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A SINGLE-MACHINE MULTI-PRODUCT LOT SCHEDULING PROBLEM WITH CONSIDERATION OF PRODUCT-DEPENDENT TRANSPORTATION

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Abstract This paper investigates the lot scheduling problem for multiple products processed on a single machine, under a cyclical production sequence and deterministic demand rate. The problem is formulated in the form of a demand and supply balance equation, with consideration of the method by which each product produced is transported from the single machine's location to spatially different multiple demand locations. The paper identifies conditions that ensure the existence of a feasible production lot schedule, and proves that the "balanced lot size" is a unique form of lot sizes for a feasible schedule over an infinite planning horizon. It is clarified that the balanced lot size is not affected by the transportation method assigned to each product. The difference in transportation method only affects the amount of initial inventory required at the beginning of the horizon. These results are applied to the problem of finding a feasible production schedule over a finite planning period, and two methods, utilization of the balanced lot size and application of the balance equation, are presented and advantages of each method are discussed.

1. Introduction

This paper investigates the fundamental properties of feasible production schedules and their lot sizes for multiple products to be processed on a single machine. Products are transported after production to different places of consumption in order to satisfy demand at each location. The paper allows different methods of transportation to be used for each product. Such a "diverging" type of a product flow, as shown in Figure 1, is frequently observed in various industries, and is increasingly common due to the recent trend toward customized products.

In such a situation, production must switch between lots to meet the given demand for each product without shortages (delays against demand), since the single machine cannot produce multiple items at one time. If some products are produced with lot sizes that are too large, it will be impossible to meet the demand of other products. If production is interrupted by machine idle times, lot sizes must be adjusted to prevent shortages during the idle periods. Thus, we must carefully determine production lot sizes to obtain a feasible schedule which can meet all the demand without any shortages.

In previous research, this problem has been formulated as an economic lot scheduling problem (ELSP) to minimize the sum of changeover cost and inventory holding cost. However, the most common approach in ELSP, the basic period approach, does not necessarily assure a feasible solution. It requires trial-and-error process [2], [5], [8], or feasibility check algorithms [3], [4], [9], neither of which guarantees feasibility. Based on this observation, Kono and Nakamura [7] identified conditions that assure the existence of a feasible schedule under common cycle assumptions. And Zipkin [12] discussed several characteristics of feasible schedules under an extended assumption of a common cycle schedule.

However, a common shortcoming in all of these previous studies is that they neglect the method by which products are transported to the places of consumption. All of them implicitly assume that the demand for each product can be satisfied as soon as the production of a lot is started. However, in practice, spatially distributed demand cannot be satisfied without transporting products to the appropriate locations. In addition, the demand period to be satisfied by each production lot is affected by the transportation method. If products are transported piece by piece, demand can be satisfied as soon as each piece is completed. On the other hand, if products are transported in batches, demand cannot be satisfied before the production of the batch is completed. Thus, the difference in the methods of transportation becomes an important factor to be considered in planning a feasible schedule of lot sizes.

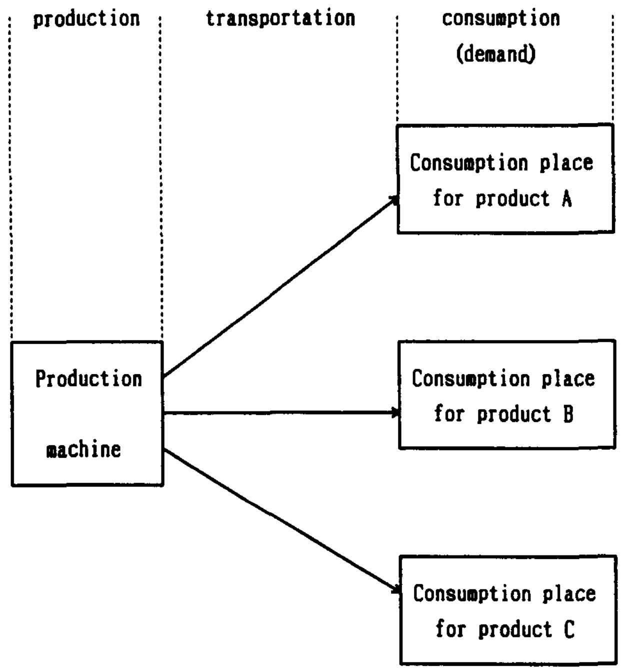

Figure 1. Product flow pattern of the problem

The main purpose of this paper is to clarify the conditions under which a feasible production schedule exists and describe the schedule's properties, reflecting such factors as the production and demand rates for each product, setup times, and the transportation methods assigned to each product. In order to better determine the relation between these relevant factors, this paper incorporates intentional machine idle time as well as physical setup time. The primary discussion is based on a model with deterministic constant demand over an infinite time horizon, employing a fixed production sequence. From a practical viewpoint, a method to obtain a feasible schedule on a finite planning horizon will also be discussed applying major results obtained in a case of infinite time horizon.

2. Model Formulation

2.1 Basic Assumptions and Notations

(1) Over an infinite time horizon $[0, \infty)$, demand rates for r products are given by constants $D_i > 0$ for product i, $i = 1, 2, \cdots, r$.

(2) Only one product can be produced at a time. The production rate of product i is given

by constant $K_i > 0$, $i = 1, 2, \cdots, r$. In order to standardize units for each product, D_i and K_i shall be scaled by the production rate K_i, so that each D_i is represented by the ratio of the demand rate to the production rate and each K_i satisfies

$$K_i = 1, \ i = 1, 2, \cdots, r. \tag{2.1}$$

And we assume that

$$D_i < 1, \ i = 1, 2, \cdots, r. \tag{2.2}$$

(3) r products are produced in lots, following a cyclic repetition of a production sequence $\pi = (1, 2, \cdots, r)$.
(4) The production lot size of product i in the k-th cycle is denoted by $x_{ik} > 0$, $i = 1, 2, \cdots, r$, $k = 0, 1, 2, \cdots, \infty$, where x_{i0} represents the initial inventory of product i held at time 0.
(5) There is no unexpected machine downtime. From the condition (2.1), x_{ik} also represents its production time.
(6) The setup time required for product i, which is not dependent upon production sequence, is denoted by $\tau_i \geq 0, i = 1, 2, \cdots, r$. Machine idle time immediately before production of product i is denoted by $u_i \geq 0$, $i = 1, 2, \cdots, r$. In order to assure adequate time for setups, the next condition must be satisfied.

$$u_i \geq \tau_i, \ i = 1, 2, \cdots, r. \tag{2.3}$$

(7) We assume that inventory capacity is limited, so that

$$0 < x_{ik} \leq M, \ i = 1, 2, \cdots, r, \ k = 0, 1, 2, \cdots, \infty. \tag{2.4}$$

(8) No shortage or advance delivery of any product is allowed, so that all production lots x_{ik}, $i = 1, 2, \cdots, r$, $k = 1, 2, \cdots, \infty$ are transported to the designated place of consumption when inventory of each product at demand location becomes 0. Transportation time is assumed to be constant, and so can be assumed to be 0 for all products.[1]

2.2 Types of Transportation Method

We shall categorize transportation methods based upon the following two factors. The first factor is transportation timing; whether each piece of product is transported immediately upon completion, or stored besides the machine after completion to be transported as a lot. The second factor is transportation lot consolidation; whether each production lot is transported separately or consolidated with other lots to be produced later in a cycle. This paper deals with the following four methods of transportation that can be derived from the possible combinations on these factors:

(1) In-lot Continuous Type (Continuous Type): Products are transported piece by piece upon coming out from the machine to the consumption places between the starting time and the completion time for the lot. This method can be observed when the machine and the consumption places are directly connected by such means as a conveyor or pipe.
(2) Single Lot Type (Lot Type): All the products in a production lot are transported in bulk as soon as the production of the lot is completed. This method is observed when small bulk carriers are used for transportation.

[1] A production schedule incorporating a positive constant transportation time for each shipment can be obtained by shifting the schedule obtained assuming 0 transportation time to reflect the actual transportation time.

(3) Multi-lot Continuous Type (Kit Type): Several lots of products (kits) are accumulated and then transported throughout the period of production of the last lot in the group. This method is often applied when distributing assembly parts in "kit" boxes.
(4) Lot Collective Type (Collective Type): Several production lots are accumulated and transported collectively after the completion of the last piece of the last product. This method is often applied when large bulk carriers, such as trucks, are utilized.

There is a clear difference among these four types of method with regard to the appropriate production timing of each lot. Products transported by the continuous type can start to meet the demand as soon as the first product in the lot is completed, while they can do so only after the completion of the entire production lot under the lot type, which then requires an earlier start in production operation. Following the same logic, production must be started much earlier under the kit and the collective type.

In order to clarify the impact of the transportation methods on a production schedule, this paper deals with a general case in which one of the four methods of transportation is arbitrarily specified for each product.

2.3 Demand and Supply Balance Equation for Each Type of the Transportation Method

In general, the k-th cycle is represented by the following sequence of r production times and r machine idle times;

$$(x_{1k}, u_2, x_{2k}, \cdots, u_i, x_{ik}, \cdots, u_j, x_{jk}, \cdots, u_r, x_{rk}, u_1).$$

Denoting by t_{ik} the length of the time period over which the demand for product i is to be satisfied by lot x_{ik}, from assumption (8):

$$x_{ik} = t_{ik} D_i, i = 1, 2, \cdots, r, \ k = 1, 2, \cdots, \infty \tag{2.5}$$

Since the left hand side denotes the production (supply) quantity and the right hand side refers to the demand to be satisfied by lot x_{ik}, we shall call (2.5) the "demand and supply balance equation" of product i for the k-th cycle.

