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A LOWER BOUND OF THE EXPECTED MAXIMUM NUMBER OF VERTEX-DISJOINT *s-t* PATHS ON PROBABILISTIC GRAPHS

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Abstract The problem of computing the expected maximum number $\Psi_{(G,p)}$ of vertex-disjoint *s-t* paths for a probabilistic graph (G,p) is considered in this paper, where G is a two-terminal graph with specified source vertex s and sink vertex $t (s \neq t)$ in which each edge has a statistically independent failure probability and each vertex is assumed to be failure-free, and p is a vector of failure probabilities of edges. This computing problem is NP-hard, even though graphs are restricted to several special classes of graphs, e.g., planar graphs, *s-t* out-in bitrees and *s-t* complete multi-stage graphs. In this paper, we propose a lower bound $\underline{\Psi}_{(G,f,p)}$ of $\Psi_{(G,p)}$ for a probabilistic graph (G,p) based on an *s-t* path number function f of G. Although the lower bound does not seem to be efficiently computed for a general probabilistic graph, we shall also give a class of probabilistic graphs for which the expected maximum number is efficiently obtained by computing the lower bound.

1. Introduction

We consider the problem of computing the expected maximum number $\Psi_{(G,p)}$ of vertex-disjoint *s-t* paths (namely, *s-t* paths sharing no vertex other than s, t,) for a probabilistic graph $(G = (V, E, s, t), p)$, where G is a two-terminal graph with specified source vertex s and sink vertex t $(s \neq t)$, $p = (p(e_1), ..., p(e_{|E|}))$ is a vector consisting of failure probabilities $p(e_i)$'s, $e_i \in E$, and edges are assumed to fail statistically independently of each other. Computing $\Psi_{(G,p)}$ for a probabilistic graph (G,p) is useful for network reliability analysis as computing the expected maximum flow for a probabilistic network [2, 7] and s,t-connectedness, namely, probability that there exists at least one operative *s-t* path in a probabilistic graph [1, 5]. Note that the problem of computing $\Psi_{(G,p)}$ contains, as a special case, the problem of computing s,t-connectedness on a probabilistic graph (G,p).

It is known that the problem of computing the expected maximum flow for a probabilistic network is NP-hard and that its lower bound which is efficiently computed is proposed by [2]. Recently, Nagamochi and Ibaraki [7] clarified that the expected maximum flow for a probabilistic monofil network [7] is efficiently computed, as a necessary and sufficient condition by which the lower bound proposed in [2] coincides with the expected maximum flow is that a network is monofil [7]. Although the problem of computing $\Psi_{(G,p)}$ for a probabilistic directed graph (G,p) is considered as a special case of the problem of computing the expected maximum flow for a probabilistic network where capacity on each edge is one, the lower bound proposed in [2] is not effective for most of such probabilistic networks as the class of probabilistic networks satisfying the necessary and sufficient condition proposed in [7] is very limited. In its introduction [7] refers to previous results on computing the expected maximum flow in a probabilistic network.

On studies of computing the expected maximum number of vertex-disjoint *s-t* paths in a probabilistic graph, it is shown by [4] that $\Psi_{(G,p)}$ for a probabilistic basically series-parallel digraph (G,p) is efficiently computed. However, it is known that the problem of computing

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$\Psi_{(G,p)}$ for a probabilistic graph (G,p) is NP-hard, even if G is restricted to several classes of graphs, e.g., s-t out-in bitrees and s-t complete multi-stage graphs [3]. Thus, it is interesting for us to find its lower bound in order to estimate $\Psi_{(G,p)}$ for a probabilistic graph (G,p).

The purpose of this paper is to propose a new lower bound of $\Psi_{(G,p)}$ for a probabilistic graph (G,p) and to discuss the effectiveness of the lower bound. This paper is organized as follows: Graph theoretic terms and notations used throughout this paper are introduced in section 2. In section 3, we first give an s-t path number function f of G, and define a lower bound $\underline{\Psi}_{(G,f,p)}$ of $\Psi_{(G,p)}$ on a probabilistic graph (G,p) using the s-t path number function f. In section 4, we evaluate the lower bound by absolute performance radio $\frac{\underline{\Psi}_{(G,f,p)}}{\Psi_{(G,p)}}$ and show the necessary and sufficient condition with respect to an s-t path number function f of G by which $\underline{\Psi}_{(G,f,p)}$ coincides with $\Psi_{(G,p)}$. Section 5 proposes an algorithm of computing the lower bound $\underline{\Psi}_{(G,f,p)}$. Although the algorithm proposed in this paper is not polynomial time in general, we shall also give a class of probabilistic graphs (G,p) for which $\Psi_{(G,p)}$ is obtained in polynomial time by computing $\underline{\Psi}_{(G,f,p)}$ using this algorithm.

2. Preliminaries

2.1 Graph Theoretic Terminologies and Notations

A two-terminal (undirected) graph $G = (V, E, s, t)$ consists of a set V of finite *vertices* and a set E of finite *edges* (unordered pairs of vertices), where s and t, called *source* and *sink*, respectively, are two distinct specified vertices of V. The set of all edges incident to some vertex v of a subset $V' \subseteq V$ is denoted by $E(V')(= \{(u,v) \in E \mid u \in V' \text{ or } v \in V'\})$.

In $G = (V, E, s, t)$, a *u-v path* π of length k from vertex u to vertex v is an alternating sequence of vertices $v_i \in V$, $0 \leq i \leq k$, and edges $(v_{i-1}, v_i) \in E$, $1 \leq i \leq k$,

$$v_0(=u), (v_0, v_1), v_1, ..., v_{k-1}, (v_{k-1}, v_k), v_k(=v),$$

where vertices v_i's, for $0 \leq i \leq k$, are distinct, and two vertices u, v are called end vertices of the u-v path. Let $E(\pi)$ be the set of all edges on a u-v path π. Let $V_I(\pi)$ be the set of all internal vertices, i.e., vertices except u, v on a u-v path π. The set of all u-v paths in G is denoted by $P_{uv}(G)$. s-t paths sharing no vertex other than s, t are called *vertex-disjoint s-t paths*, and the maximum number of vertex-disjoint s-t paths in G is denoted by $\kappa_{st}(G)$.

In $G = (V, E, s, t)$, the subgraph obtained by removing all edges in $U(\subseteq E)$ is denoted by $G - U(= (V, E - U, s, t))$, and the subgraph obtained by removing all vertices in $V'(\subseteq V - \{s,t\})$ and all edges incident to some vertex v in V' is denoted by $G - V'(= (V - V', E - E(V'), s, t))$. A subset V' is called *an s-t vertex-cutset* if $G - V'$ has no s-t path. An s-t path π is said to be *an s-t vertex-cut-path* if $V_I(\pi)$ is an s-t vertex-cutset. By the well-known Menger's Theorem [6], the minimum cardinality of s-t vertex-cutset is equal to the maximum number of vertex-disjoint s-t paths for any G.

2.2 A Probabilistic Graph

A probabilistic graph, denoted by $(G = (V, E, s, t), p)$ or (G, p), for short, is defined as follows:

(i) $G = (V, E, s, t)$ is a two-terminal graph, and p is an $|E|$ dimensional vector consisting of failure probabilities $p(e)$'s of edges $e \in E$.

(ii) Each edge e of E is in either of the following two states: failed or operative (not failed), having known independent failure probability $p(e)$, $0 \leq p(e) < 1$ (or operative probability $q(e) = 1 - p(e)$).

(iii) No failure is assumed to arise at each vertex v of V.

For a probabilistic graph $(G=(V,E,s,t),p)$, let the subgraph $G-U(\subseteq E)$ correspond to the event $\mathcal{E}_U$ that all edges of U are failed and all edges of $E-U$ are operative. Clearly, the probability $\rho(G-U)$ of arising subgraph $G-U(\subseteq E)$ is computed by the following formula:

$$\rho(G-U)=\prod_{e\in U}p(e)\prod_{e\in E-U}q(e)(=1-p(e)).$$

It is clear that $\sum_{U\subseteq E}\rho(G-U)=1$.

Now, we define the *expected maximum number* $\Psi_{(G,p)}$ of vertex-disjoint s-t paths on a probabilistic graph $(G=(V,E,s,t),p)$ by the following formula:

$$(1)\qquad \Psi_{(G,p)}\equiv\sum_{U\subseteq E}\kappa_{st}(G-U)\rho(G-U).$$

It is known that the problem of computing $\Psi_{(G,p)}$ for a probabilistic graph (G,p) is NP-hard, even if G is restricted to several special classes of graphs like planar graphs, s-t out-in bitrees and s-t multi-stage complete graphs, etc. [3]. Thus, it is interesting for us to consider a lower bound of $\Psi_{(G,p)}$ in order to estimate $\Psi_{(G,p)}$.

3. A Lower Bound of $\Psi_{(G,p)}$

We shall define a lower bound of the expected maximum number of vertex-disjoint s-t paths in a probabilistic graph in this section. For this aim, we need more notations.

For a graph $G=(V,E,s,t)$, an *s-t path number function* f of G is a one-to-one integral function $f: P_{st}(G)\mapsto\{1,...,|P_{st}(G)|\}$. For a graph G and an s-t path number function f of G, an s-t path π_k with $f(\pi)=k$ is called an *s-t path of s-t path number* k. Let $\pi_{m(G-V',f)}$ denote the s-t path with the minimum s-t path number $m(G-V',f)$ in $G-V'(\subseteq V)$ with respect to f, and let $\pi_{m(G-E',f)}$ denote the s-t path with the minimum s-t path number $m(G-E',f)$ in $G-E'(\subseteq E)$ with respect to f.

