^{i+b-1}. \end{aligned}$$

By the above recursive relation, we have

$$\text{(3.18)}\quad p_{i0}(0) = \frac{C}{1-\gamma^b}(\gamma^{a-1} - \gamma^i) + \frac{p_0}{\mu(1-\alpha^b) - \nu}((\mu - \nu)\alpha^{a-1} - \mu\alpha^{b+i}), \ 0 \le i \le a-2.$$

Next we determine the constants C, p_0 and p_i $(a \le i < b)$. For this purpose, we will find $b-a+2$ equations involving C, p_0 and p_i $(a \le i < b)$, among which $b-a+1$ equations come from boundary conditions (3.2d) and other equation comes from the following relation

$$\lambda = \sum_{i=0}^{\infty} p_{i0}(0) + \sum_{i=0}^{\infty} \sum_{j=a}^{b} p_{ij}(0). \tag{3.19}$$

Thus from (3.19) we obtain one equation;

$$\begin{aligned} \lambda =& \sum_{i=0}^{a-2} \left(\frac{C(\gamma^{a-1}-\gamma^i)}{1-\gamma^b} + \frac{p_0((\mu-\nu)\alpha^{a-1}-\mu\alpha^{b+i})}{\mu(1-\alpha^b)-\nu} \right) \\ &+ \sum_{i=a-1}^{\infty} p_0\alpha^i + \sum_{i=0}^{\infty}\sum_{j=a}^{b-1} p_j\omega^i \\ &+ \sum_{i=0}^{\infty} \left(C\gamma^i + \frac{p_0\nu\alpha^{i+b}}{\mu(1-\alpha^b)-\nu} - \sum_{j=a}^{b-1} p_j\omega^i \right) \\ =& \frac{C}{1-\gamma^b}\left[(a-1)\gamma^{a-1} + \frac{\gamma^{a-1}-\gamma^b}{1-\gamma}\right] \\ &+ \frac{p_0(\mu-\nu)}{\mu(1-\alpha^b)-\nu}\left[(a-1)\alpha^{a-1} + \frac{\alpha^{a-1}-\alpha^b}{1-\alpha}\right] \end{aligned} \tag{3.20}$$

Letting $\theta = \mu$ in (3.2d) yields

$$p_{0i}(0) = \sum_{j=a}^{b} \mu p_{ij}^*(\mu) + \nu p_{i0}^*(\mu), \quad a \le i \le b. \tag{3.21}$$

By substituting (3.4), (3.5), (3.6), (3.13) and (3.16) into (3.21), we have other equations;

$$\frac{C}{\gamma} + \frac{p_0\nu\alpha^{b-1}}{\mu(1-\alpha^b)-\nu} - \sum_{j=a}^{b-1}\frac{p_j}{\omega} = 0, \tag{3.22}$$

$$p_i = C(\gamma-\omega)\gamma^{i-b-1} + \frac{p_0\nu(\alpha-\omega)}{\mu(1-\alpha^b)-\nu}\alpha^{i-1}, \quad a \le i \le b-1. \tag{3.23}$$

By solving the simultaneous system of $b-a+2$ linear equations (3.20), (3.22) and (3.23) for unknown constants C, p_0 and p_i $(a \le i \le b-1)$, we have

(3.24a)

$$C = \lambda f(\alpha)\left\{\frac{f(\alpha)g(\gamma)}{1-\gamma^b} + \frac{(\nu-\mu)f(\gamma)g(\alpha)}{\gamma\nu\alpha^{b-1}}\right\}^{-1},$$

(3.24b)

$$p_0 = \frac{\lambda(\nu-\mu(1-\alpha^b))f(\gamma)}{\gamma\nu\alpha^{b-1}}\left\{\frac{f(\alpha)g(\gamma)}{1-\gamma^b} + \frac{(\nu-\mu)f(\gamma)g(\alpha)}{\gamma\nu\alpha^{b-1}}\right\}^{-1},$$