Here, the relative positioning of a demand period of length t_{ik} and the corresponding lot production period x_{ik} on the time axis differs according to the transportation type (see Figure 2). If product i is of the continuous type, t_{ik} is the period between production starting time of lot x_{ik} and that of $x_{i,k+1}$ as in Figure 2-(1). Then,

$$\begin{aligned} x_{ik} = \{&(x_{ik} + x_{i+1,k} + \cdots + x_{rk} + x_{1,k+1} + x_{2,k+1} + \cdots + x_{i-1,k+1}) \\ &+ (u_{i+1} + \cdots + u_r + u_1 + u_2 + \cdots + u_i)\} D_i, \ k = 1, 2, \cdots, \infty. \end{aligned} \tag{2.6}$$

Shifting the terms representing consumption in the k-th cycle to the left hand side, we obtain

$$\begin{aligned} &x_{ik} - (x_{ik} + x_{i+1,k} + \cdots + x_{rk}) D_i - (u_{i+1} + \cdots + u_r + u_1) D_i \\ &= (x_{1,k+1} + x_{2,k+1} + \cdots + x_{i-1,k+1}) D_i + (u_2 + \cdots + u_i) D_i, \ k = 1, 2, \cdots, \infty. \end{aligned} \tag{2.7.i}$$

Since there is no demand before time $0, x_{i0}$ is given by

$$x_{i0} = (x_{11} + x_{21} + \cdots + x_{i-1,1}) D_i + (u_2 + \cdots + u_i) D_i. \tag{2.7.ii}$$

In the same manner, we obtain the followings for the other transportation types:

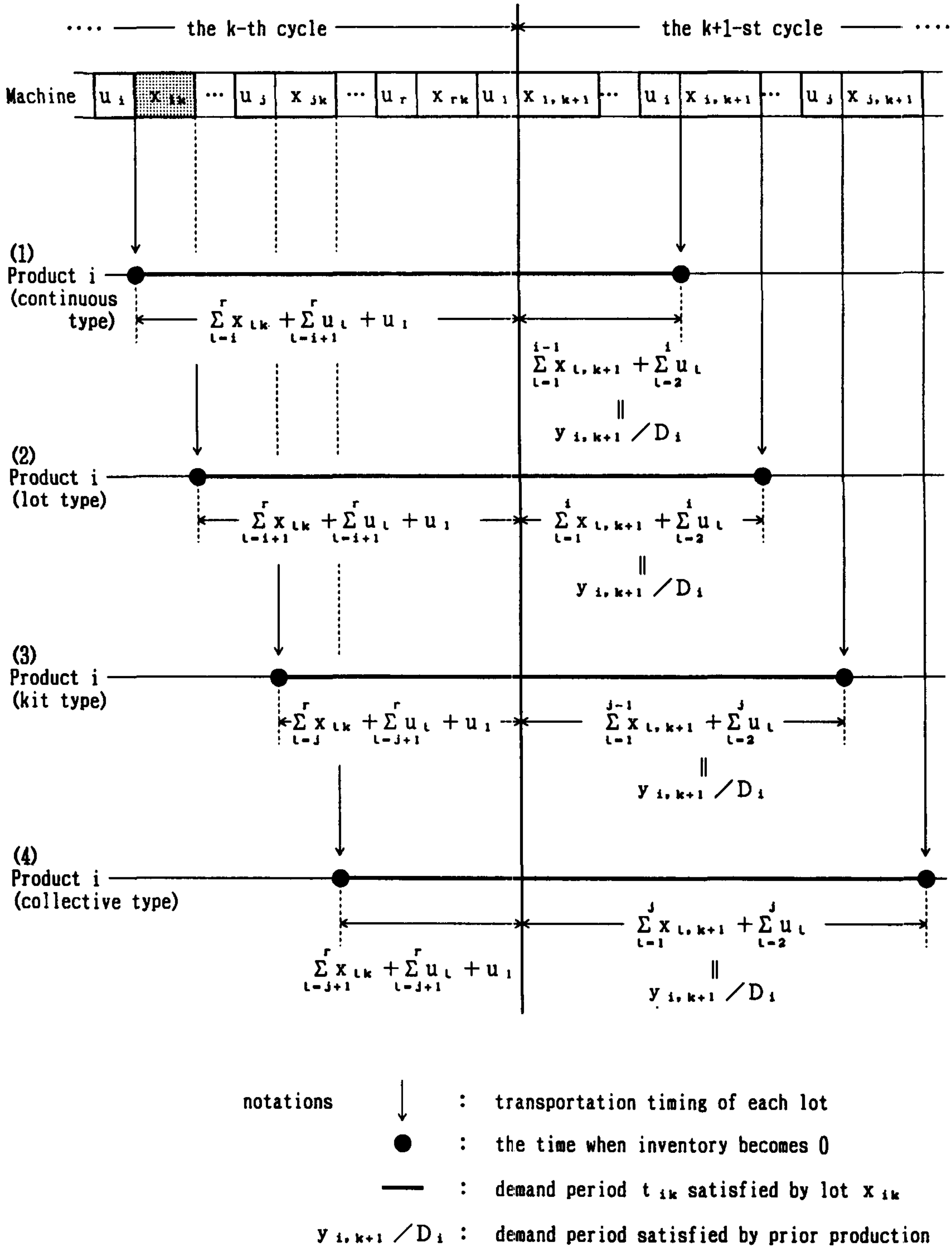

Figure 2. Demand period t_{ik} for each transportation type

Lot Type:

$$x_{ik} - (x_{i+1,k} + x_{i+2,k} + \cdots + x_{rk})D_i - (u_{i+1} + \cdots + u_r + u_1)D_i$$
$$= (x_{1,k+1} + x_{2,k+1} + \cdots + x_{i,k+1})D_i + (u_2 + \cdots + u_i)D_i, \quad k = 1,2,\cdots,\infty. \quad (2.8.i)$$
$$x_{i0} = (x_{11} + x_{21} + \cdots + x_{i1})D_i + (u_2 + \cdots + u_i)D_i. \quad (2.8.ii)$$

Kit Type: to be transported together with product $j > i$

$$x_{ik} - (x_{jk} + x_{j+1,k} + \cdots + x_{rk})D_i - (u_{j+1} + \cdots + u_r + u_1)D_i$$
$$= (x_{1,k+1} + x_{2,k+1} + \cdots + x_{j-1,k+1})D_i + (u_2 + \cdots + u_j)D_i, \quad k = 1,2,\cdots,\infty. \quad (2.9.i)$$
$$x_{i0} = (x_{11} + x_{21} + \cdots + x_{j-1,1})D_i + (u_2 + \cdots + u_j)D_i. \quad (2.9.ii)$$

Collective Type: to be transported together with product $j > i$

$$x_{ik} - (x_{j+1,k} + x_{j+2,k} + \cdots + x_{rk})D_i - (u_{j+1} + \cdots + u_r + u_1)D_i$$
$$= (x_{1,k+1} + x_{2,k+1} + \cdots + x_{j,k+1})D_i + (u_2 + \cdots + u_j)D_i, \quad k = 1,2,\cdots,\infty. \quad (2.10.i)$$
$$x_{i0} = (x_{11} + x_{21} + \cdots + x_{j1})D_i + (u_2 + \cdots + u_j)D_i. \quad (2.10.ii)$$

In equations (2.7) through (2.10), the left hand side describes the carried-over inventory at the end of the k-th cycle (initial inventory when $k = 0$), and the right hand side describes the quantity produced in advance in the k-th cycle for the consumption in the $k+1$-st cycle, which shall be called the "prior production quantity" of product i for the $k+1$-st cycle and denoted by $y_{i,k+1}$. Coefficients of $x_{ik}, x_{i,k+1}$, and u_i, $i = 1,2,\cdots r$, on the both sides of (2.7) through (2.10) are listed for each transportation type in Table 1.

2.4 Balance Equation for r Products with Mixed Transportation Methods

We shall represent a set of production lot sizes (initial inventory when $k = 0$) and machine idle times in each cycle by vectors

$$\mathbf{x}_k = [x_{1k}, x_{2k}, \cdots, x_{rk}]^T, \quad k = 0,1,2,\cdots,\infty, \quad \mathbf{u} = [u_2, u_3, \cdots, u_r, u_1]^T.$$

And we shall denote the coefficient matrix of $\mathbf{x}_k$ on the left hand side in equations (2.7.i) through (2.10.i) by $\mathbf{E}$, that of $\mathbf{x}_{k+1}$ on the right hand side by $\mathbf{F}$, and those of $\mathbf{u}$ on the left and right hand sides by $\mathbf{D}_0$ and $\mathbf{D}_1$, respectively. These matrices can be determined referring to Table 1, corresponding to the transportation type of each product.

As an example, we shall consider a case of $r = 6$, where products 1 and 6 are of the collective type, products 2 and 4 of the kit type, product 3 of the lot type and product 5 of the continuous type. Then;

$$\mathbf{E} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -D_2 & -D_2 & -D_2 \\ 0 & 0 & 1 & -D_3 & -D_3 & -D_3 \\ 0 & 0 & 0 & 1-D_4 & -D_4 & -D_4 \\ 0 & 0 & 0 & 0 & 1-D_5 & -D_5 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \mathbf{F} = \begin{bmatrix} D_1 & D_1 & D_1 & D_1 & D_1 & D_1 \\ D_2 & D_2 & D_2 & 0 & 0 & 0 \\ D_3 & D_3 & D_3 & 0 & 0 & 0 \\ D_4 & D_4 & D_4 & 0 & 0 & 0 \\ D_5 & D_5 & D_5 & D_5 & 0 & 0 \\ D_6 & D_6 & D_6 & D_6 & D_6 & D_6 \end{bmatrix}$$

$$\mathbf{D}_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & D_1 \\ 0 & 0 & 0 & D_2 & D_2 & D_2 \\ 0 & 0 & D_3 & D_3 & D_3 & D_3 \\ 0 & 0 & 0 & D_4 & D_4 & D_4 \\ 0 & 0 & 0 & 0 & D_5 & D_5 \\ 0 & 0 & 0 & 0 & 0 & D_6 \end{bmatrix} \quad \mathbf{D}_1 = \begin{bmatrix} D_1 & D_1 & D_1 & D_1 & D_1 & 0 \\ D_2 & D_2 & D_2 & 0 & 0 & 0 \\ D_3 & D_3 & 0 & 0 & 0 & 0 \\ D_4 & D_4 & D_4 & 0 & 0 & 0 \\ D_5 & D_5 & D_5 & D_5 & 0 & 0 \\ D_6 & D_6 & D_6 & D_6 & D_6 & 0 \end{bmatrix}$$

Table 1. Coefficients of $x_{ik}, x_{i,k+1}$ and u_i under $\pi = (1, 2, \cdots, r)$ for each transportation type

(1) Coefficients of x_{ik} in the k-th cycle

Type	x_{1k}	$\cdots$	$x_{i-1,k}$	x_{ik}	$x_{i+1,k}$	$\cdots$	$x_{j-1,k}$	x_{jk}	$x_{j+1,k}$	$\cdots$	$x_{r-1,k}$	x_{rk}
Continuous	0	$\cdots$	0	$1-D_i$	$-D_i$	$\cdots$	$-D_i$	$-D_i$	$-D_i$	$\cdots$	$-D_i$	$-D_i$
Lot	0	$\cdots$	0	1	$-D_i$	$\cdots$	$-D_i$	$-D_i$	$-D_i$	$\cdots$	$-D_i$	$-D_i$
Kit	0	$\cdots$	0	1	0	$\cdots$	0	$-D_i$	$-D_i$	$\cdots$	$-D_i$	$-D_i$
Collective	0	$\cdots$	0	1	0	$\cdots$	0	0	$-D_i$	$\cdots$	$-D_i$	$-D_i$