3.1 Finding an s-t Path Number Function

In this subsection we shall demonstrate that an s-t path number function is constructively obtained for a graph G. To do this, we first show that given a graph $G=(V,E,s,t)$, the s-t path set $P_{st}(G)$ of G can be constructively obtained by a combinatorial method, namely, by enumerating all possible vertex sequences where each vertex is distinct. This procedure is described as follows.

Procedure Find-$P_{st}(G)$
Input: A graph $G=(V=\{s,v_1,v_2,\cdots,v_n,t\},E,s,t)$.
Output: The s-t path set $P_{st}(G)$.
BEGIN
L1. $P_{st}(G):=\phi$; $P:=\phi$;
L2. FOR $i:=1$ TO n DO $s(i):=1$;
{$s(i)$ denotes the number of the ith vertex in a vertex sequence.}
L3. FOR $i:=1$ TO n DO $Mark(i):=0$;
L4. FOR $Length=1$ TO n DO
BEGIN
L5. FOR $i:=Length$ TO 1 DO
BEGIN
L6. IF $Mark(i)=1$ THEN BEGIN $s(i):=s(i)+1$; $Mark(i):=0$ END;
L7. IF $s(1)=n+1$ THEN GOTO L12;
L8. IF $s(i)=n+1$ THEN BEGIN $s(i):=1$; $Mark(i-1):=1$ END;

L9. $P := v_{s(i)} \cdot P$
END;
L10. IF $\pi : s \cdot P \cdot t$ is an s-t path THEN $P_{st}(G) := P_{st}(G) \cup \{\pi\}$;
L11. $Mark(Length) := 1$; $P := \phi$; GOTO L5;
L12. $s(1) := 1$;
END;
L13. Output $P_{st}(G)$
END. □

Clearly, all possible vertex sequences with k vertices, where $k = Length$, are enumerated by L5-L11 in **Procedure Find-$P_{st}(G)$**. Thus, **Procedure Find-$P_{st}(G)$** correctly finds the s-t path set $P_{st}(G)$ for a graph $G = (V, E, s, t)$. In general, finding $P_{st}(G)$ for a graph G requires exponential time with respect to the size of G. However, there may exist some efficient method of finding $P_{st}(G)$ for a special graph G. In fact, for a one-layered s-t graph G whose definition will be given in the subsection 5.2, its s-t path set $P_{st}(G)$ is obtained in polynomial time with respect to the size of G. Furthermore, if the s-t path set $P_{st}(G)$ for a graph G is given, we can obtain an s-t path number function f by the following procedure.

Procedure Find-f
Input: The s-t path set $P_{st}(G)$ of a graph $G = (V, E, s, t)$.
Output: An s-t path number function f.
BEGIN
$i := 1$;
WHILE $P_{st}(G) \neq \phi$ DO
BEGIN
Select an s-t path π from $P_{st}(G)$;
$f(\pi) := i$; $i := i + 1$;
Delete the s-t path π from $P_{st}(G)$;
END;
Output f
END. □

As the number of s-t paths in a graph G is an exponential function with respect to the size of G, finding an s-t path number function f requires exponential time in general.

3.2 Finding Vertex-disjoint s-t Paths

First, we give procedure **FVDP** to find vertex-disjoint s-t paths in a graph $G = (V, E, s, t)$, based on an s-t path number function f of G.

Procedure FVDP
Input: A graph $G = (V, E, s, t)$ and an s-t path number function f of G.
Output: The set of vertex-disjoint s-t paths $FVDP(G, f)$.
BEGIN
$G' := G$; $FVDP(G, f) := \phi$;
WHILE $P_{st}(G') \neq \phi$ DO
BEGIN
Find $\pi_{m(G',f)}$ from $P_{st}(G')$;
$FVDP(G, f) := FVDP(G, f) \cup \{\pi_{m(G',f)}\}$;
$G' := G' - V_I(\pi_{m(G',f)})$ {This implies that we modify $P_{st}(G')$ by deleting all

s-t paths from $P_{st}(G')$ having at least one common vertex with $V_I(\pi_{m(G',f)})$.}
END;
Output $FVDP(G,f)$
END. □

It is clear that $FVDP(G,f)$ obtained by **Procedure FVDP** is a set of vertex-disjoint s-t paths in G. Namely,

$$(2) \qquad |FVDP(G,f)| \leq \kappa_{st}(G), \quad for\ any\ G\ and\ f\ of\ G.$$

Note that we may obtain different $FVDP(G,f)$ using different s-t path number function f of G for a fixed graph G, and $FVDP(G,f)$ for an s-t path number function f of G is not necessarily a set of vertex-disjoint s-t paths of the maximum number in G.

In addition, when we implement the procedure, s-t path set $P_{st}(G)$ is realized by a list structure where each element corresponds to each s-t path in $P_{st}(G)$ and stores the information including the s-t path number, all vertices on the s-t path and the pointer indicating the next element, and having head pointer indicating the first element in the list. Thus, based on the list structure for $P_{st}(G)$, **Procedure FVDP** is easily implemented. Certainly, more efficient structures representing $P_{st}(G)$ like balanced tree may be also available. We have shown here that **Procedure FVDP** is actually executable, although there may be more efficient implementation of the procedure.

3.3 Definition of a Lower Bound

For a probabilistic graph $(G=(V,E,s,t),p)$ and an s-t path number function f of G, we define a value $\underline{\Psi}_{(G,f,p)}$ by the following formula:

$$(3) \qquad \underline{\Psi}_{(G,f,p)} \equiv \sum_{U\subseteq E} |FVDP(G-U,f)|\rho(G-U).$$

By (2), $\underline{\Psi}_{(G,f,p)}$ is a lower bound of $\Psi_{(G,p)}$ for a probabilistic graph (G,p) and an s-t path number function f, namely,

$$\underline{\Psi}_{(G,f,p)} \leq \Psi_{(G,p)}, \quad for\ any\ (G,p)\ and\ f\ of\ G.$$

4. Evaluation of the Lower Bound

4.1 An Absolute Performance Ratio

We now evaluate the lower bound by $\frac{\underline{\Psi}_{(G,f,p)}}{\Psi_{(G,p)}}$, called *absolute performance ratio.* Firstly, as $\underline{\Psi}_{(G,f,p)}$ is a lower bound of $\Psi_{(G,p)}$, we immediately obtain the following formula.

$$\frac{\underline{\Psi}_{(G,f,p)}}{\Psi_{(G,p)}} \leq 1$$

For a probabilistic graph $(G=(V,E,s,t),p)$ and an s-t path number function f of G, we define $\mathcal{U}_0 \equiv \{U \subseteq E \mid FVDP(G-U,f) = \kappa_{st}(G-U)\}$ and $\mathcal{U}_1 \equiv \{U \subseteq E \mid FVDP(G-U,f) < \kappa_{st}(G-U)\}$. Clearly,

$$\mathcal{U}_0 \cap \mathcal{U}_1 = \phi, \qquad \mathcal{U}_0 \cup \mathcal{U}_1 = \{U : U \subseteq E\}$$

and

$$\sum_{U\in\mathcal{U}_1} \kappa_{st}(G-U)\rho(G-U) > \sum_{U\in\mathcal{U}_1} FVDP(G-U,f)\rho(G-U).$$

By the definition of $\mathcal{U}_1$, $\mathcal{U}_1 = \phi$ implies that $\underline{\Psi}_{(G,f,p)} = \Psi_{(G,p)}$ for a probabilistic graph (G,p) and an s-t path number function f of G. A necessary and sufficient condition to satisfy $\mathcal{U}_1 = \phi$ will be shown in the next subsection. Now, let us assume that $\mathcal{U}_1 \neq \phi$. Thus,

$$\begin{aligned}
\frac{\underline{\Psi}_{(G,f,p)}}{\Psi_{(G,p)}} &= \frac{\sum_{U\in\mathcal{U}_0} FVDP(G-U,f)\rho(G-U) + \sum_{U\in\mathcal{U}_1} FVDP(G-U,f)\rho(G-U)}{\sum_{U\in\mathcal{U}_0} \kappa_{st}(G-U)\rho(G-U) + \sum_{U\in\mathcal{U}_1} \kappa_{st}(G-U)\rho(G-U)} \\
&= \frac{\sum_{U\in\mathcal{U}_0} \kappa_{st}(G-U)\rho(G-U) + \sum_{U\in\mathcal{U}_1} FVDP(G-U,f)\rho(G-U)}{\sum_{U\in\mathcal{U}_0} \kappa_{st}(G-U)\rho(G-U) + \sum_{U\in\mathcal{U}_1} \kappa_{st}(G-U)\rho(G-U)} \\
&\quad (\text{ by the definition of } \mathcal{U}_0 \text{ }) \\
&> \frac{\sum_{U\in\mathcal{U}_1} FVDP(G-U,f)\rho(G-U)}{\sum_{U\in\mathcal{U}_1} \kappa_{st}(G-U)\rho(G-U)} \\
&\quad (\text{ by } \textstyle\sum_{U\in\mathcal{U}_1} \kappa_{st}(G-U)\rho(G-U) > \sum_{U\in\mathcal{U}_1} FVDP(G-U,f)\rho(G-U) \text{ }) \\
&\geq \frac{\sum_{U\in\mathcal{U}_1} \rho(G-U)}{\kappa(G) \sum_{U\in\mathcal{U}_1} \rho(G-U)} \\
&\quad (\text{ by } FVDP(G-U,f) \geq 1,\ \kappa_{st}(G) \geq \kappa_{st}(G-U),\ for\ U \in \mathcal{U}_1 \text{ }) \\
&= \frac{1}{\kappa_{st}(G)}
\end{aligned}$$