(3.24c)

$$p_i = \frac{\lambda}{\gamma}\left\{\frac{f(\alpha)(\gamma-\omega)\gamma^{i-1}}{r^{b-1}} - \frac{f(\gamma)(\alpha-\omega)\alpha^{i-1}}{\alpha^{b-1}}\right\} \cdot\left\{\frac{f(\alpha)g(\gamma)}{1-\gamma^b} + \frac{(\nu-\mu)f(\gamma)g(\alpha)}{\gamma\nu\alpha^{b-1}}\right\}^{-1}, \quad a \le i \le b-1,$$

where $g(x) = (a-1)x^{a-1} + \frac{x^{a-1}-x^b}{1-x}$ and $f(x) = 1 - \frac{(x-\omega)(1-x^{b-a})}{\omega(1-x)x^{b-a}}$.

Thus we have obtained the following main results.

Theorem 3.3. (i) The steady state probabilities $p_{i0}^{(a)}$ ($p_{ij}^{(a)}$) that an arrival sees i customers in the queue and the server is on vacation (busy with j customers in a batch, respectively) are given by

$$p_{i0}^{(a)} = \frac{1}{\lambda}\left\{\frac{C}{1-\gamma^b}(\gamma^{a-1}-\gamma^i) + \frac{p_0}{\mu(1-\alpha^b)-\nu}((\mu-\nu)\alpha^{a-1} - \mu\alpha^{b+i})\right\}, \quad 0 \le i \le a-2,$$

$$p_{i0}^{(a)} = \frac{1}{\lambda}p_0\alpha^i, \quad i \ge a-1,$$

$$p_{ij}^{(a)} = \frac{1}{\lambda}p_j\omega^i, \quad i \ge 0,\ a \le j \le b-1,$$

$$p_{ib}^{(a)} = \frac{1}{\lambda}\left\{C\gamma^i + \frac{p_0\nu}{\mu(1-\alpha^b)-\nu}\alpha^{i+b} - \sum_{j=a}^{b-1} p_j\omega^i\right\}, \quad i \ge 0.$$

(ii) The steady state probabilities $p_{i0}^*(0)$ ($p_{ij}^*(0)$) that there are i customers in the system and the server is on vacation (busy with j customers in a batch, respectively) at arbitrary time points are given by

$$p_{i0}^*(0) = \frac{p_0}{\nu}(1-\alpha)\alpha^{i-1}, \quad i \ge a,$$

$$p_{ij}^*(0) = \frac{p_j}{\mu}(1-\omega)\omega^{i-1} \quad i \ge 1,\ a \le j \le b-1,$$

$$p_{0j}^*(0) = \frac{1}{\mu}\left\{\frac{C(1-\gamma)}{1-\gamma^b}\gamma^{j-1} + \frac{p_0(1-\alpha)(\mu-\nu)}{\mu(1-\alpha^b)-\nu}\alpha^{j-1} + p_j\right\}, \quad a \le j \le b-1,$$

$$p_{ib}^*(0) = \frac{C(1-\gamma)}{\mu(1-\gamma^b)}\gamma^{i-1} + \frac{p_0(1-\alpha)}{\mu(1-\alpha^b)-\nu}\alpha^{i+b-1} - \sum_{j=a}^{b-1}\frac{p_j(1-\omega)}{\mu}\omega^{i-1}, \quad i \ge 1,$$

$$p_{0b}^*(0) = \frac{1}{\mu}\left\{\frac{C}{1-\gamma^b}(\gamma^{b-1}-1) + \frac{p_0(\mu(1-\alpha)-\nu)}{\mu(1-\alpha^b)-\nu}\alpha^{b-1} + \sum_{j=a}^{b-1}p_j\right\},$$

$p^*_{i0}(0)$ $(0 \le i \le a-1)$ are obtained from (3.2a) and (3.2b). The constants C, p_0 and p_i $(a \le i \le b-1)$ are given by (3.24).

Proof. Since $p^{(a)}_{ij} = \frac{1}{\lambda}p_{ij}(0)$, (i) follows from (3.18), (3.3), (3.5) and (3.13). (ii) follows from (3.4), (3.6), (3.2d) and (3.16) by letting $\theta = 0$.