(2) Coefficients of $x_{i,k+1}$ in the $k+1$-st cycle

Type	$x_{1,k+1}$	$\cdots$	$x_{i-1,k+1}$	$x_{i,k+1}$	$x_{i+1,k+1}$	$\cdots$	$x_{j-1,k+1}$	$x_{j,k+1}$	$x_{j+1,k+1}$	$\cdots$	$x_{r-1,k+1}$	$x_{r,k+1}$
Continuous	D_i	$\cdots$	D_i	0	0	$\cdots$	0	0	0	$\cdots$	0	0
Lot	D_i	$\cdots$	D_i	D_i	0	$\cdots$	0	0	0	$\cdots$	0	0
Kit	D_i	$\cdots$	D_i	D_i	D_i	$\cdots$	D_i	0	0	$\cdots$	0	0
Collective	D_i	$\cdots$	D_i	D_i	D_i	$\cdots$	D_i	D_i	0	$\cdots$	0	0

(3) Coefficients of u_i in the k-th cycle

Type	u_2	$\cdots$	u_i	u_{i+1}	u_{i+2}	$\cdots$	u_j	u_{j+1}	u_{j+2}	$\cdots$	u_r	u_1
Continuous	0	$\cdots$	0	D_i	D_i	$\cdots$	D_i	D_i	D_i	$\cdots$	D_i	D_i
Lot	0	$\cdots$	0	D_i	D_i	$\cdots$	D_i	D_i	D_i	$\cdots$	D_i	D_i
Kit	0	$\cdots$	0	0	0	$\cdots$	0	D_i	D_i	$\cdots$	D_i	D_i
Collective	0	$\cdots$	0	0	0	$\cdots$	0	D_i	D_i	$\cdots$	D_i	D_i

(4) Coefficients of u_i in the $k+1$-st cycle

Type	u_2	$\cdots$	u_i	u_{i+1}	u_{i+2}	$\cdots$	u_j	u_{j+1}	u_{j+2}	$\cdots$	u_r	u_1
Continuous	D_i	$\cdots$	D_i	0	0	$\cdots$	0	0	0	$\cdots$	0	0
Lot	D_i	$\cdots$	D_i	0	0	$\cdots$	0	0	0	$\cdots$	0	0
Kit	D_i	$\cdots$	D_i	D_i	D_i	$\cdots$	D_i	0	0	$\cdots$	0	0
Collective	D_i	$\cdots$	D_i	D_i	D_i	$\cdots$	D_i	0	0	$\cdots$	0	0

Diagonal elements in matrix $\mathbf{E}$ always include value 1, which signifies the production of a lot, and $-D_i$ in matrix $\mathbf{E}$ describes the consumption in the k-th cycle, while D_i in matrix $\mathbf{F}$ means consumption in the $k+1$-st cycle. And D_i in matrices $\mathbf{D}_0$ and $\mathbf{D}_1$ represent consumption during machine idle times in the k-th and $k+1$-st cycles, respectively.

Demand and supply balance equation for r products can be represented with these matrices by

$$\mathbf{E}\mathbf{x}_k - \mathbf{D}_0\mathbf{u} = \mathbf{F}\mathbf{x}_{k+1} + \mathbf{D}_1\mathbf{u}, \ k = 1, 2, \cdots, \infty. \tag{2.11}$$

Shifting the vector $\mathbf{D}_0\mathbf{u}$ to the right hand side, we get

$$\mathbf{E}\mathbf{x}_k = \mathbf{F}\mathbf{x}_{k+1} + \mathbf{D}\mathbf{u}, \ k = 1, 2, \cdots, \infty, \tag{2.12.i}$$

$$\text{where } \mathbf{D} = \mathbf{D}_0 + \mathbf{D}_1 = \begin{bmatrix} D_1 & D_1 & \cdots & D_1 \\ D_2 & D_2 & \cdots & D_2 \\ \vdots & \vdots & & \vdots \\ D_r & D_r & \cdots & D_r \end{bmatrix}.$$

While the initial inventory vector is given by

$$\mathbf{x}_0 = \mathbf{F}\mathbf{x}_1 + \mathbf{D}_1\mathbf{u}. \tag{2.12.ii}$$

The problem to be investigated in this paper is to find all solutions $\{\mathbf{x}_k\}$, $k = 0, 1, 2, \cdots, \infty$, to equations (2.12.i) and (2.12.ii) under the condition (2.4).

2.5 Fundamental Properties of Matrices E and F

It is clear from Table 1, under an arbitrary mixture of transportation types, that matrix $\mathbf{E} = [e_{ij}]$ is a triangular matrix whose elements are

$$\begin{aligned} e_{ij} &= 0 \quad \text{or} \quad -D_i, \quad i < j. \\ e_{ij} &= 1 \quad \text{or} \quad 1 - D_i, \ i = j. \\ e_{ij} &= 0, \qquad\qquad\quad i > j. \end{aligned} \tag{2.13}$$

Therefore:

Property 2.1 Regardless of the mixture of transportation types, matrix $\mathbf{E}$ is of full rank.

It is clear from the matrix inversion procedure that matrix $\mathbf{E}^{-1} = [e'_{ij}]$ satisfies the next property.

Property 2.2 Regardless of the mixture of transportation types, matrix $\mathbf{E}^{-1} = [e'_{ij}]$ is a non-negative matrix with

$$\begin{aligned} e'_{ij} &\geq 0, & i < j. \\ e'_{ij} &= 1 \text{ or } 1/(1 - D_i) \geq 1, & i = j. \\ e'_{ij} &= 0, & i > j. \end{aligned} \tag{2.14}$$

On the other hand, matrix $\mathbf{F} = [f_{ij}]$ is with

$$\begin{aligned} f_{ij} &= D_i \text{ or } 0, & i \leq j. \\ f_{ij} &= D_i, & i > j. \end{aligned} \tag{2.15}$$

Matrix $\mathbf{F}$ is not always of full rank. All elements on the first row of $\mathbf{F}$ are 0 if product 1 is of the continuous type. All rows for products in the same group of the kit or the collective type are linearly dependent. It is of full rank only when all products are of the lot type. It is then clear that the production lot sizes can be determined by the backward induction in (2.12.i).

Since lot x_{ik} is consumed during r successive lot production periods in the k-th and k+1-st cycles, independent of the transportation type of product i, either of the corresponding elements in matrix $\mathbf{E}$ or $\mathbf{F}$ becomes $-D_i$ or D_i, and the other becomes 0. The value 1 appears in all diagonal elements of matrix $\mathbf{E}$. Therefore:

Property 2.3 Independent of the mixture of transportation types for r products,

$$\mathbf{E} - \mathbf{F} = \mathbf{I} - \mathbf{D}, \text{ where } \mathbf{I} \text{ is the unit matrix.} \tag{2.16}$$

These properties play an important role in investigating feasible solutions for (2.12).

2.6 Relationship between Lot Size and Prior Production Quantity

We shall represent the prior production quantity which is produced in the $k-1$-st cycle for the consumption in the k-th cycle by a vector

$$\mathbf{y}_k = [y_{1k}, y_{2k}, \cdots, y_{rk}]^T, \ k = 1, 2, \cdots, \infty.$$

Then,

$$\mathbf{y}_k = \mathbf{F}\mathbf{x}_k + \mathbf{D}_1\mathbf{u}, \ k = 1, 2, \cdots, \infty. \tag{2.17}$$

And obviously,

$$\mathbf{y}_1 = \mathbf{x}_0. \tag{2.18}$$

Here, multiplying equation (2.11) on the left by matrix $\mathbf{FE}^{-1}$, adding $\mathbf{D}_1\mathbf{u}$ to the both sides, and applying the relation (2.17), the next equation is derived.

$$\mathbf{y}_k = \mathbf{G}\mathbf{y}_{k+1} + \mathbf{G}\mathbf{D}_0\mathbf{u} + \mathbf{D}_1\mathbf{u}, \ k = 1, 2, \cdots, \infty, \tag{2.19}$$

where

$$\mathbf{G} = \mathbf{FE}^{-1}. \tag{2.20}$$

Matrix $\mathbf{G}$ is not of full rank in general since it is a product with matrix $\mathbf{F}$. Therefore, the values of vectors $\mathbf{y}_k$ are determined also by the backward induction in (2.19).

From equations (2.17) and (2.18), for a solution $\mathbf{x}_k$, $k = 0, 1, 2, \cdots, \infty$, of (2.12), we can confirm that $\mathbf{y}_k$, $k = 1, 2, \cdots, \infty$, are uniquely determined and are a solution for equation (2.19). On the other hand, equations (2.11) and (2.17) yield that

$$\mathbf{x}_k = \mathbf{E}^{-1}(\mathbf{y}_{k+1} + \mathbf{D}_0\mathbf{u}), \ k = 1, 2, \cdots, \infty. \tag{2.21}$$

Statements (2.18) and (2.21) tell us that $\mathbf{x}_k$, $k = 0, 1, 2, \cdots, \infty$, are uniquely determined for a solution $\mathbf{y}_k$, $k = 1, 2, \cdots, \infty$, of (2.19), and constitute a solution for equation (2.12).

From this discussion, we can conclude that there is one-to-one correspondence between the solutions $\{\mathbf{x}_k\}$ for (2.12) and $\{\mathbf{y}_k\}$ for (2.19). Since matrix $\mathbf{G}$ defined in (2.20) has an attractive property as will be described in the next chapter, we shall use equation (2.19) and its solution $\{\mathbf{y}_k\}$ when convenient in analysis.

3. Feasible Solutions

In this section, we shall examine the problem of finding all feasible solutions $\{\mathbf{x}_k\}$ in (2.12), taking into consideration the relative size of the total demand for all products $\sum_{1\leq i\leq r} D_i$ against the production rate $K_i = 1$. Under the condition $\sum_{1\leq i\leq r} D_i > 1$, the production capacity is insufficient to meet the given total demand over an infinite horizon, so there exists no feasible solution.[2] Therefore, the other two cases, $\sum_{1\leq i\leq r} D_i < 1$ and $\sum_{1\leq i\leq r} D_i = 1$, are investigated.

[2] If we change the inequality sign "<" into ">" in the proof of Lemmas 3.1 and 3.2, matrix $\mathbf{G} = [g_{ij}]$ satisfies;

$$\sum_{1\leq i\leq r} D_i > 1 \Rightarrow \sum_{1\leq i\leq r} g_{ij} > 1, \ 1 \leq j \leq r \Rightarrow \lim_{N\to\infty} \mathbf{G}^N = \mathbf{Z},$$

where all elements in $\mathbf{Z}$ diverge to $+\infty$ for sufficiently large N. Then statements (3.35) and (2.17) yield that there exists no feasible solution in (2.12) under the condition (2.4).