is obtained. Based on the above discussion, we obtain the following Theorem 1.
Theorem 1. For any probabilistic graph (G,p) and any s-t path number function f of G, we have

$$\underline{\Psi}_{(G,f,p)} = \Psi_{(G,p)}$$

when $\mathcal{U}_1 = \phi$, and we have

$$1 > \frac{\underline{\Psi}_{(G,f,p)}}{\Psi_{(G,p)}} > \frac{1}{\kappa_{st}(G)}$$

when $\mathcal{U}_1 \neq \phi$. □

4.2 A Necessary and Sufficient Condition

Now, we give the necessary and sufficient condition by which $\underline{\Psi}_{(G,f,p)}$ coincides with $\Psi_{(G,p)}$ for a probabilistic graph (G,p).
Lemma 1. For a probabilistic graph $(G=(V,E,s,t),p)$, $\underline{\Psi}_{(G,f,p)} = \Psi_{(G,p)}$ if, and only if, the s-t path number function f of G satisfies

$$|FVDP(G-U,f)| = \kappa_{st}(G-U), \quad for\ any\ U \subseteq E. \tag{4}$$

Proof. The *if* part is obvious by (1),(2) and (3). Suppose that $|FVDP(G-U,f)| < \kappa_{st}(G-U)$ for some $U \subseteq E$. Then $\underline{\Psi}_{(G,f,p)} < \Psi_{(G,p)}$ by (1),(2) and (3), which, however, contradicts the assumption that $\underline{\Psi}_{(G,f,p)} = \Psi_{(G,p)}$. Thus, we have proved the *only-if* part. □
Definition 1. An s-t path number function f of a graph G is said to be *exact* if f satisfies (4). □

A graph $G = (V, E, s, t)$ is said to be *s-t k-connected* if $\kappa_{st}(G) \geq k$. A graph G is called *an s-t path-cut graph* if there is an s-t vertex-cut-path in G. A π-cut s-t 2-connected graph G is the graph where π is an s-t vertex-cut-path in G and G is s-t 2-connected. A π-cut s-t 2-connected graph $G = (V, E, s, t)$ is said to be *minimal* if, for any edge $e \in E - E(\pi)$, the subgraph $G - \{e\}$ is not π-cut s-t 2-connected. In $G = (V, E, s, t)$, the set of all minimal π-cut s-t 2-connected subgraphs of G for an s-t path $\pi \in P_{st}(G)$ is denoted by $\mathcal{B}(G, \pi)$. The following Lemma 2 holds by the definitions.

Lemma 2. For a graph $G = (V, E, s, t)$ and an s-t path $\pi \in P_{st}(G)$, $\mathcal{B}(G, \pi) \neq \phi$ if, and only if, there exist two vertex-disjoint s-t paths π', $\pi'' \in P_{st}(G)$ and two vertices $v, v' \in V_I(\pi)$ $(v \neq v')$ such that $v \in V_I(\pi')$ and $v' \in V_I(\pi'')$. □

Lemma 3. If there exists an s-t path π satisfying $\mathcal{B}(G, \pi) = \phi$ in a graph $G = (V, E, s, t)$, then we have

$$\kappa_{st}(G - V_I(\pi)) = \kappa_{st}(G) - 1.$$

Proof. Clearly, $\kappa_{st}(G - V_I(\pi)) \leq \kappa_{st}(G) - 1$. Assume that $\kappa_{st}(G - V_I(\pi)) < \kappa_{st}(G) - 1$. Then by Menger's Theorem [6], the subgraph $G - V_I(\pi)$ has an s-t vertex-cutset V^* of the minimum cardinality satisfying $|V^*| \leq \kappa_{st}(G) - 2$. Furthermore, π is an s-t vertex-cut-path of $G - V^*$. Moreover, let V' be an s-t vertex-cutset of the minimum cardinality in $G - V^*$. Clearly, $V' \cup V^*$ is an s-t vertex-cutset of G. Since $|V^*| \leq \kappa_{st}(G) - 2$, $|V'| = \kappa_{st}(G - V^*)$ and $\kappa_{st}(G)$ is equal to the minimum cardinality of s-t vertex-cutset in G by Menger's Theorem [6], we have

$$\kappa_{st}(G) \leq |V' \cup V^*| = |V'| + |V^*| \leq |V'| + \kappa_{st}(G) - 2,$$

namely, $|V'| = \kappa_{st}(G - V^*) \geq 2$. This implies that there exist at least two s-t vertex-disjoint s-t paths π', π'' in $G - V^*$ and $v, v' \in V_I(\pi)(v \neq v')$ (note that π is an s-t vertex-cut-path of $G - V^*$) such that $v \in V_I(\pi')$ and $v' \in V_I(\pi'')$. Hence, $\mathcal{B}(G, \pi) \neq \phi$ by Lemma 2, which, however, contradicts the assumption of this lemma that $\mathcal{B}(G, \pi) = \phi$.
□

Lemma 4. In a graph $G = (V, E, s, t)$, an s-t path number function f of G is exact if, and only if, for any $U \subseteq E$, $\mathcal{B}(G - U, \pi_{m(G-U,f)}) = \phi$.

Proof. Necessity: Assume that an s-t path number function f of G is exact and that for some $U \subseteq E$, $\mathcal{B}(G - U, \pi_{m(G-U,f)}) \neq \phi$. By $\mathcal{B}(G - U, \pi_{m(G-U,f)}) \neq \phi$, $G - U$ has a subgraph $G' \in \mathcal{B}(G - U, \pi_{m(G-U,f)})$. $\kappa_{st}(G') = 2$ by the definition of $\mathcal{B}(G - U, \pi_{m(G-U,f)})$. As $\pi_{m(G-U,f)}$ is the s-t path with the minimum number and an s-t vertex-cut-path in G', we have $FVDP(G', f) = \{\pi_{m(G-U,f)}\}$ by **FVDP**. Hence, $|FVDP(G', f)|(= 1) < \kappa_{st}(G')(= 2)$, contradicting the fact that f is exact.

Sufficiency: Assume that, for any $U \subseteq E$, $\mathcal{B}(G - U, \pi_{m(G-U,f)}) = \phi$ holds. Then it is easy to prove that $|FVDP(G - U, f)| = \kappa_{st}(G - U)$ for any $U \subseteq E$ by iteratively applying Lemma 3. □

By Lemma 1 and Lemma 4, we obtain the following Theorem 2.

Theorem 2. For a probabilistic graph (G, p), $\underline{\Psi}_{(G,f,p)} = \Psi_{(G,p)}$ if, and only if, the s-t path number function f of G satisfies $\mathcal{B}(G - U, \pi_{m(G-U,f)}) = \phi$ for any $U \subseteq E$. □

Note that there may not necessarily exist an exact s-t path number function in any graph. For example, the graph G illustrated in Fig.1 has no exact s-t path number function f, as $\mathcal{B}(G, \pi) \neq \phi$ for any s-t path π.

5. Computation of the Lower Bound

By (3), we can obtain an algorithm of computing $\underline{\Psi}_{(G,f,p)}$ for a probabilistic graph (G, p) and an s-t path number function f of G. However the time complexity of the algorithm is

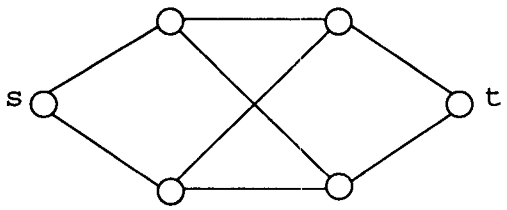

Figure 1: A two-terminal graph G no having exact $s-t$ path number function.

$O(2^{|E|})$, i.e., exponential with respect to $|E|$, where $|E|$ is a number of all edges in G. Thus we give another algorithm of computing $\Psi_{(G,f,p)}$ for a probabilistic graph (G,p) and an s-t path number function f of G. We first wish to recall the procedure **FVDP**.

5.1 An Algorithm of Computing $\Psi_{(G,f,p)}$

For a probabilistic graph $(G=(V,E,s,t),p)$ and an s-t path number function f of G, let $\mathcal{U}_{f,\pi_i}$ denote the set of all $U \subseteq E$ for which s-t path π_i is in the set of vertex-disjoint s-t paths $FVDP(G-U,f)$, namely, $\mathcal{U}_{f,\pi_i} = \{U \subseteq E \mid \pi_i \in FVDP(G-U,f)\}$. Let $p(\mathcal{E}_U)$ be the probability of the event $\mathcal{E}_U$ that all edges of $U(\subseteq E)$ are failed and all edges of $E-U(\subseteq E)$ are operative, namely, $p(\mathcal{E}_U)=\rho(G-U)$. Let $p(\mathcal{E}_{f,\pi_i})$ be the probability of the event $\mathcal{E}_{f,\pi_i}$ that at least one event $\mathcal{E}_U$, for all $U \in \mathcal{U}_{f,\pi_i}$, arises. Thus, we have

$$
(5) \qquad \begin{aligned}
\Psi_{(G,f,p)} &= \sum_{U\subseteq E} |FVDP(G-U,f)|\rho(G-U) \\
&= \sum_{i=1}^{|P_{st}(G)|} \sum_{U\in\mathcal{U}_{f,\pi_i}} \rho(G-U) \\
&= \sum_{i=1}^{|P_{st}(G)|} \sum_{U\in\mathcal{U}_{f,\pi_i}} p(\mathcal{E}_U) \\
&= \sum_{i=1}^{|P_{st}(G)|} p(\mathcal{E}_{f,\pi_i}).
\end{aligned}
$$

Clearly, we can compute the lower bound by formula (5) instead of formula (3). From the viewpoint of computational complexity, formula (5) is essentially different from formula (3), as the time complexity of computing $\Psi_{(G,f,p)}$ for any class of probabilistic graphs $(G=(V,E,s,t),p)$ is at least $O(2^{|E|})$ by formula (3) and it is not necessarily exponential time by formula (5). In fact, for probabilistic one-layered s-t graphs shown in the next subsection, this complexity is polynomial time.