4. Special cases.

(i) Since our model for $a = b = 1$ becomes $G/M/1$ queue with exponential vacation, we will show that Theorem 3.3 matches with known result for $G/M/1$ queue with exponential vacation (Choi and Park[2], Tian et al.[12]). From (3.24a) and (3.24b), we obtain

$$C = \lambda\gamma(1-\gamma)\sigma\beta, \tag{4.1a}$$

$$p_0 = \lambda(1-\gamma)\sigma, \tag{4.1b}$$

where $\sigma = \frac{\mu(1-\alpha)-\nu}{\mu(1-\gamma)-\nu}$ and $\beta = \frac{-\nu}{\mu(1-\alpha)-\nu}$. Then by substuting C and p_0 into Theorem 3.3(i), we obtain

$$p^{(a)}_{i0} = (1-\gamma)\sigma\alpha^i, \quad i \ge 0, \tag{4.2a}$$

$$p^{(a)}_{i1} = (1-\gamma)\sigma\beta(\gamma^{i+1} - \alpha^{i+1}), \quad i \ge 0. \tag{4.2b}$$

Similarly we have from Theorem 3.3(ii) that

$$p^*_{i0}(0) = \frac{1}{\nu}(1-\alpha)(1-\gamma)\sigma\alpha^{i-1}, \quad i \ge 1, \tag{4.3a}$$

$$p^*_{i1}(0) = \lambda(1-\gamma)\sigma\beta[\frac{\gamma^i}{\mu} - \frac{(1-\alpha)\alpha^i}{\nu}], \quad i \ge 1, \tag{4.3b}$$

$$p^*_{01}(0) = \frac{\lambda}{\mu}(1-\gamma)\sigma. \tag{4.3c}$$

The above result agrees with the one for $G/M/1$ with exponential vacation (Choi and Park[2], Tian et al.[12]).

(ii) As the rate of exponential vacation time tends to infinite, queue size distribution for queueing system with vacation approaches to that for queueing system without vacation. We show that the above facts hold for $G/M^{1,b}/1$ queue with $a = 1$(Chaudhry and Templeton[1]). Let $C^{(o)} = \lim_{\nu\to\infty} C$, $p^{(o)}_0 = \lim_{\nu\to\infty} p_0$ and $p^{(o)}_i = \lim_{\nu\to\infty} p_i$. Using $\lim_{\nu\to\infty}\alpha = \lim_{\nu\to\infty} a^*(\nu) = 0$, we obtain from (3.24a), (3.24b) and (3.24c) that

$$C^{(o)} = \frac{\lambda\omega(1-\gamma)\gamma^b}{\omega+\gamma^b-\gamma}, \tag{4.4a}$$

$$p^{(o)}_0 = \lambda\left(1 - \frac{\gamma^b\omega}{\omega+\gamma^b-\gamma}\right), \tag{4.4b}$$

$$p^{(o)}_1 = \frac{\lambda\omega^2\gamma(1-\gamma^{b-1})}{\omega+\gamma^b-\gamma}, \tag{4.4c}$$

$$p^{(o)}_i = \frac{\lambda\omega(1-\gamma)(\gamma-\omega)\gamma^{i-1}}{\omega+\gamma^b-\gamma}, \quad 2 \le i \le b-1. \tag{4.4d}$$

Let $p_{00}^{(ao)} = \lim_{\nu\to\infty} p_{00}^{(a)}$ and $p_{ij}^{(ao)} = \lim_{\nu\to\infty} p_{ij}^{(a)}$. Then by Theorem 3.3(i), we obtain

$$p_{00}^{(ao)} = 1 - \frac{\gamma^b \omega}{\omega + \gamma^b - \gamma}, \tag{4.5a}$$

$$p_i^{(ao)} \equiv \sum_{j=1}^{b} p_{ij}^{(ao)} = \frac{\omega(1-\gamma)\gamma^{i+b}}{\omega + \gamma^b - \gamma}, \quad i \geq 0. \tag{4.5b}$$

Note that $p_i^{(ao)}$ is the probability that there are i customers in the queue and server is busy at arrival time points. The above result agrees with the one for ordinary $G/M^{1,b}/1$ queue (Chaudhry and Templeton[1], page 292).

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