3.1 Feasible Solution in the Case $\sum_{1\le i\le r} D_i < 1$

3.1.1 A Stationary Solution when Machine Idle Time Is Required

We first investigate the case $\sum_{1\le i\le r} D_i < 1$ under the condition that machine idle time is required, namely $\sum_{1\le i\le r} u_i > 0$.

We assume that there exists a stationary solution $\{\mathbf{x}_0^*, \mathbf{x}^*, \cdots, \mathbf{x}^*, \cdots\}$ in (2.12). Then,

$$\mathbf{E}\mathbf{x}^* = \mathbf{F}\mathbf{x}^* + \mathbf{D}\mathbf{u}. \tag{3.1.i}$$
$$\mathbf{x}_0^* = \mathbf{F}\mathbf{x}^* + \mathbf{D}_1\mathbf{u}. \tag{3.1.ii}$$

Applying Property 2.3 to (3.1.i),

$$(\mathbf{I} - \mathbf{D})\mathbf{x}^* = \mathbf{D}\mathbf{u}. \tag{3.2}$$

Since matrix $(\mathbf{I} - \mathbf{D})$ is of full rank in the case $\sum_{1\le i\le r} D_i < 1$, $\mathbf{x}^*$ and $\mathbf{x}_0^*$ are determined by the following equations (refer to Appendix A).

$$\mathbf{x}^* = t^*\mathbf{d}, \tag{3.3.i}$$

and from (3.1.ii),

$$\mathbf{x}_0^* = t^*\mathbf{F}\mathbf{d} + \mathbf{D}_1\mathbf{u}, \tag{3.3.ii}$$

where

$$t^* = \sum_{1\le i\le r} u_i/(1 - \sum_{1\le i\le r} D_i), \tag{3.4}$$

and $\mathbf{d} = [D_1, D_2, \cdots, D_r]^T$.

We shall hereafter call the lot size determined by (3.3.i) the "balanced lot size". The prior production quantity $\mathbf{y}^*$ under the balanced lot size is determined by (2.17):

$$\mathbf{y}^* = t^*\mathbf{F}\mathbf{d} + \mathbf{D}_1\mathbf{u}, \tag{3.5}$$

which is equivalent to the initial inventory given by (3.3.ii).

3.1.2 Uniqueness of the Solution

Next, we shall prove that $\{\mathbf{x}_0^*, \mathbf{x}^*, \cdots, \mathbf{x}^*, \cdots\}$ is a unique solution for (2.12) in the case $\sum_{1\le i\le r} D_i < 1$ and $\sum_{1\le i\le r} u_i > 0$. We assume that there exists another solution $\{\mathbf{x}_k^o\}$, $k = 0, 1, 2, \cdots, \infty$. We shall denote the difference between the two solutions by

$$\bar{\mathbf{x}}_0 = \mathbf{x}_0^o - \mathbf{x}_0^*. \tag{3.6.i}$$
$$\bar{\mathbf{x}}_k = \mathbf{x}_k^o - \mathbf{x}^*, \; k = 1, 2, \cdots, \infty. \tag{3.6.ii}$$

From (2.12),

$$\bar{\mathbf{x}}_0 = \mathbf{F}\bar{\mathbf{x}}_1. \tag{3.7.i}$$
$$\mathbf{E}\bar{\mathbf{x}}_k = \mathbf{F}\bar{\mathbf{x}}_{k+1}, \; k = 1, 2, \cdots, \infty. \tag{3.7.ii}$$

If $\bar{\mathbf{x}}_n = \mathbf{0}$ for some finite number n, then $\mathbf{E}\bar{\mathbf{x}}_{n-1} = \mathbf{0}$ from (3.7.ii). It follows from Property 2.1 that $\bar{\mathbf{x}}_{n-1} = \mathbf{0}$. Repeating the same procedure, we can derive $\bar{\mathbf{x}}_k = \mathbf{0}$ for all k, $0 \le k \le n$. Therefore, if $\{\mathbf{x}_k^o\}$ exists, there must exist a finite number n such that

$$\bar{\mathbf{x}}_k = \mathbf{0}, \; k = 0, 1, 2, \cdots, n-1, \text{ and } \bar{\mathbf{x}}_k \ne \mathbf{0}, \; k = n, n+1, \cdots, \infty. \tag{3.8}$$

Here we put

$$\bar{\mathbf{y}}_k = \mathbf{F}\bar{\mathbf{x}}_k,\ k = 1, 2, \cdots, \infty. \tag{3.9}$$

Multiplied by matrix **G** defined in (2.20) from the left and applying (3.9), statement (3.7.ii) can be restated as

$$\bar{\mathbf{y}}_k = \mathbf{G}\bar{\mathbf{y}}_{k+1},\ k = 1, 2, \cdots, \infty. \tag{3.10}$$

Further

$$\bar{\mathbf{y}}_k = \mathbf{G}^N\bar{\mathbf{y}}_{k+N},\ N \geq 1. \tag{3.11}$$

Hence, from (3.7) and Property 2.1, the conditions in (3.8) are equivalent to the following statements.

$$\mathbf{F}\bar{\mathbf{x}}_{k+1} = \mathbf{0},\ k = 0, 1, 2, \cdots, n-1, \text{ and } \mathbf{F}\bar{\mathbf{x}}_{k+1} \neq \mathbf{0},\ k = n, n+1, \cdots, \infty. \tag{3.12}$$

From (3.9), we obtain:

Proposition 3.1 If the solution $\{\mathbf{x}_k^o\}$ exists, then there must exist a finite number n such that

$$\bar{\mathbf{y}}_k = \mathbf{0},\ k = 1, 2, \cdots, n, \text{ and } \bar{\mathbf{y}}_k \neq \mathbf{0},\ k = n+1, n+2, \cdots, \infty. \tag{3.13}$$

Under the condition (2.4), $\{\bar{\mathbf{y}}_k\}$ is obviously bounded by some finite value vector **M** so that

$$|\bar{\mathbf{y}}_k| \leq \mathbf{M},\ k = 1, 2, \cdots, \infty. \tag{3.14}$$

Therefore, statement (3.11) can be stated for the finite number $n+1$ in Proposition 3.1 as

$$|\bar{\mathbf{y}}_{n+1}| = \mathbf{G}^N|\bar{\mathbf{y}}_{n+1+N}| \leq \mathbf{G}^N\mathbf{M},\ N \geq 1. \tag{3.15}$$

Hence, if it can be proven that

$$\lim_{N\to\infty} \mathbf{G}^N = \mathbf{0}, \tag{3.16}$$

then statement $|\bar{\mathbf{y}}_{n+1}| = \mathbf{0}$, which is contradictory to (3.13), can be derived. Therefore, we shall investigate the convergence of $\lim_{N\to\infty} \mathbf{G}^N$.

Lemma 3.1 The next proposition holds for the sum of column elements of $\mathbf{G} = [g_{ij}]$.

(III.1) $$\sum_{1\leq i\leq r} D_i < 1 \Rightarrow \sum_{1\leq i\leq r} g_{ij} \leq \sum_{1\leq i\leq r} D_i < 1,\ j = 1, 2, \cdots, r.$$[3]

Proof: Refer to Appendix B.

Lemma 3.2 The next proposition concerns the power matrix of **G**.

(III.2) $$\sum_{1\leq i\leq r} g_{ij} < 1,\ j = 1, 2, \cdots, r \Rightarrow \lim_{N\to\infty} \mathbf{G}^N = \mathbf{0}.$$

Proof: Refer to Appendix C.

From Lemmas 3.1 and 3.2, the next proposition is obtained.

(III.3) $$\sum_{1\leq i\leq r} D_i < 1 \Rightarrow \lim_{N\to\infty} \mathbf{G}^N = \mathbf{0}.$$

[3] Equality holds in the case that the method of transportation for all products is of a single group of the collective type.

It follows from (3.15) and (3.16) that

$$|\bar{\mathbf{y}}_{n+1}| \leq \lim_{N\to\infty} \mathbf{G}^N\mathbf{M} = \mathbf{0}. \tag{3.17}$$

This result is contradictory to the assumption in Proposition 3.1.

Summarizing the above analysis, we have the next theorem.

Theorem 3.1 Independent of the mixture of transportation types for r products, the form of the balanced lot size in (3.3.i) and the associated initial inventory in (3.3.ii) is a unique solution for (2.12) in the case $\sum_{1\leq i\leq r} D_i < 1$ and $\sum_{1\leq i\leq r} u_i > 0$.

3.1.3 A Case where Machine Idle Time Is Not Required

In the case $\sum_{1\leq i\leq r} D_i < 1$, since the production rate is greater than the total demand, there is no feasible solution under the condition $\sum_{1\leq i\leq r} u_i = 0$, causing over-production. Since $\sum_{1\leq i\leq r} u_i = 0$ yields $u_i = 0$ for all i and thereby $\mathbf{u} = \mathbf{0}$, we get from (2.12) that

$$\mathbf{x}_0 = \mathbf{F}\mathbf{x}_1. \tag{3.18.i}$$
$$\mathbf{E}\mathbf{x}_k = \mathbf{F}\mathbf{x}_{k+1},\ k = 1, 2, \cdots, \infty. \tag{3.18.ii}$$

Because (3.18) is the same structure as (3.7), $\mathbf{x}_k = \mathbf{0}$ for all k can be proven in the same manner as Section 3.1.2. Thereby there is no feasible solution in (3.18).

Therefore, in the case $\sum_{1\leq i\leq r} D_i < 1$, we can conclude that a feasible solution is obtained if and only if $\sum_{1\leq i\leq r} u_i > 0$. When setup time is required for some products, a feasible solution is obtained uniquely for arbitrary $\sum_{1\leq i\leq r} u_i$, provided $u_i \geq \tau_i$, $i = 1, 2, \cdots, r$. If setup time is 0 for all products, a feasible solution is uniquely obtained, provided $\sum_{1\leq i\leq r} u_i$ is positive. This result implies that any reduction in setup time increases the feasible range of $\sum_{1\leq i\leq r} u_i$, making it easier to create a feasible schedule.

3.2 Feasible Solution in the Case $\sum_{1\leq i\leq r} D_i = 1$

3.2.1 A Stationary Solution when Machine Idle Time Is Not Required

We shall investigate the case $\sum_{1\leq i\leq r} D_i = 1$ under the condition that machine idle time is not required for all products, so that $\sum_{1\leq i\leq r} u_i = 0$. From (2.3), the condition $\sum_{1\leq i\leq r} u_i = 0$ yields $u_i = 0$, and thereby $\tau_i = 0$ for all products.