In order to find the event $\mathcal{E}_{f,\pi_i}$, we need more notations. For a graph $G=(V,E,s,t)$ and an s-t path number function f of G, let $Q_{f,\pi_i} = \{\ \pi_j \in P_{st}(G) \mid V_I(\pi_i)\cap V_I(\pi_j) \neq \phi \ and\ f(\pi_j)=j < f(\pi_i)=i\}$. The following Lemma 5 obviously holds by the definitions.

Lemma 5. For a graph $G=(V,E,s,t)$ and an s-t path number function f of G,

$$
\mathcal{U}_{f,\pi_i} = \begin{cases} \{\ U\subseteq E \mid E(\pi_i)\subseteq E-U\ \} & if \quad Q_{f,\pi_i}=\phi, \\ \{\ U\subseteq E \mid E(\pi_i)\subseteq E-U \ and\ U \notin \mathcal{U}_{f,\pi_j} \ for\ all\ \pi_j\in Q_{f,\pi_i}\ \} & if \quad Q_{f,\pi_i}\neq\phi, \end{cases}
$$

where $1 \le j < i \le |P_{st}(G)|$. □

For some events $\mathcal{E}_1, \mathcal{E}_2, ..., \mathcal{E}_m$, we denote by $\mathcal{E}_1\mathcal{E}_2\cdots\mathcal{E}_m$, or, for short, $\prod_{i=1}^m \mathcal{E}_i$ the event that all events $\mathcal{E}_1, \mathcal{E}_2, ..., \mathcal{E}_m$ simultaneously arise, and by $\mathcal{E}_1 + \mathcal{E}_2 + \cdots + \mathcal{E}_m$, or, for short, $\sum_{i=1}^m \mathcal{E}_i$ the event that at least one event among $\mathcal{E}_1, \mathcal{E}_2, ..., \mathcal{E}_m$ arises. The event Ω called *whole event* satisfies $p(\Omega) = 1$, and the event $\emptyset$ called *empty event* satisfies $p(\emptyset) = 0$. Let $\overline{\mathcal{E}}$ denote the *complementary event* of $\mathcal{E}$. Clearly, the following formulas hold for any two events $\mathcal{E}_i$, $\mathcal{E}_j$.

$$\mathcal{E}_i\overline{\overline{\mathcal{E}_i}\mathcal{E}_j} = \mathcal{E}_i\overline{\mathcal{E}_j}, \tag{6}$$

$$\mathcal{E}_i\overline{\mathcal{E}_i\overline{\mathcal{E}_j}} = \mathcal{E}_i. \tag{7}$$

For a probabilistic graph $(G = (V, E, s, t), p)$, we denote by $\mathbf{e}$ the event that edge $e \in E$ is operative (not failed). Then $\overline{\mathbf{e}}$ is the event that edge $e \in E$ is failed. Namely, $p(\overline{\mathbf{e}}) = p(e)$ and $p(\mathbf{e}) = q(e)(= 1 - p(e))$. Thus the event $\mathcal{E}_U$ for any $U \subseteq E$ is described as follows:

$$\mathcal{E}_U = \prod_{e\in U} \overline{\mathbf{e}} \prod_{e\in E-U} \mathbf{e}. \tag{8}$$

For any $E' \subseteq E$, we can easily prove the following formula:

$$\sum_{U\subseteq E'} \mathcal{E}_U = \sum_{U\subseteq E'} \Big(\prod_{e\in U} \overline{\mathbf{e}} \prod_{e\in E'-U} \mathbf{e}\Big) = \Omega. \tag{9}$$

For a probabilistic graph $(G = (V, E, s, t), p)$ and an s-t path number function f, when $Q_{f,\pi_i} = \phi$, we have

$$\begin{aligned} \mathcal{E}_{f,\pi_i} &= \sum_{U\in\mathcal{U}_{f,\pi_i}} \mathcal{E}_U \\ &= \sum_{U\subseteq E \text{ and } E(\pi_i)\subseteq E-U} \mathcal{E}_U \quad (\text{ by Lemma 5 }) \\ &= \Big(\prod_{e\in E(\pi_i)} \mathbf{e}\Big) \sum_{U'\subseteq E-E(\pi_i)} \mathcal{E}_{U'} \quad (\text{ by (8) }) \\ &= \prod_{e\in E(\pi_i)} \mathbf{e} \quad (\text{ by (9) }). \end{aligned} \tag{10}$$

Note that $p(\mathcal{E}_{f,\pi_i})(= \prod_{e\in E(\pi_i)} p(\mathbf{e}))$ is efficiently computed. However, when $Q_{f,\pi_i} \neq \phi$, we have

$$\begin{aligned} \mathcal{E}_{f,\pi_i} &= \sum_{U\in\mathcal{U}_{f,\pi_i}} \mathcal{E}_U \\ &= \sum_{U\subseteq E \text{ and } E(\pi_i)\subseteq E-U \text{ and } U\notin\mathcal{U}_{f,\pi_j} \text{ for all } \pi_j\in Q_{f,\pi_i}} \mathcal{E}_U \quad (\text{ by Lemma 5 }) \\ &= \Big(\sum_{U\subseteq E \text{ and } E(\pi_i)\subseteq E-U} \mathcal{E}_U\Big)\Big(\sum_{U\subseteq E \text{ and } U\notin\mathcal{U}_{f,\pi_j} \text{ for all } \pi_j\in Q_{f,\pi_i}} \mathcal{E}_U\Big) \\ &= \Big[\Big(\prod_{e\in E(\pi_i)} \mathbf{e}\Big) \sum_{U'\subseteq E-E(\pi_i)} \mathcal{E}_{U'}\Big]\Big(\prod_{\pi_j\in Q_{f,\pi_i}} \sum_{U\notin\mathcal{U}_{f,\pi_j} \text{ and } U\subseteq E} \mathcal{E}_U\Big) \quad (\text{ by (8) }) \\ &= \prod_{e\in E(\pi_i)} \mathbf{e} \prod_{\pi_j\in Q_{f,\pi_i}} \overline{\mathcal{E}}_{f,\pi_j} \quad (\text{ by (9) and the definition of } \mathcal{E}_{f,\pi_i}). \end{aligned} \tag{11}$$

By (5), (10) and (11), we thus obtain an algorithm of computing $\underline{\Psi}_{(G,f,p)}$ for a probabilistic graph $(G = (V, E, s, t), p)$ and a given s-t path number function f of G.

Note that, in general, the time complexity of the algorithm is not polynomial for a probabilistic graph (G,p) and a given s-t path number function f of G, as the events $\mathcal{E}_{f,\pi_j}$ for all $\pi_j \in Q_{f,\pi_i}$ are not necessarily independent of each other, i.e., the probability $p(\mathcal{E}_{f,\pi_i})$ where $Q_{f,\pi_i} \neq \phi$ does not seem to be efficiently computed in general, and in addition to this, $|P_{st}(G)|$ is not polynomial in the size of a general graph G.

However, in the next subsection, we give a class of probabilistic graphs for which the lower bound is efficiently computed by the algorithm.

5.2 An Example of a Probabilistic Graph where $\underline{\Psi}_{(G,f,p)}$ is Polynomially Computable

Definition 2. A graph $G = (V, E, s, t)$ is called *one-layered s-t graph* if $G - \{s,t\}$ exactly consists of a simple path, i.e., $V = \{ s, v_1, ..., v_n, t \}$ and $E = C(= \{c_i = (v_i, v_{i+1}) \mid i = 1, ..., n-1 \}) \cup A(= \{a_i = (s, v_i) \mid \textit{for some } i,\ 1 \leq i \leq n \}) \cup B(= \{b_i = (v_i, t) \mid \textit{for some } i,\ 1 \leq i \leq n \})$. □

The graph illustrated in Fig.2 is an example of a one-layered s-t graph. We shall show that, for a probabilistic one-layered s-t graph, the expected maximum number of vertex-disjoint s-t paths is efficiently obtained by computing the lower bound.

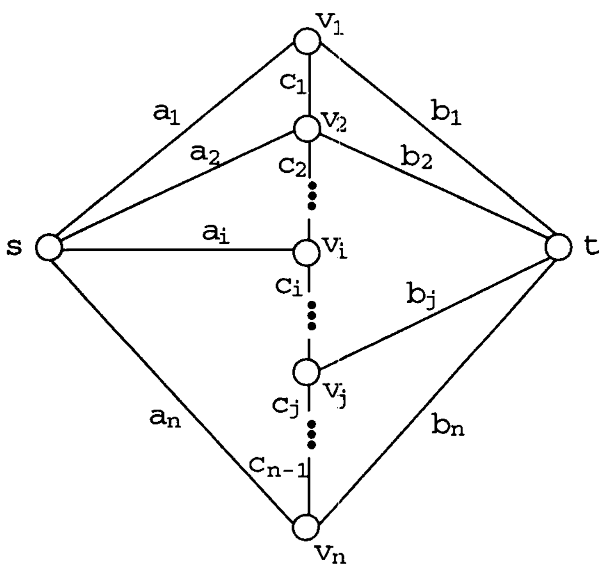

Figure 2: An example of a one-layered $s-t$ graph.