We assume that there exists a stationary solution $\{\mathbf{x}_0^*, \mathbf{x}^*, \cdots, \mathbf{x}^*, \cdots\}$ in (2.12). Then applying $\mathbf{u} = \mathbf{0}$,

$$\mathbf{E}\mathbf{x}^* = \mathbf{F}\mathbf{x}^*. \tag{3.19.i}$$
$$\mathbf{x}_0^* = \mathbf{F}\mathbf{x}^*. \tag{3.19.ii}$$

Applying Property 2.3 to (3.19.i),

$$(\mathbf{I} - \mathbf{D})\mathbf{x}^* = \mathbf{0}. \tag{3.20}$$

Since matrix $(\mathbf{I} - \mathbf{D})$ has rank $r-1$ under the condition $\sum_{1\leq i\leq r} D_i = 1$, there exist infinitely many non-trivial solutions (refer to Appendix A), which can be represented by

$$x_i^* = tD_i,\ i = 1, 2, \cdots, r. \tag{3.21}$$

Applying the condition that $\sum_{1\leq i\leq r} D_i = 1$ and $x_i^* > 0$ for all i,

$$t = \sum_{1\leq i\leq r} x_i^* > 0, \tag{3.22}$$

where t represents the length of one complete cycle.

Therefore, in the case $\sum_{1\leq i\leq r} D_i = 1$ and $\sum_{1\leq i\leq r} u_i = 0$, there exist infinitely many solutions which can be represented by

$$\mathbf{x}^* = t\mathbf{d}, \tag{3.23.i}$$
$$\mathbf{x}_0^* = t\mathbf{F}\mathbf{d}, \tag{3.23.ii}$$

where $t > 0$ and $\mathbf{d} = [D_1, D_2, \cdots, D_r]^T$.

The lot size determined by (3.23.i) shall be also called the "balanced lot size" whose value is determined depending upon the arbitrary length of the cycle time t. The prior production quantity $\mathbf{y}^*$ under the balanced lot size is determined from (2.17) by

$$\mathbf{y}^* = t\mathbf{F}\mathbf{d}, \text{ where } t > 0, \tag{3.24}$$

which is equivalent to the initial inventory given by (3.23.ii).

3.2.2 Uniqueness of the Solution

We assume that there exists another solution $\{\mathbf{x}_k^o\}$, $k = 1, 2, \cdots, \infty$, in the case $\sum_{1\leq i\leq r} D_i = 1$ and $\sum_{1\leq i\leq r} u_i = 0$. From (2.12), $\{\mathbf{x}_k^o\}$ satisfies

$$\mathbf{x}_0^o = \mathbf{F}\mathbf{x}_1^o. \tag{3.25.i}$$
$$\mathbf{E}\mathbf{x}_k^o = \mathbf{F}\mathbf{x}_{k+1}^o, \ k = 1, 2, \cdots, \infty. \tag{3.25.ii}$$

If $\mathbf{x}_n^o = t\mathbf{d}$ for some finite number n, then from (3.25.ii) and Property 2.1 it holds that $\mathbf{x}_{n-1}^o = t\mathbf{E}^{-1}\mathbf{F}\mathbf{d}$. Applying Property 2.3, we have $\mathbf{x}_{n-1}^o = t\mathbf{d}$. Repeating the same procedure, we obtain $\mathbf{x}_k^o = t\mathbf{d}$ for all k, $1 \leq k \leq n$, and $\mathbf{x}_0^o = t\mathbf{F}\mathbf{d}$. Therefore, if $\{\mathbf{x}_k^o\}$ exists, a finite number n must exist such that

$$\mathbf{x}_0^o = t\mathbf{F}\mathbf{d}, \ \mathbf{x}_k^o = t\mathbf{d}, \ k = 1, 2, \cdots, n-1, \text{ and } \mathbf{x}_k^o \neq t\mathbf{d}, \ k = n, n+1, \cdots, \infty. \tag{3.26}$$

Here, we put

$$\mathbf{y}_k^o = \mathbf{F}\mathbf{x}_k^o, \ k = 1, 2, \cdots, \infty. \tag{3.27}$$

Multiplied by matrix $\mathbf{G}$ from the left and applying (3.27), statement (3.25.ii) can be represented by

$$\mathbf{y}_k^o = \mathbf{G}\mathbf{y}_{k+1}^o, \ k = 1, 2, \cdots, \infty. \tag{3.28}$$

Further

$$\mathbf{y}_k^o = \mathbf{G}^N\mathbf{y}_{k+N}^o, \ N \geq 1. \tag{3.29}$$

From (3.25), Properties 2.1 and 2.3, the conditions in (3.26) are equivalent to the next statements.

$$\mathbf{F}\mathbf{x}_1^o = t\mathbf{F}\mathbf{d}, \ \mathbf{F}\mathbf{x}_{k+1}^o = t\mathbf{F}\mathbf{d}, \ k = 1, 2, \cdots, n-1, \text{ and}$$
$$\mathbf{F}\mathbf{x}_{k+1}^o \neq t\mathbf{F}\mathbf{d}, \ k = n, n+1, \cdots, \infty. \tag{3.30}$$

Then from (3.27):

Proposition 3.2 If the solution $\{\mathbf{x}_k^o\}$ exists, then there must exist a finite number n such that

$$\mathbf{y}_k^o = t\mathbf{F}\mathbf{d}, \ k = 1, 2, \cdots, n, \text{ and } \mathbf{y}_k^o \neq t\mathbf{F}\mathbf{d}, \ k = n+1, n+2, \cdots, \infty. \tag{3.31}$$

In order to derive a contradiction to (3.31), we shall investigate the property of matrix $\mathbf{G}$ under the condition $\sum_{1\leq i\leq r} D_i = 1$.

Lemma 3.3 The next proposition holds for the sum of column elements of $\mathbf{G} = [g_{ij}]$.

(III.4) $$\sum_{1\le i\le r} D_i = 1 \Rightarrow \sum_{1\le i\le r} g_{ij} = 1,\ j = 1,2,\cdots,r.$$

Proof: The same as the proof of Lemma 3.1, except one changes the inequality into equality.

Lemma 3.4 The next proposition holds about the power matrix of **G**.

(III.5) $$\sum_{1\le i\le r} g_{ij} = 1,\ 1 \le j \le r \Rightarrow \lim_{N\to\infty} \mathbf{G}^N = \mathbf{C} \equiv \begin{bmatrix} c_1 & c_1 & \cdots & c_1 \\ \vdots & \vdots & & \vdots \\ c_r & c_r & \cdots & c_r \end{bmatrix}$$

where $c_i \ge 0$,[4] $1 \le i \le r$, and $\sum_{1\le i\le r} c_i = 1$.

Proof: Refer to Appendix D.

Lemma 3.5 Matrix **C** is given by the next statement.

$$\mathbf{C} = (1/a)\mathbf{FD}, \tag{3.32}$$

where a is the sum of column elements in matrix **FD** and $a > 0$.

Proof: Refer to Appendix E.

Lemma 3.6 The next statement holds for the sum of elements in $\mathbf{y}_N^o$.

$$\lim_{N\to\infty} \sum_{1\le i\le r} y_{iN} = b,\ b \text{ is a positive constant}, \tag{3.33}$$

where $\mathbf{y}_N^o = [y_{1N}, y_{2N}, \cdots, y_{rN}]^T$.

Proof: Refer to Appendix F.

From Lemmas 3.3 through 3.5,

(III.6) $$\sum_{1\le i\le r} D_i = 1 \Rightarrow \lim_{N\to\infty} \mathbf{G}^N = (1/a)\mathbf{FD}.$$

It follows from Lemma 3.6 for the finite number $n+1$ in Proposition 3.2 that

$$\begin{aligned} \mathbf{y}_{n+1}^o &= \lim_{N\to\infty} \mathbf{G}^N \mathbf{y}_{n+1+N}^o \\ &= \{(1/a)\mathbf{FD}\} \lim_{N\to\infty} \mathbf{y}_{n+1+N}^o \\ &= (b/a)\mathbf{Fd},\ b/a > 0. \end{aligned} \tag{3.34}$$

This result is contradictory to the assumption in Proposition 3.2.
We summarize the above analysis into the following theorem.

Theorem 3.2 Independent of the mixture of transportation types for r products, the form of the balanced lot size in (3.23.i) and the associated initial inventory in (3.23.ii) is a unique

[4] Equality holds only for c_1 when product 1 is of the continuous type.

solution for (2.12) in the case $\sum_{1\le i\le r} D_i = 1$ and $\sum_{1\le i\le r} u_i = 0$. There exist infinitely many solutions since cycle time is arbitrary, although it must be consistent in all cycles.

3.2.3 A Case where Machine Idle Time Is Required

In the case $\sum_{1\le i\le r} D_i = 1$, it is predicted that a feasible solution cannot be obtained if idle time is positive for some product, namely $\sum_{1\le i\le r} u_i > 0$, causing insufficient net production capacity. In fact, it follows from (2.19) for arbitrary k that

$$\mathbf{y}_k = \mathbf{G}^N\mathbf{y}_{k+N}+(\mathbf{G}+\mathbf{G}^2+\cdots+\mathbf{G}^N)\mathbf{D}_0\mathbf{u}+(\mathbf{I}+\mathbf{G}+\mathbf{G}^2+\cdots+\mathbf{G}^{N-1})\mathbf{D}_1\mathbf{u},\ N\ge 1. \quad (3.35)$$

All elements of $\mathbf{y}_k$ in (3.35) diverge to $+\infty$ for sufficiently large N. It follows from (2.17) that at least one element of $\mathbf{x}_k$ diverges for sufficiently large N and there exists no feasible solution in (2.12).

From the above analysis, we can summarize, in the case $\sum_{1\le i\le r} D_i = 1$, that a feasible solution can be obtained uniquely for arbitrarily determined cycle time $t > 0$ under the condition of continuous production operation. It follows that there is no feasible solution if setup time is required for some product.

4. Properties of the Balanced Lot Size

In the case $\sum_{1\le i\le r} D_i < 1$, a feasible solution, called the balanced lot size, is obtained uniquely for arbitrary $\sum_{1\le i\le r} u_i > 0$ which satisfies the condition $u_i \ge \tau_i$, $i = 1, 2, \cdots, r$. In the case $\sum_{1\le i\le r} D_i = 1$, a feasible solution, also called the balanced lot size, is obtained in a unique form for arbitrary cycle time $t > 0$. And in the case $\sum_{1\le i\le r} D_i > 1$, there is no feasible solution. The balanced lot size satisfies the following properties:

<u>Property 4.1</u> The balanced lot size is determined in proportion to the demand rate D_i of each product.

<u>Property 4.2</u> The balanced lot size does not depend upon production sequence.

<u>Property 4.3</u> In the case $\sum_{1\le i\le r} D_i < 1$, the balanced lot size is determined in proportion to the sum of machine idle time $\sum_{1\le i\le r} u_i$ in each cycle, and in inverse proportion to $1 - \sum_{1\le i\le r} D_i$ which means production capacity surplus. The balanced lot size can be enlarged by increasing the value of $\sum_{1\le i\le r} u_i$, whereas its minimum is obtained when $\sum_{1\le i\le r} u_i = \sum_{1\le i\le r} \tau_i$.

<u>Property 4.4</u> In the case $\sum_{1\le i\le r} D_i = 1$, there are infinite number of alternative values in the lot size, since the cycle time is arbitrary. There is no bound on its minimum.

<u>Property 4.5</u> The balanced lot size does not depend upon the method of transportation for each product. The difference in transportation method appears on a schedule as the shift of transportation timing for the same length in each cycle. It then affects the necessary amount of initial inventory at the beginning of the planning horizon.

We can utilize these properties in creating a schedule in practice. For example, Properties 4.3 and 4.4 imply how to cope with restrictions such as maximum lot size or predetermined cycle length. Similarly, Property 4.2 indicates that we can choose a desirable production sequence due to the situation. Thus, the balanced lot size is a practical guide to obtain a feasible schedule on an infinite time horizon.

5. Application to the Problem on a Finite Planning Horizon

Major results obtained in the case of infinite planning horizon can be practical guides for obtaining a feasible schedule on a finite planning period $[0, T]$. In this section, we shall clarify two scheduling methods; direct application of the balanced lot size, and the utilization of the balance equation.

5.1 A Case of Constant Demand Rate

First, a case of constant demand rate over the horizon shall be examined with a numerical example of a two-stage automobile tire manufacturing process, where soft rubber rings shaped in a single forming machine are transported to and processed at one of three sulphuration machines. We assume three types of products, and sulphuration machines to process products 1 and 2 are directly connected with the forming machine by conveyors, while product 3 is supplied to the third machine by a bulk carrier. Then, products 1 and 2 are of the continuous type, and product 3 becomes the lot type. The following production speed and machine idle times are assumed:

	Forming	Sulphuration	
Product 1	150	15	(pieces/hour)
Product 2	120	24	(pieces/hour)
Product 3	90	36	(pieces/hour)

$$\text{Machine idle time}: u_1 = u_2 = u_3 = 1 \text{ hour}.$$

And we assume that 75 pieces for product 1 and 180 pieces for product 3 must be prepared at the end of a planning period of one week. The ending inventory for product i shall be denoted by Q_i. Then, by scaling each product by the amount of hourly forming quantity, we obtain:

Product 1	$K_1 = 1$	$D_1 = 0.1$	$Q_1 = 0.5$
Product 2	$K_2 = 1$	$D_2 = 0.2$	$Q_2 = 0$
Product 3	$K_3 = 1$	$D_3 = 0.4$	$Q_3 = 2.0$

A feasible production schedule under a cyclic production sequence $\pi = (1, 2, 3)$ over the planning horizon of 50 hours (one week) shall be investigated.

5.1.1 Application of the Balanced Lot Size

First, cycle time is determined to be $t^* = 10$ hours from (3.4). The balanced lot size is given from (3.3.i) as

$$x_1^* = 1.0, \qquad x_2^* = 2.0, \qquad x_3^* = 4.0 \text{ units}.$$

Initial inventory is calculated by (3.3.ii) as

$$x_{10}^* = 0, \qquad x_{20}^* = 0.4, \qquad x_{30}^* = 3.6 \text{ units}.$$

The planning horizon of 50 hours is comprised of 5 cycles. In the last cycle, the lot size of each product must be adjusted so that the inventory at the end of the horizon satisfies the given requirement for each product. Denoting the supply timing of each product in the last cycle by $z_{i,5}$, the modified lot size $x_{i,5}$ is obtained by

$$x_{i,5} = (50 - z_{i,5})D_i + Q_i, \ i = 1, 2, 3, \tag{5.1}$$

and the difference between x_i^* and $x_{i,5}$ in the last cycle is converted into additional machine idle times. The obtained schedule appears in Figure 3. Converting the lot sizes back into the original pieces of tires, we obtain the following results:

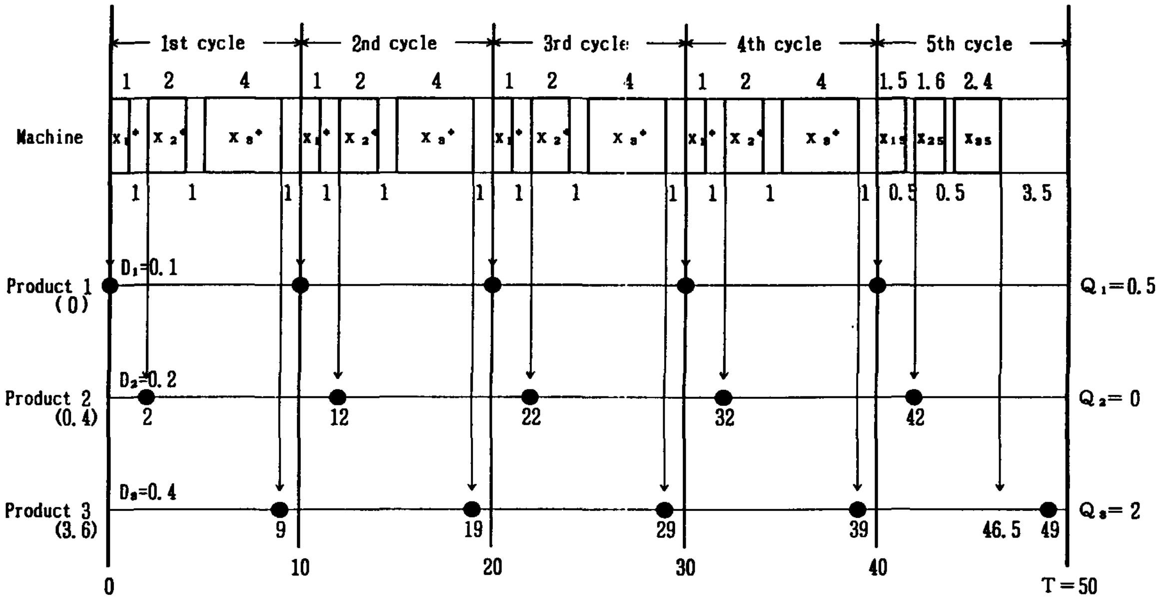

Figure 3. A feasible schedule (application of the balanced lot size)

	Pdt 1	Pdt 2	Pdt 3	
Initial inventory	0	48	324	(pieces)
1st-4th cycles	150	240	360	(pieces)
5th cycle	225	192	216	(pieces)
Total	825	1200	1980	(pieces)

Property 5.1 The schedule obtained by applying the balanced lot size satisfies:

1) Simple repetition of the same lot size in each cycle except for the last cycle. Thereby, cycle time is consistent in all cycles.
2) Schedule is not affected by the transportation type of each product.
3) Lot size of each product is not affected by the production sequence.

5.1.2 Application of the Balance Equation

We shall denote the number of production cycles on $[0, T]$ by n. As discussed in Section 2.5, a solution for the balance equation is determined by the backward induction if $\mathbf{x}_n$ is determined. It then becomes necessary to guarantee a schedule comprised of n cycles, in which production of the first lot begins exactly at time 0. Therefore, we need to introduce an additional idle time that works as a parameter. Here in this paper, we shall put a lag between production completion time of the last lot and the end of the planning horizon T, which is called stop lag and denoted by R.

The balance equation on $[0, T]$ can be represented as follows. First, that for the last cycle is given by

$$\mathbf{E}\mathbf{x}_n - \mathbf{D}_0\mathbf{u}_n = \mathbf{Q}, \tag{5.2}$$

where $\mathbf{Q} = [Q_1, Q_2, \cdots, Q_r]^T$, $\mathbf{u}_n = [u_2, u_3, \cdots, u_r, R]^T$.
Secondly, the balance equation between the carried-over inventory and the prior production quantity is given by

$$\mathbf{E}\mathbf{x}_k - \mathbf{D}_0\mathbf{u} = \mathbf{F}\mathbf{x}_{k+1} + \mathbf{D}_1\mathbf{u},\ k = 1, 2, \cdots, n-1. \tag{5.3}$$

And, the initial inventory is given by

$$\mathbf{x}_0 = \mathbf{F}\mathbf{x}_1 + \mathbf{D}_1\mathbf{u}. \tag{5.4}$$

Since matrix **E** if of full rank, equations (5.2) and (5.3) can be restated as follows.

$$\mathbf{x}_n = \mathbf{E}^{-1}(\mathbf{D}_0\mathbf{u}_n + \mathbf{Q}). \tag{5.5.i}$$

$$\mathbf{x}_k = \mathbf{E}^{-1}\mathbf{F}\mathbf{x}_{k+1} + \mathbf{E}^{-1}\mathbf{D}\mathbf{u},\ k = 1, 2, \cdots, n-1. \tag{5.5.ii}$$

For some determined values of n and the ending inventory $\mathbf{Q}$, $\mathbf{x}_k$ is determined from the last cycle with R as a parameter. Then, we can decide the value of R to create a schedule in which production of the first lot x_{11} begins at time 0 by solving

$$\sum_{1\le k\le n}\sum_{1\le i\le r} x_{ik} + n\sum_{1\le i\le r} u_i - u_1 + R = T. \tag{5.6}$$

By solving equations (5.5) and (5.6), we can obtain a schedule shown in Figure 4 for the numerical example above. Converting the lot sizes back into the original values, we obtain:

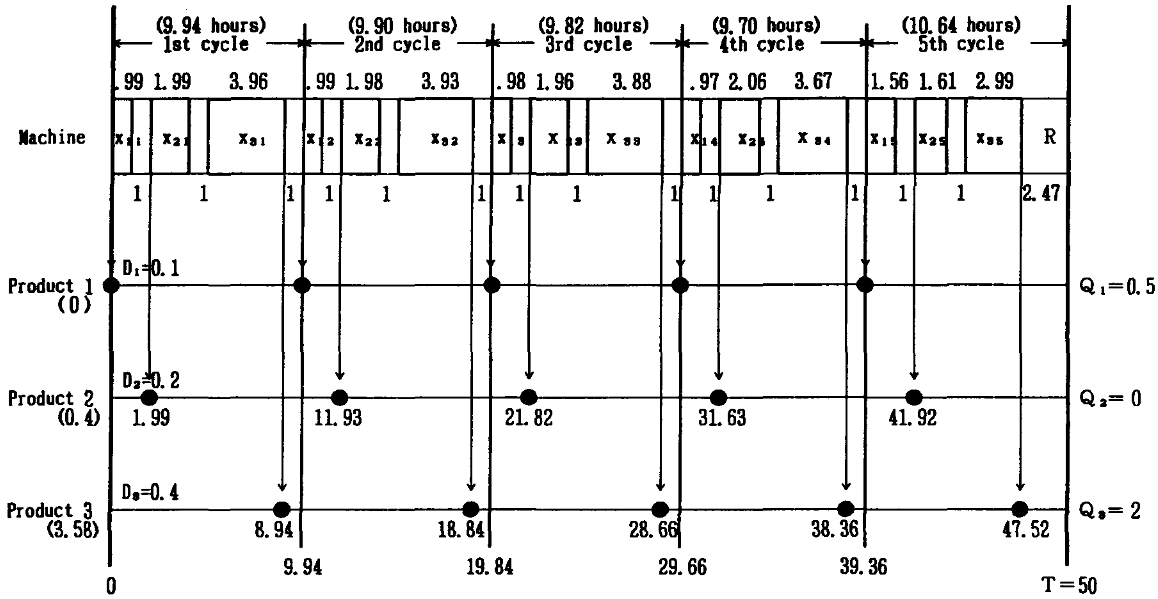

Figure 4. A feasible schedule (application of the balance equation)