5.2.1 Some Properties of a One-layered s-t Graph

In a one-layered s-t graph $G = (V = \{s, v_1, ..., v_n, t\}, E, s, t)$, we have

$$(12) \qquad \pi = \begin{cases} a_i, v_i, b_i & if \quad i = j, \\ a_i, v_i, c_i, v_{i+1}, c_{i+1}, \cdots, v_{j-1}, c_{j-1}, v_j, b_j & if \quad i < j, \\ a_i, v_i, c_{i-1}, v_{i-2}, c_{i-2}, \cdots, v_{j+1}, c_{j+1}, v_j, b_j & if \quad i > j \end{cases}$$

for any s-t path $\pi \in P_{st}(G)$, where $1 \leq i, j \leq n$, and an s-t path $\pi \in P_{st}(G)$ is denoted by $\pi(a_i, b_j)$, furthermore, for short, by π_{ij}. The following lemmas obviously hold.

Lemma 6. For a one-layered s-t graph $G = (V = \{s, v_1, ..., v_n, t\}, E, s, t)$, $|P_{st(G)}| \leq n^2$. □

Lemma 7. In a one-layered s-t graph $G = (V, E, s, t)$, for any two vertex-disjoint s-t paths $\pi_{i'j'}$, $\pi_{i''j''} \in P_{st}(G)$, either $\max\{i', j'\} < \min\{i'', j''\}$ or $\min\{i', j'\} > \max\{i'', j''\}$. □

Definition 3. In a one-layered s-t graph $G = (V, E, s, t)$, for two s-t paths π_{ij}, $\pi_{i'j'} \in P_{st}(G)$, π_{ij} is *above* $\pi_{i'j'}$ (or $\pi_{i'j'}$ is *below* π_{ij}) if one of the following conditions holds:

(i) $\min\{i, j\} < \min\{i', j'\}$.
(ii) $\min\{i, j\} = i = j' = \min\{i', j'\}$.
(iii) $\min\{i, j\} = i = i' = \min\{i', j'\}$ *and* $j \leq j'$.
(iv) $\min\{i, j\} = j = j' = \min\{i', j'\}$ *and* $i \leq i'$.

□

Namely, if all s-t paths in G are arranged as follows:

$$(13)\qquad \begin{cases} \pi_{11},\ \pi_{12},\ \pi_{13},\ \ldots,\ \pi_{1n-1},\ \pi_{1n},\ \pi_{21},\ \pi_{31},\ \ldots,\ \pi_{n-11},\ \pi_{n1}, \\ \quad \pi_{22},\ \pi_{23},\ \ldots,\ \pi_{2n-1},\ \pi_{2n},\ \pi_{32},\ \ldots,\ \pi_{n-12},\ \pi_{n2}, \\ \qquad \ldots\ldots, \\ \qquad\quad \pi_{n-1n-1},\ \pi_{n-1n},\ \pi_{nn-1}, \\ \qquad\qquad \pi_{nn}, \end{cases}$$

then, for two s-t paths π_{ij}, $\pi_{i'j'}$ in the same row (namely, either (iii) or (iv) in Definition 3), π_{ij} is above $\pi_{i'j'}$ if π_{ij} is to the left of $\pi_{i'j'}$, and for two s-t paths π_{ij}, $\pi_{i'j'}$ in two different rows (namely, either (i) or (ii) in Definition 3) π_{ij} is above $\pi_{i'j'}$ if the row of π_{ij} is above the row of $\pi_{i'j'}$.

Lemma 8. In a one-layered s-t graph $G = (V, E, s, t)$, for an s-t path $\pi_{ij} \in P_{st}(G)$, suppose that $\mathcal{B}(G, \pi_{ij}) \neq \phi$. Let $\pi_{i'j'}$, $\pi_{i''j''}$ be two vertex-disjoint s-t paths of $G' \in \mathcal{B}(G, \pi_{ij})$. Then, $\pi_{i'j'}$ (or $\pi_{i''j''}$) is above π_{ij}, and $\pi_{i''j''}$ (or $\pi_{i'j'}$) is below π_{ij}.

Proof. For two vertex-disjoint s-t paths $\pi_{i'j'}$, $\pi_{i''j''}$ of $G' \in \mathcal{B}(G, \pi_{ij})$, without loss of generality, assume that $\max\{i', j'\} < \min\{i'', j''\}$ by Lemma 7. By $G' \in \mathcal{B}(G, \pi_{ij})$, for the s-t path π_{ij}, we have $a_i = a_{i'}$ and $b_j = b_{j''}$ (or $a_i = a_{i''}$ and $b_j = b_{j'}$, respectively). Otherwise, G' is not a minimal π-cut s-t 2-connected subgraph. As $j = j'' \geq \min\{i'', j''\} > \max\{i', j'\} \geq i' = i$ and $i' = i \geq \min\{i', j'\}$, namely, $\min\{i, j\} = i \geq \min\{i', j'\}$, and as $j = j'' \geq \min\{i'', j''\} > \max\{i', j'\} \geq j'$ namely, $j \geq j'$, $\pi_{i'j'}$ is above π_{ij} by conditions (i) and (iii) in Definition 3. Furthermore, as $\min\{i, j\} = i = i' \leq \max\{i', j'\} < \min\{i'', j''\}$, namely, $\min\{i, j\} < \min\{i'', j''\}$, $\pi_{i''j''}$ is below π_{ij} by condition (i) in Definition 3. □

Lemma 9. For a one-layered s-t graph $G = (V, E, s, t)$, if the s-t path number function f^* of G satisfies the following Condition 1, then for any $U \subseteq E$, f^* satisfies $\mathcal{B}(G - U, \pi_{m(G-U,f^*)}) = \phi$.

Condition 1: For any two s-t paths $\pi, \pi' \in P_{st}(G)$, $f^*(\pi) < f^*(\pi')$ if, and only if, π is above π'.

Proof. Let f^* be an s-t path number function of G satisfying Condition 1, and let $\pi_{m(G-U,f^*)} : a_i, ..., b_j$ be the s-t path with the minimum number in $G - U$ with respect to f^*. Suppose that f^* satisfies $\mathcal{B}(G - U, \pi_{m(G-U,f^*)}) \neq \phi$ for some $U \subseteq E$. By Lemma 8, there exist two vertex-disjoint s-t paths π', π'' in the subgraph $G' \in \mathcal{B}(G - U, \pi_{m(G-U,f^*)})$ such that π' (or π'') is above $\pi_{m(G-U,f^*)}$. By Condition 1, $f^*(\pi') < f^*(\pi_{m(G-U,f^*)})$, which, however, is a contradiction. □

By the above Lemma 9 and Theorem 2, the following Theorem 3 is immediately obtained.

Theorem 3. In a probabilistic one-layered s-t graph $(G = (V, E, s, t), p)$, for the s-t path number function f^* of G satisfying Condition 1, $\underline{\Psi}_{(G,f^*,p)} = \Psi_{(G,p)}$. □

5.2.2 Computation of $\Psi_{(G,f^*,p)}$

For a probabilistic one-layered s-t graph $(G=(V,E,s,t),p)$, where

$$(14)\qquad \begin{aligned} V &= \{\, s, v_1, ..., v_n, t \,\}, \\ E &= C(=\{c_i=(v_i,v_{i+1}) \mid i=1,...,n-1\}) \\ &\quad \cup A(=\{a_i=(s,v_i) \mid for\ all\ i,\ 1\le i\le n\}) \\ &\quad \cup B(=\{b_i=(v_i,t) \mid for\ all\ i,\ 1\le i\le n\}), \end{aligned}$$

let f^* be the s-t path number function satisfying Condition 1. Furthermore, for short, let $\mathcal{E}_{ij}$ denote the event $\mathcal{E}_{f^*,\pi_{ij}}$ that at least one event $\mathcal{E}_U$ arises, where U is in $\mathcal{U}_{f^*,\pi_{ij}}$ with respect to f^* of G. We use $Q_{f^*,\pi_{ij}}$ to denote the set of all s-t paths $\pi_{i'j'}$'s satisfying $V(\pi_{ij})\cap V(\pi_{i'j'})\neq\phi$ and $f^*(\pi_{i'j'})<f^*(\pi_{ij})$.

By Condition 1 and the definitions, we have $f^*(\pi_{11})=1$. Clearly, $Q_{f^*,\pi_{11}}=\phi$. Thus, by (10),(12), we have

$$(15)\qquad \mathcal{E}_{11} = \prod_{e\in E(\pi_{11})} \mathbf{e} = \mathbf{a}_1\mathbf{b}_1.$$

Lemma 10. For a probabilistic one-layered s-t graph $(G=(V,E,s,t),p)$, where V, E satisfy (14), let f^* be an s-t path number function of G satisfying Condition 1.

$$(16)\qquad \mathcal{E}_{1j} = \mathbf{a}_1\left(\prod_{w=1}^{j-1}\mathbf{c}_w\right)\mathbf{b}_j\left(\prod_{w=1}^{j-1}\overline{\mathbf{b}}_w\right),\quad for\ 2\le j\le n,$$

and

$$(17)\qquad \mathcal{E}_{i1} = \mathbf{a}_i\left(\prod_{w=1}^{i-1}\mathbf{c}_w\right)\mathbf{b}_1\left(\prod_{w=1}^{i-1}\overline{\mathbf{a}}_w\right),\quad for\ 2\le i\le n.$$

Proof. We prove (16) by induction with respect to j.