	Pdt 1	Pdt 2	Pdt 3	
Initial inventory	0	48.0	321.8	(pieces)
1st cycle	149.1	238.8	356.3	(pieces)
2nd cycle	148.5	237.6	353.6	(pieces)
3rd cycle	147.3	235.2	349.1	(pieces)
4th cycle	145.4	247.2	330.1	(pieces)
5th cycle	234.7	193.2	269.1	(pieces)
Total	825	1200	1980	(pieces)

There are various alternative schedules obtainable depending upon the values R and n, as well as where to put the lag time to adjust the production staring time of the first lot to time 0.

Property 5.2 The schedule obtained by solving the balance equation satisfies:

1) Lot sizes for each product take different values in each cycle. Thereby, cycle time is not consistent for each cycle.
2) Schedule is affected by the transportation type of each product.
3) Lot sizes of each product is affected by the production sequence.

5.1.3 Comparison of the Two Methods

We can state that the balanced lot method is quite easy to apply and has a practical advantage of regulatory repetition of the same pattern in each cycle. However, it has a disadvantage that a feasible schedule cannot be obtained if $\sum_{1\leq i\leq r} D_i > 1$, while the backward induction can cope with the case, although lot sizes become quite large if the planning period is long enough. There is another shortcoming in the balanced lot application in that the feasibility of lot size adjustment in the last cycle depends upon the required amount of the ending inventory for each product, while the backward method assures a feasible schedule for any condition on the ending inventory. Thus, it is valuable to prepare these two methods and to choose a desirable one according to the situation.

5.2 A Case of Fluctuating Demand Rate

The two scheduling methods clarified in Section 5.1 can easily be extended to a situation where demand rate for each product fluctuates period by period over a certain finite planning horizon. A typical illustration is a problem on a planning horizon of a month in which demand rate changes every week.

5.2.1 Application of the Balanced Lot Size

This method is quite simple in its principle. According to the demand rate of each product on each period, the values of the balanced lot size can be determined independently for each period. In order to prevent any shortage and surplus inventory at the end of each period, the lot sizes in the last cycle of each period must be adjusted to balance with the required amount of initial inventory for the next period given by (3.3.ii) or (3.23.ii). The latitude for the lot size adjustment increases as the sum of machine idle time over a certain period is enlarged, while the required amount of initial inventory also increases and the lot size adjustment in the previous period becomes difficult. Thus, determination of appropriate values of idle time in each period requires a judgement of a schedule planner.

5.2.2 Application of the Balance Equation

The demand and supply balance equations are determined depending upon the demand rate of each product on each period. Following the backward induction, we should first solve (5.5.i) for the last cycle in the last period incorporating stop lag as a parameter. Then, using (5.5.ii), lot sizes for all cycles in the last period are obtained including a parameter in their form. Applying the statement (5.6), the value of the stop lag is determined so that the production of the first lot starts just at the beginning of the period.[5]

[5] The right hand side of (5.6) should be the length of the period instead of T.

The key in this method is the conjunction between adjacent periods. The required amount of initial inventory at the beginning of the last period is obtained by (5.4), whose value becomes the condition for the ending inventory $\mathbf{Q}$ in (5.5.i) for the immediately previous period. Given the value $\mathbf{Q}$, the lot sizes over that period can then be determined using equations (5.5.i), (5.5.ii), and (5.6). In this manner, a production schedule over the entire planning horizon is obtained by the manner of backward induction. According to the values of stop lag over each period. there are various schedules obtainable.

5.2.3 Effect of Change in Transportation Types

It sometimes becomes necessary to change the transportation method for some product in the middle of the planning horizon, due to such reasons as change in the place of consumption or in transportation conditions. We can easily cope with such a situation, distinguishing the planning horizon into independent periods at the time when transportation type change occurs. Then, the change in transportation types can be treated in the same manner as the change in demand rate. In case that such a change is of an unexpected one, the method to apply the balanced lot size is easier to cope with the case, since the value of the balanced lot size itself does not depend upon the transportation types. On the other hand, the entire schedule prior to the change in transportation types are affected in the method of solving the balance equation, reflecting its property of the backward induction. Still, since pros and cons of the two methods discussed in Section 5.1.3 hold in this case, it is meaningful to prepare the two methods for the choice according to the situation.

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Appendix A Rank and inverse of matrix $(\mathbf{I}-\mathbf{D})$

(1) Rank of matrix $(\mathbf{I}-\mathbf{D})$

Matrix $(\mathbf{I}-\mathbf{D})$ is converted into a diagonal matrix using the following fundamental operation matrices on rows and columns. For the sake of simplification, we take the case of $r = 4$.

$$\mathbf{L} \equiv \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \mathbf{R}_1 \equiv \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & -1 & -1 & 1 \end{bmatrix} \mathbf{R}_2 \equiv \begin{bmatrix} 1 & 0 & 0 & D_1 \\ 0 & 1 & 0 & D_2 \\ 0 & 0 & 1 & D_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Then,

$$(\mathbf{I}-\mathbf{D})\mathbf{R}_1\mathbf{R}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & -1 & -1 & 1-\sum_{1\le i\le r} D_i \end{bmatrix} \equiv \mathbf{P}. \tag{A.1}$$

Since both $\mathbf{R}_1$ and $\mathbf{R}_2$ are regular, $\text{rank}(\mathbf{I}-\mathbf{D}) = \text{rank}(\mathbf{P})$. Therefore,

$$\text{rank}(\mathbf{I}-\mathbf{D}) = r, \quad \text{if } \sum_{1\le i\le r} D_i \neq 1.$$
$$\text{rank}(\mathbf{I}-\mathbf{D}) = r-1, \quad \text{if } \sum_{1\le i\le r} D_i = 1.$$

(2) The stationary solution $\mathbf{x}^*$ in the case $\sum_{1\le i\le r} D_i < 1$ and $\sum_{1\le i\le r} u_i > 0$

Matrix $(\mathbf{I}-\mathbf{D})$ can be converted into a diagonal matrix by

$$\mathbf{LP} = \mathbf{L}(\mathbf{I}-\mathbf{D})\mathbf{R}_1\mathbf{R}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1-\sum_{1\le i\le r} D_i \end{bmatrix} \equiv \mathbf{Q}. \tag{A.2}$$

It follows

$$(\mathbf{I}-\mathbf{D})^{-1} = \mathbf{R}_1\mathbf{R}_2\mathbf{Q}^{-1}\mathbf{L}$$
$$= \{1/(1-\sum_{1\le i\le r} D_i\}$$
$$\begin{bmatrix} 1-\sum_{1\le i\le r} D_i + D_1 & D_1 & D_1 & D_1 \\ D_2 & 1-\sum_{1\le i\le r} D_i + D_2 & D_2 & D_2 \\ D_3 & D_3 & 1-\sum_{1\le i\le r} D_i + D_3 & D_3 \\ D_4 & D_4 & D_4 & 1-\sum_{1\le i\le r} D_i + D_4 \end{bmatrix} \tag{A.3}$$

Therefore, for the general case of r products,

$$\mathbf{x}^* = (\mathbf{I}-\mathbf{D})^{-1}\mathbf{D}\mathbf{u} = \{\sum_{1\le i\le r} u_i/(1-\sum_{1\le i\le r} D_i)\}\mathbf{d} = t^*\mathbf{d}. \tag{A.4}$$

(3) The stationary solution $\mathbf{x}^*$ in the case $\sum_{1\le i\le r} D_i = 1$ and $\sum_{1\le i\le r} u_i = 0$

We shall represent matrix $(\mathbf{I}-\mathbf{D})$ by a set of column vectors such that $(\mathbf{I}-\mathbf{D}) = [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4]$. Then $(\mathbf{I}-\mathbf{D})\mathbf{x}^* = \mathbf{0}$ can be represented by

$$x_1^*\mathbf{a}_1 + x_2^*\mathbf{a}_2 + x_3^*\mathbf{a}_3 + x_4^*\mathbf{a}_4 = \mathbf{0}. \tag{A.5}$$

From (A.1),

$$(\mathbf{I}-\mathbf{D})\mathbf{R}_1\mathbf{R}_2 = [\mathbf{a}_1 - \mathbf{a}_4, \mathbf{a}_2 - \mathbf{a}_4, \mathbf{a}_3 - \mathbf{a}_4, D_1\mathbf{a}_1 + D_2\mathbf{a}_2 + D_3\mathbf{a}_3 + D_4\mathbf{a}_4] \tag{A.6}$$

It follows from (A.1),

$$D_1\mathbf{a}_1 + D_2\mathbf{a}_2 + D_3\mathbf{a}_3 + D_4\mathbf{a}_4 = \mathbf{0}. \tag{A.7}$$

From (A.5) and (A.7),

$$\{x_1^* - (D_1/D_4)x_4^*\}\mathbf{a}_1 + \{x_2^* - (D_2/D_4)x_4^*\}\mathbf{a}_2 + \{x_3^* - (D_3/D_4)x_4^*\}\mathbf{a}_3 = \mathbf{0}. \tag{A.8}$$

Since $\mathrm{rank}(\mathbf{I}-\mathbf{D}) = r-1 = 3$, $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ are linearly independent. Therefore,

$$x_i^* = tD_i,\ i = 1,2,\cdots,r, \tag{A.9}$$

where t is an arbitrary positive value.