When $j=2$, $Q_{f^*,\pi_{12}}=\{\pi_{11}\}$ holds by (13) and the definitions. Thus,

$$\begin{aligned} \mathcal{E}_{12} &= \prod_{e\in E(\pi_{12})}\mathbf{e}\prod_{\pi_{xy}\in Q_{f^*,\pi_{12}}}\overline{\mathcal{E}}_{xy} \quad (\text{ by (11) }) \\ &= \mathbf{a}_1\mathbf{c}_1\mathbf{b}_2\overline{\mathbf{b}}_1 \quad (\text{ by (15),(6) }) \end{aligned}$$

is obtained and (16) holds.

Suppose that for $j<k$,

$$(18)\qquad \mathcal{E}_{1j} = \mathbf{a}_1\left(\prod_{w=1}^{j-1}\mathbf{c}_w\right)\mathbf{b}_j\left(\prod_{w=1}^{j-1}\overline{\mathbf{b}}_w\right),$$

and we consider the case of $j=k$. By (13) and the definitions,

$$(19)\qquad Q_{f^*,\pi_{1k}} = \{\ \pi_{11},\pi_{12},...,\pi_{1k-1}\ \}.$$

Thus, we have

$$\begin{aligned} \mathcal{E}_{1k} &= \prod_{e\in E(\pi_{1k})}\mathbf{e}\prod_{\pi_{xy}\in Q_{\pi_{1k}}}\overline{\mathcal{E}}_{xy} \quad (\text{ by (11) }) \\ &= \mathbf{a}_1\left(\prod_{w=1}^{k-1}\mathbf{c}_w\right)\mathbf{b}_k\left(\overline{\mathbf{a}_1\mathbf{b}_1}\right)\left(\prod_{j=2}^{k-1}\overline{\mathcal{E}}_{1j}\right) \quad (\text{ by (12),(15),(19) }) \\ &= \mathbf{a}_1\left(\prod_{w=1}^{k-1}\mathbf{c}_w\right)\mathbf{b}_k\overline{\mathbf{b}}_1\left[\prod_{j=2}^{k-1}\overline{\mathbf{a}_1\left(\prod_{w=1}^{j-1}\mathbf{c}_w\right)\mathbf{b}_j\left(\prod_{w=1}^{j-1}\overline{\mathbf{b}}_w\right)}\right] \quad (\text{ by (6),(18) }) \\ &= \mathbf{a}_1\left(\prod_{w=1}^{k-1}\mathbf{c}_w\right)\mathbf{b}_k\overline{\mathbf{b}}_1\left[\prod_{j=2}^{k-1}\overline{\mathbf{b}_j\left(\prod_{w=1}^{j-1}\overline{\mathbf{b}}_w\right)}\right] \quad (\text{ by (6) }). \end{aligned}$$

By the following formula

$$\overline{\mathbf{b}}_1\left[\prod_{j=2}^{k-1}\overline{\mathbf{b}_j\left(\prod_{w=1}^{j-1}\overline{\mathbf{b}}_w\right)}\right] = \overline{\mathbf{b}}_1\overline{\mathbf{b}_2\overline{\mathbf{b}}_1}\cdots\overline{\mathbf{b}_{k-1}\overline{\mathbf{b}}_1\overline{\mathbf{b}}_2\cdots\overline{\mathbf{b}}_{k-2}} = \prod_{w=1}^{k-1}\overline{\mathbf{b}}_w, \quad (\text{ by (6) })$$

we have

$$\mathcal{E}_{1k} = \mathbf{a}_1\left(\prod_{w=1}^{k-1}\mathbf{c}_w\right)\mathbf{b}_k\left(\prod_{w=1}^{k-1}\overline{\mathbf{b}}_w\right)$$

for $j = k$. Namely, (16) holds by induction.

Similarly, we can prove (17) by the method similar to the proof of (16). □

Furthermore, we define $\mathcal{E}^*_{ij}$ to be the event satisfying

$$(20) \qquad p(\mathcal{E}_{ij}) = \begin{cases} p(\mathbf{a}_i)p(\mathbf{b}_i)p(\mathcal{E}^*_{ii}) & if \quad i=j, \\ p(\mathbf{b}_j)p(\mathcal{E}^*_{ij}) & if \quad i<j, \\ p(\mathbf{a}_i)p(\mathcal{E}^*_{ij}) & if \quad i>j. \end{cases}$$

Clearly, $\mathcal{E}^*_{11}$ is Ω, and by Lemma 10, we have

$$(21) \qquad \mathcal{E}^*_{1j} = \mathbf{a}_1\left(\prod_{w=1}^{j-1}\mathbf{c}_w\right)\left(\prod_{w=1}^{j-1}\overline{\mathbf{b}}_w\right), \quad for\ 2 \le j \le n,$$

$$(22) \qquad \mathcal{E}^*_{i1} = \left(\prod_{w=1}^{i-1}\mathbf{c}_w\right)\mathbf{b}_1\left(\prod_{w=1}^{i-1}\overline{\mathbf{a}}_w\right), \quad for\ 2 \le i \le n.$$

Lemma 11. For a probabilistic one-layered s-t graph $(G = (V, E, s, t), p)$, where V, E satisfy (14), let f^* be an s-t path number function of G satisfying Condition 1. For any i, j, where $2 \le i, j \le n$, we have

$$(23) \qquad \mathcal{E}_{ij} = \begin{cases} \mathbf{a}_i\left(\prod_{w=i}^{j-1}\mathbf{c}_w\right)\mathbf{b}_j\left[\prod_{h=1}^{i-1}\overline{\mathcal{E}^*_{hi}\overline{\mathcal{E}^*_{ih}}}\right]\left(\prod_{w=i}^{j-1}\overline{\mathbf{b}}_w\right) & if \quad i<j, \\ \mathbf{a}_i\left(\prod_{w=j}^{i-1}\mathbf{c}_w\right)\mathbf{b}_j\left[\prod_{h=1}^{j-1}\overline{\mathcal{E}^*_{hj}\overline{\mathcal{E}^*_{jh}}}\right]\left(\prod_{w=j}^{i-1}\overline{\mathbf{a}}_w\right) & if \quad i>j, \\ \mathbf{a}_i\mathbf{b}_i\left[\prod_{h=1}^{i-1}\overline{\mathcal{E}^*_{hi}}\ \overline{\mathcal{E}^*_{ih}}\right] & if \quad i=j \end{cases}$$

and

$$(24) \qquad \mathcal{E}^*_{ij} = \begin{cases} \mathbf{a}_i\left(\prod_{w=i}^{j-1}\mathbf{c}_w\right)\left[\prod_{h=1}^{i-1}\overline{\mathcal{E}^*_{hi}\overline{\mathcal{E}^*_{ih}}}\right]\left(\prod_{w=i}^{j-1}\overline{\mathbf{b}}_w\right) & if \quad i<j, \\ \left(\prod_{w=j}^{i-1}\mathbf{c}_w\right)\mathbf{b}_j\left[\prod_{h=1}^{j-1}\overline{\mathcal{E}^*_{hj}\overline{\mathcal{E}^*_{jh}}}\right]\left(\prod_{w=j}^{i-1}\overline{\mathbf{a}}_w\right) & if \quad i>j, \\ \left[\prod_{h=1}^{i-1}\overline{\mathcal{E}^*_{hi}}\ \overline{\mathcal{E}^*_{ih}}\right] & if \quad i=j. \end{cases}$$

□

The proof of this lemma is lengthy and is shown in appendix.

Lemma 12. The event $\mathcal{E}^*_{i'j'}$, where $1 \le i', j' \le n$, is independent of each of events $\mathbf{a}_i$, $\mathbf{c}_i$ and $\mathbf{b}_i$, respectively, for all $n \ge i \ge \max\{i', j'\}$.

Proof. We prove that the event $\mathcal{E}^*_{i'j'}$ is independent of each of events $\mathbf{a}_i$, $\mathbf{c}_i$ and $\mathbf{b}_i$, respectively, for all $n \ge i \ge \max\{i', j'\}$, by induction with respect to $i' + j'$.

When $i' + j' = 2$, since i', $j' \ge 1$, we have $i' = j' = 1$. As $\mathcal{E}^*_{11} = \Omega$, therefore, $\mathcal{E}^*_{11}$ is independent of each of events $\mathbf{a}_i$, $\mathbf{c}_i$ and $\mathbf{b}_i$ for all $n \ge i \ge \max\{i', j'\} = 1$.

Assume that $\mathcal{E}^*_{i'j'}$ is independent of each of events $\mathbf{a}_i$, $\mathbf{c}_i$ and $\mathbf{b}_i$ for all $n \ge i \ge \max\{i', j'\}$ for $i' + j' < k$, and we consider the case of $i' + j' = k$. Assume that $i' > j'$. By (24),

$$\mathcal{E}^*_{i'j'} = \mathbf{c}_{i'-1}\cdots\mathbf{c}_{j'+1}\mathbf{b}_{j'}\prod_{h=1}^{j'-1}\left(\overline{\mathcal{E}^*_{hj'}}\ \overline{\mathcal{E}^*_{j'h}}\right)\prod_{w=j'}^{i'-1}\overline{\mathbf{a}}_w$$

As $j'-1 \geq h$ and $i' > j'$, we have $j'+h < i'+j' = k$. By the assumption, each of events $\mathbf{a}_i$, $\mathbf{c}_i$ and $\mathbf{b}_i$ for $i \geq \max\{i', j'\}$ is independent of $\prod_{h=1}^{j'-1}\left(\overline{\mathcal{E}^*}_{hj'}\ \overline{\mathcal{E}^*}_{j'h}\right)$, and of $\mathbf{c}_{i'-1}\cdots\mathbf{c}_{j'+1}\mathbf{b}_{j'}\overline{\mathbf{a}}_{j'}\cdots\overline{\mathbf{a}}_{i'-1}$. Hence, $\mathcal{E}^*_{i'j'}$ is independent of each of events $\mathbf{a}_i$, $\mathbf{c}_i$ and $\mathbf{b}_i$ for all $n \geq i \geq \max\{i', j'\}$. Similarly, assuming $i' < j'$ or $i' = j'$, we can also prove this result by the same method. □

Lemma 13. For any i, j', j'', where $1 \leq i, j', j'' \leq n$, $i > j', j''$ and $j' \neq j''$, we have the following facts.

$$\mathcal{E}^*_{ij'}\mathcal{E}^*_{ij''} = \emptyset,\quad \mathcal{E}^*_{j'i}\mathcal{E}^*_{j''i} = \emptyset \quad \text{and} \quad \mathcal{E}^*_{j'i}\mathcal{E}^*_{ij''} = \emptyset.$$

Proof. Without loss of generality, assume that $j' > j''$. As $i > j' > j''$, by (24),

$$\begin{aligned}\mathcal{E}^*_{ij''} &= \mathbf{c}_{i-1}\cdots\mathbf{c}_{j'-1}\left[\mathbf{c}_{j'}\cdots\mathbf{c}_{j''+1}\mathbf{b}_{j''}\prod_{h=1}^{j''-1}\left(\overline{\mathcal{E}^*}_{hj''}\ \overline{\mathcal{E}^*}_{j''h}\right)\overline{\mathbf{a}}_{j''}\cdots\overline{\mathbf{a}}_{j'-1}\right]\overline{\mathbf{a}}_{j'}\cdots\overline{\mathbf{a}}_{i-1}\\ &= \mathbf{c}_{i-1}\cdots\mathbf{c}_{j'-1}\mathcal{E}^*_{j'j''}\overline{\mathbf{a}}_{j'}\cdots\overline{\mathbf{a}}_{i-1} \qquad (\text{ by } (24)\ )\end{aligned}$$

is obtained. As $\mathcal{E}^*_{j'j''}\left[\prod_{h=1}^{j'-1}\overline{\mathcal{E}^*}_{hj'}\overline{\mathcal{E}^*}_{j'h}\right] = \emptyset$ for $j' > j'' > 1$, by (24),

$$\begin{aligned}\mathcal{E}^*_{ij'}\mathcal{E}^*_{ij''} &= \mathbf{a}_i\left(\prod_{w=j'}^{i-1}\mathbf{c}_w\right)\mathbf{b}_{j'}\left[\prod_{h=1}^{j'-1}\overline{\mathcal{E}^*}_{hj'}\overline{\mathcal{E}^*}_{j'h}\right]\left(\prod_{w=j'}^{i-1}\overline{\mathbf{a}}_w\right)\mathbf{c}_{i-1}\cdots\mathbf{c}_{j'-1}\mathcal{E}^*_{j'j''}\overline{\mathbf{a}}_{j'}\cdots\overline{\mathbf{a}}_{i-1}\\ &= \emptyset.\end{aligned}$$

By a similar method, we can also prove that $\mathcal{E}^*_{j'i}\mathcal{E}^*_{j''i} = \emptyset$.

Furthermore, by (24), we have that for $j' > j''$, $\mathbf{b}_{j''}$ appears as a term in $\mathcal{E}^*_{ij''}$ and $\overline{\mathbf{b}}_{j''}$ appears as a term in $\mathcal{E}^*_{j'i}$, and that for $j' < j''$, $\mathbf{a}_{j'}$ appears as a term in $\mathcal{E}^*_{j'i}$ and $\overline{\mathbf{a}}_{j'}$ appears as a term in $\mathcal{E}^*_{ij''}$. Thus, we can show that $\mathcal{E}^*_{j'i}\mathcal{E}^*_{ij''} = \emptyset$. □

By Lemmas 10,11,12,13 and the definition of a probabilistic graph, the probability $p(\mathcal{E}_{ij})$ is computed as follows:

(25) $p(\mathcal{E}_{ij})$

$$= \begin{cases} q(a_1)q(b_1) & if\ \ i=j=1,\\ q(a_1)\Big(\prod_{w=1}^{j-1} q(c_w)\Big)q(b_j)\Big(\prod_{w=1}^{j-1} p(b_w)\Big) & if\ \ n \geq j \geq 2,\\ q(a_i)\Big(\prod_{w=1}^{i-1} q(c_w)\Big)q(b_1)\Big(\prod_{w=1}^{i-1} p(a_w)\Big) & if\ \ n \geq i \geq 2,\\ q(a_i)\Big(\prod_{w=i}^{j-1} q(c_w)\Big)q(b_j)\Big[1-\sum_{h=1}^{i-1}\Big(p(\mathcal{E}^*_{hi}) + p(\mathcal{E}^*_{ih})\Big)\Big]\Big(\prod_{w=i}^{j-1} p(b_w)\Big) & if\ \ 2 \leq i < j \leq n,\\ q(a_i)\Big(\prod_{w=j}^{i-1} q(c_w)\Big)q(b_j)\Big[1-\sum_{h=1}^{j-1}\Big(p(\mathcal{E}^*_{hi}) + p(\mathcal{E}^*_{ih})\Big)\Big]\Big(\prod_{w=j}^{i-1} p(a_w)\Big) & if\ \ n \geq i > j \geq 2,\\ q(a_i)q(b_i)\Big[1-\sum_{h=1}^{i-1}\Big(p(\mathcal{E}^*_{hi}) + p(\mathcal{E}^*_{ih})\Big)\Big] & if\ \ n \geq i = j \geq 2. \end{cases}$$

By (25) and (5), $\underline{\Psi}_{(G,f^*,p)}$ is obtained by computing $p(\mathcal{E}_{ij})$ from $p(\mathcal{E}_{11})$ to $p(\mathcal{E}_{nn})$ for a probabilistic one-layered *s-t* graph (G,p) and the *s-t* path number function f^* of G satisfying Condition 1.

5.2.3 Complexity of Computing $\underline{\Psi}_{(G,f^*,p)}$

For a probabilistic one-layered *s-t* graph (G,p), we analyze the time complexity of computing $\underline{\Psi}_{(G,f^*,p)}$.

By Definition 2, whether a graph $G = (V, E, s, t)$ is a one-layered s-t graph or not can be decided in $O(|V|)$ time. Moreover, note that by (20), $p(\mathcal{E}^*_{ij})$ is obtained in a constant time by the known $p(\mathcal{E}_{ij})$. Therefore, by (25), $p(\mathcal{E}_{ij})$ is computed in $O(n)$ time. As $|P_{st}(G)| \leq n^2$ by Lemma 6, $\Psi_{(G,f^*,p)}$ is computed in $O(n^3)$ time by (5).

Note that for a probabilistic one-layered s-t graph $(G = (V, E, s, t), p)$, where $0 \leq p(e) \leq 1, e \in E$, all of the above results also hold. The fact that $p(e) = 1, e \in E$ signifies that E does not contain the edge e. i.e., that G satisfies (14) imposes no restriction for computing $\underline{\Psi}_{(G,f^*,p)}$ in a probabilistic one-layered s-t graph (G, p). Thus, we immediately obtain the following Theorem 4.

Theorem 4. For a probabilistic one-layered s-t graph $(G = (V, E, s, t), p)$, $\Psi_{(G,p)}(= \underline{\Psi}_{(G,f^*,p)})$ is computed in $O(|V|^3)$ time. □

6. Concluding Remarks

For a probabilistic graph, we proposed a lower bound for estimating the expected maximum number of vertex-disjoint s-t paths. Although this lower bound does not seem to be efficiently computed for a general probabilistic graph, we can efficiently obtain the expected maximum number of vertex-disjoint s-t paths for a probabilistic one-layered s-t graph using this lower bound, as the lower bound is efficiently computed and is equal to the expected maximum number in this case.

Note that all of the above proofs are irrelevant to the direction of edges in a graph. Hence, all of the above obtained results are also valid for a probabilistic directed graph.

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Appendix : Proof of Lemma 11

We prove this lemma by induction with respect to s-t path number $f^*(\pi_{ij})$, where $2 \leq i, j \leq n$.

When $f^*(\pi_{ij}) = 2n$, namely, $i = 2$ and $j = 2$, by (13) and the definitions,

$$Q_{f^*,\pi_{22}} = \{ \pi_{12}, ..., \pi_{1n}, \pi_{21}, ..., \pi_{n1} \}. \tag{26}$$

Thus, we have

$$
\begin{aligned}
&\mathcal{E}_{22} \\
=\ & \prod_{e \in E(\pi_{22})} \mathbf{e} \prod_{\pi_{xy} \in Q_{f^*,\pi_{22}}} \overline{\mathcal{E}}_{xy} \quad (\text{ by } (11)\ ) \\
=\ & \mathbf{a}_2\mathbf{b}_2\left(\prod_{j=2}^{n} \overline{\mathcal{E}}_{1j}\right)\left(\prod_{i=2}^{n} \overline{\mathcal{E}}_{i1}\right) \quad (\text{ by } (12),(26)\ ) \\
=\ & \mathbf{a}_2\mathbf{b}_2\left[\prod_{j=2}^{n} \overline{\mathbf{a}_1\left(\prod_{w=1}^{j-1} \mathbf{c}_w\right)\mathbf{b}_j\left(\prod_{w=1}^{j-1} \overline{\mathbf{b}}_w\right)}\right]\left[\prod_{i=2}^{n} \overline{\mathbf{a}_i\left(\prod_{w=1}^{i-1} \mathbf{c}_w\right)\mathbf{b}_1\left(\prod_{w=1}^{i-1} \overline{\mathbf{a}}_w\right)}\right] \quad (\text{ by } (16),(17)\ ) \\
=\ & \mathbf{a}_2\mathbf{b}_2\left[\ \overline{\mathbf{a}_1 \prod_{w=1}^{2-1} \mathbf{c}_w\ \mathbf{b}_2 \prod_{w=1}^{2-1} \overline{\mathbf{b}}_w}\ \right]\left[\ \overline{\mathbf{a}_2\ \prod_{w=1}^{2-1} \mathbf{c}_w\ \mathbf{b}_1\ \prod_{w=1}^{2-1} \overline{\mathbf{a}}_w}\ \right] \quad (\text{ by } (7)\ ) \\
=\ & \mathbf{a}_2\mathbf{b}_2\ \overline{\mathcal{E}^*}_{12}\overline{\mathcal{E}^*}_{21}, \quad (\text{ by } (21),(22)\ )
\end{aligned}
$$

and (23),(24) hold.