<u>Appendix B</u> Proof of Lemma 3.1

Multiplied by matrix $\mathbf{E}^{-1}$ from the right to (2.16),

$$\mathbf{G} = \mathbf{D}\mathbf{E}^{-1} - \mathbf{E}^{-1} + \mathbf{I}. \tag{B.1}$$

We can describe (B.1) by row vectors as

$$\mathbf{g}_i = D_i \sum_{1\le i\le r} \mathbf{e}_i - \mathbf{e}_i + \mathbf{u}_i,\ i = 1,2,\cdots,r, \tag{B.2}$$

where $\mathbf{g}_i$ and $\mathbf{e}_i$ are the i-th row vectors of matrices $\mathbf{G}$ and $\mathbf{E}^{-1}$, and $\mathbf{u}_i$ is the r dimensional unit vector. Clearly,

$$\mathbf{e}_i - \mathbf{u}_i \ge \mathbf{0},\ i = 1,2,\cdots,r. \tag{B.3}$$

Applying (B.3) and the relation $\sum_{1\le i\le r} D_i < 1$ to (B.2),

$$\begin{aligned}\sum_{1\le i\le r} \mathbf{g}_i &= (\sum_{1\le i\le r} D_i) \sum_{1\le i\le r} \mathbf{e}_i - \sum_{1\le i\le r} (\mathbf{e}_i - \mathbf{u}_i) \\ &\le (\sum_{1\le i\le r} D_i)\{\sum_{1\le i\le r} \mathbf{e}_i - \sum_{1\le i\le r} (\mathbf{e}_i - \mathbf{u}_i)\} \\ &= [\sum_{1\le i\le r} D_i, \sum_{1\le i\le r} D_i, \cdots, \sum_{1\le i\le r} D_i].\end{aligned}$$

Therefore,

$$\sum_{1\le i\le r} g_{ij} \le \sum_{1\le i\le r} D_i < 1,\ 1 \le j \le r.$$

<u>Appendix C</u> Proof of Lemma 3.2

Let $\mathbf{g}_i$ be the i-th row vector and $\mathbf{h}_j$ the j-th column vector of matrix $\mathbf{G}$. We shall denote $\mathbf{M}_i = [M_i, M_i, \cdots, M_i]$, where $M_i = \max_{1 \leq j \leq r} g_{ij}$. Applying Lemma 3.1, we can derive that

$$\mathbf{G}^2 = \begin{bmatrix} (\mathbf{g}_1, \mathbf{h}_1), & (\mathbf{g}_1, \mathbf{h}_2), & \cdots, & (\mathbf{g}_1, \mathbf{h}_r) \\ \vdots & \vdots & & \vdots \\ (\mathbf{g}_r, \mathbf{h}_1), & (\mathbf{g}_r, \mathbf{h}_2), & \cdots, & (\mathbf{g}_r, \mathbf{h}_r) \end{bmatrix}$$
$$\leq \begin{bmatrix} (\mathbf{M}_1, \mathbf{h}_1), & (\mathbf{M}_1, \mathbf{h}_2), & \cdots, & (\mathbf{M}_1, \mathbf{h}_r) \\ \vdots & \vdots & & \vdots \\ (\mathbf{M}_r, \mathbf{h}_1), & (\mathbf{M}_r, \mathbf{h}_2), & \cdots, & (\mathbf{M}_r, \mathbf{h}_r) \end{bmatrix}$$
$$= \begin{bmatrix} M_1 \sum_{1 \leq i \leq r} D_i, & M_1 \sum_{1 \leq i \leq r} D_i, & \cdots, & M_1 \sum_{1 \leq i \leq r} D_i \\ \vdots & \vdots & & \vdots \\ M_r \sum_{1 \leq i \leq r} D_i, & M_r \sum_{1 \leq i \leq r} D_i, & \cdots, & M_r \sum_{1 \leq i \leq r} D_i \end{bmatrix}$$

Similarly

$$\mathbf{G}^N \leq \mathbf{A}_N, \ N \geq 2,$$

where

$$\mathbf{A}_N = \begin{bmatrix} M_1(\sum_{1 \leq i \leq r} D_i)^{N-1}, & M_1(\sum_{1 \leq i \leq r} D_i)^{N-1}, & \cdots, & M_1(\sum_{1 \leq i \leq r} D_i)^{N-1} \\ \vdots & \vdots & & \vdots \\ M_r(\sum_{1 \leq i \leq r} D_i)^{N-1}, & M_r(\sum_{1 \leq i \leq r} D_i)^{N-1}, & \cdots, & M_r(\sum_{1 \leq i \leq r} D_i)^{N-1} \end{bmatrix}$$

Since $\lim_{N \to \infty} \mathbf{A}_N = \mathbf{0}$, it is clear that $\lim_{N \to \infty} \mathbf{G}^N = \mathbf{0}$.

Appendix D Proof of Lemma 3.4

We denote;

$$\mathbf{G}^N = [g_{ij}^{(N)}] \ (N \text{ shall be omitted when } N = 1),$$
$$M_i^{(N)} = \max_{1 \leq j \leq r} g_{ij}^{(N)}, \ m_i^{(N)} = \min_{1 \leq j \leq r} g_{ij}^{(N)}, \ i = 1, 2, \cdots, r.$$

It is clear for each row i that

$$g_{ij}^{(N+1)} = \sum_{1 \leq k \leq r} g_{ik}^{(N)} g_{kj} \leq M_i^{(N)} \sum_{1 \leq i \leq r} g_{ij} = M_i^{(N)}, \ 1 \leq j \leq r. \tag{D.1}$$

Then $M_i^{(N)}$ satisfies

$$M_i^{(N)} \geq M_i^{(N+1)}, \text{ for any } N \geq 1, \ i = 1, 2, \cdots, r. \tag{D.2}$$

Similarly

$$m_i^{(N)} \leq m_i^{(N+1)}, \text{ for any } N \geq 1, \ i = 1, 2, \cdots, r. \tag{D.3}$$

We shall denote $\max_{1 \leq j \leq r} g_{ij}^{(N)}$ by $g_{ia}^{(N)}$ and $\min_{1 \leq j \leq r} g_{ij}^{(N)}$ by $g_{ib}^{(N)}$. Since $\sum_{1 \leq i \leq r} g_{ij} = 1$ for all columns j under the condition $\sum_{1 \leq i \leq r} D_i = 1$:

$$\sum_{g_{ia} > g_{ib}} (g_{ia} - g_{ib}) = - \sum_{g_{ia} < g_{ib}} (g_{ia} - g_{ib}). \tag{D.4}$$
$$\sum_{g_{ia} > g_{ib}} (g_{ia} - g_{ib}) \leq \sum_{g_{ia} > g_{ib}} g_{ia} \equiv \delta_i < 1. \tag{D.5}$$

Applying (D.4) and (D.5), we can derive that

$$\begin{aligned} M_i^{(N)} - m_i^{(N)} &= \max_{1\leq j\leq r} g_{ij}^{(N)} - \min_{1\leq j\leq r} g_{ij}^{(N)} \\ &= \sum_{i\leq j\leq r} g_{ij}^{(N-1)}(g_{ja} - g_{jb}) \\ &= \sum_{g_{ja}>g_{jb}} g_{ij}^{(N-1)}(g_{ja} - g_{jb}) - \sum_{g_{ja}>g_{jb}} g_{ij}^{(N-1)}(g_{ja} - g_{jb}) \\ &\leq M_i^{(N-1)} \sum_{g_{ja}>g_{jb}} (g_{ja} - g_{jb}) - m_i^{(N-1)} \sum_{g_{ja}>g_{jb}} (g_{ja} - g_{jb}) \\ &= (M_i^{(N-1)} - m_i^{(N-1)}) \sum_{g_{ja}>g_{jb}} (g_{ja} - g_{jb}) \\ &< (M_i^{(N-1)} - m_i^{(N-1)})\ \delta_j, \end{aligned} \tag{D.6}$$

where $0 < \delta_j < 1,\ j = 2, 3, \cdots, N$. Then,

$$\lim_{N\to\infty} (M_i^{(N)} - m_i^{(N)}) = (M_i^{(1)} - m_i^{(1)}) \lim_{N\to\infty} \Pi_{2\leq j\leq N}\delta_j = 0,\ i = 1, 2, \cdots, r. \tag{D.7}$$

Here from Lemma 3.3,

$$\boldsymbol{\gamma}\mathbf{G}^N = \boldsymbol{\gamma}\mathbf{G}^{N-1} = \cdots = \boldsymbol{\gamma}\mathbf{G} = \boldsymbol{\gamma}. \tag{D.8}$$

where $\boldsymbol{\gamma} = [1, 1, \cdots, 1]$. Then,

$$\sum_{1\leq i\leq r} c_i = 1. \tag{D.9}$$

Appendix E Proof of Lemma 3.5

Multiplying (2.16) in Property 2.3 by matrix $\mathbf{D}$ from the right and applying $\mathbf{DD} = \mathbf{D}$ in the case $\sum_{1\leq i\leq r} D_i = 1$,

$$\mathbf{FD} = \mathbf{ED} \tag{E.1}$$

Repeatedly multiplied by matrix $\mathbf{G}$ from the left and applying Lemma 3.4,

$$\mathbf{CFD} = \mathbf{FD} \tag{E.2}$$

Since elements in each row of $\mathbf{FD}$ are constant, it follows

$$\mathbf{CFD} = a\mathbf{C} \tag{E.3}$$

where a is the sum of column elements of matrix $\mathbf{FD}$ and $a > 0$. Then,

$$\mathbf{C} = (1/a)\mathbf{FD},\ a > 0. \tag{E.4}$$

Appendix F Proof of Lemma 3.6

From (3.29),

$$\mathbf{y}_k^o = \mathbf{G}^N \mathbf{y}_{k+N}^o = \mathbf{G}^{N+1} \mathbf{y}_{k+N+1}^o. \tag{F.1}$$

Then

$$\mathbf{G}^{N+1}\mathbf{y}_{k+N+1}^o - \mathbf{G}^N \mathbf{y}_{k+N}^o = \mathbf{0}. \tag{F.2}$$

Since $\lim_{N\to\infty}\mathbf{G}^N$ converges to matrix $\mathbf{C}$ and $\{\mathbf{y}_k^o\}$ is bounded under the condition (2.4), we get

$$\begin{aligned}&\lim_{N\to\infty}(\mathbf{G}^{N+1}\mathbf{y}_{k+N+1}^o - \mathbf{G}^N\mathbf{y}_{k+N}^o)\\ =&\lim_{N\to\infty}(\mathbf{C}\mathbf{y}_{k+N+1}^o - \mathbf{C}\mathbf{y}_{k+N}^o)\\ =&[c_1, c_2, \cdots, c_r]^T\lim_{N\to\infty}(\sum_{1\le i\le r} y_{i,k+N+1} - \sum_{1\le i\le r} y_{i,k+N}) = \mathbf{0}.\end{aligned} \tag{F.3}$$

Lemma 3.4 tells that the vector $[c_1, c_2, \cdots, c_r]^T \neq \mathbf{0}$. Then (F.3) implies statement (3.33).

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