Assume that

$$
(27) \qquad \mathcal{E}_{ij} = \begin{cases} \mathbf{a}_i\left(\prod_{w=i}^{j-1} \mathbf{c}_w\right)\mathbf{b}_j\left[\prod_{h=1}^{i-1} \overline{\mathcal{E}^*}_{hi}\overline{\mathcal{E}^*}_{ih}\right]\left(\prod_{w=i}^{j-1} \overline{\mathbf{b}}_w\right) & if \quad i < j, \\ \mathbf{a}_i\left(\prod_{w=j}^{i-1} \mathbf{c}_w\right)\mathbf{b}_j\left[\prod_{h=1}^{j-1} \overline{\mathcal{E}^*}_{hj}\overline{\mathcal{E}^*}_{jh}\right]\left(\prod_{w=j}^{i-1} \overline{\mathbf{a}}_w\right) & if \quad i > j, \\ \mathbf{a}_i\mathbf{b}_i\left[\prod_{h=1}^{i-1} \overline{\mathcal{E}^*}_{hi}\ \overline{\mathcal{E}^*}_{ih}\right] & if \quad i = j \end{cases}
$$

and

$$
(28) \qquad \mathcal{E}^*_{ij} = \begin{cases} \mathbf{a}_i\left(\prod_{w=i}^{j-1} \mathbf{c}_w\right)\left[\prod_{h=1}^{i-1} \overline{\mathcal{E}^*}_{hi}\ \overline{\mathcal{E}^*}_{ih}\right]\left(\prod_{w=i}^{j-1} \overline{\mathbf{b}}_w\right) & if \quad i < j, \\ \left(\prod_{w=j}^{i-1} \mathbf{c}_w\right)\mathbf{b}_j\left[\prod_{h=1}^{j-1} \overline{\mathcal{E}^*}_{hj}\overline{\mathcal{E}^*}_{jh}\right]\left(\prod_{w=j}^{i-1} \overline{\mathbf{a}}_w\right) & if \quad i > j, \\ \left[\prod_{h=1}^{i-1} \overline{\mathcal{E}^*}_{hi}\ \overline{\mathcal{E}^*}_{ih}\right] & if \quad i = j \end{cases}
$$

for $f^*(\pi_{ij}) < k$, and we consider the case of $f^*(\pi_{ij}) = k$. Assume that $i < j$. By (13) and the definitions,

$$
\begin{aligned}
(29)\ Q_{f^*,\pi_{ij}} = \{ & \pi_{1i},\ \pi_{1i+1}, \dots, \pi_{1n},\ \pi_{2i},\ \pi_{2i+1}, \dots, \pi_{2n},\ \dots\ ,\ \pi_{i-1i},\ \pi_{i-1i+1},\ \dots,\ \pi_{i-1n}, \\
& \pi_{ii},\ \pi_{ii+1},\ \dots,\ \pi_{ij-1}, \\
& \pi_{i1},\ \pi_{i+11},\ \dots,\ \pi_{n1},\ \pi_{i2},\ \pi_{i+12}, \dots, \pi_{n2},\ \dots\ ,\ \pi_{ii-1},\ \pi_{i+1i-1},\ \dots,\ \pi_{ni-1}\ \}.
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
(30)\ \mathcal{E}_{ij} &= \prod_{e \in E(\pi_{ij})} \mathbf{e} \prod_{\pi_{xy} \in Q_{f^*,\pi_{ij}}} \overline{\mathcal{E}}_{xy} \quad (\text{ by } (11)\ ) \\
&= \mathbf{a}_i\left(\prod_{w=i}^{j-1} \mathbf{c}_w\right)\mathbf{b}_j\left[\prod_{h=1}^{i-1}\prod_{g=i}^{n} \overline{\mathcal{E}}_{hg}\right]\left[\prod_{h=1}^{i-1}\prod_{g=i}^{n} \overline{\mathcal{E}}_{gh}\right]\left[\prod_{w=i}^{j-1} \overline{\mathcal{E}}_{iw}\right]. (\text{ by } (12),(29)\ )
\end{aligned}
$$

For any h, g, where $1 \le h \le i-1$ and $j+1 \le g \le n$, since $i < j$, we have $h \le j \le g-1$. Thus,

$$
(31) \qquad \prod_{w=h}^{g-1} \overline{\mathbf{b}}_w = \overline{\mathbf{b}}_h \cdots \overline{\mathbf{b}}_j \cdots \overline{\mathbf{b}}_{g-1}.
$$

Since $\pi_{hg} \subseteq Q_{f^*,\pi_{ij}}$, by this assumption, we have

$$
\begin{aligned}
(32) \qquad \mathbf{b}_j\overline{\mathcal{E}}_{hg} &= \mathbf{b}_j\ \overline{\mathbf{a}_h\left(\prod_{w=h}^{g-1} \mathbf{c}_w\right)\mathbf{b}_g\left[\prod_{h'=h}^{h-1} \overline{\mathcal{E}^*}_{hh'}\overline{\mathcal{E}^*}_{hh'}\right]\ \left(\prod_{w=h}^{g-1} \overline{\mathbf{b}}_w\right)} \quad (\text{ by } (27)\ ) \\
&= \mathbf{b}_j \qquad (\text{ by } (31),(7)\ ).
\end{aligned}
$$

Moreover, by (30),(32),

$$
(33) \qquad \mathcal{E}_{ij} = \mathbf{a}_i\left(\prod_{w=i}^{j-1} \mathbf{c}_w\right)\mathbf{b}_j\left[\prod_{h=1}^{i-1}\prod_{g=i}^{j} \overline{\mathcal{E}}_{hg}\right]\left[\prod_{h=1}^{i-1}\prod_{g=i}^{n} \overline{\mathcal{E}}_{gh}\right]\left[\prod_{w=i}^{j-1} \overline{\mathcal{E}}_{iw}\right].
$$

Note that by iteratively applying $\mathcal{E}_i\overline{\mathcal{E}}_j + \overline{\mathcal{E}}_i\overline{\mathcal{E}}_j = \overline{\mathcal{E}}_j$,

$$\begin{aligned}(34)\qquad \Omega &= \mathbf{b}_i + \mathbf{b}_{i+1}\overline{\mathbf{b}}_i + \cdots + \mathbf{b}_{j-1}\overline{\mathbf{b}}_i\,\overline{\mathbf{b}}_{i+1}\cdots\overline{\mathbf{b}}_{j-2} + \overline{\mathbf{b}}_i\,\overline{\mathbf{b}}_{i+1}\cdots\overline{\mathbf{b}}_{j-2}\,\overline{\mathbf{b}}_{j-1}\\ &= \mathbf{b}_i + \textstyle\sum_{g=i+1}^{j-1}\left[\mathbf{b}_g\left(\prod_{w=i}^{g-1}\overline{\mathbf{b}}_w\right)\right] + \prod_{w=i}^{j-1}\overline{\mathbf{b}}_w.\end{aligned}$$

For $1 \le h \le i-1$, we have

$$\begin{aligned}&\left(\textstyle\prod_{w=i}^{j-1}\mathbf{c}_w\right)\mathbf{b}_j\left[\prod_{g=i}^{j}\overline{\mathcal{E}}_{hg}\right]\\ =\;&\left(\textstyle\prod_{w=i}^{j-1}\mathbf{c}_w\right)\mathbf{b}_j\left[\prod_{g=i}^{j}\overline{\mathbf{a}_h\left(\prod_{w=h}^{g-1}\mathbf{c}_w\right)\mathbf{b}_g\left[\prod_{h'=1}^{h-1}\overline{\mathcal{E}^*}_{hh'}\overline{\mathcal{E}^*}_{h'h}\right]\left(\prod_{w=h}^{g-1}\overline{\mathbf{b}}_w\right)}\right] \quad (\text{ by } (27))\\ =\;&\left(\textstyle\prod_{w=i}^{j-1}\mathbf{c}_w\right)\mathbf{b}_j\;\overline{\mathcal{E}'\left\{\overline{\mathbf{b}}_i + \sum_{g=i+1}^{j-1}\left[\mathbf{b}_g\left(\prod_{w=h}^{g-1}\overline{\mathbf{b}}_w\right)\right] + \left(\prod_{w=i}^{j-1}\overline{\mathbf{b}}_w\right)\right\}} \quad (\text{ by } (6))\\ =\;&\left(\textstyle\prod_{w=i}^{j-1}\mathbf{c}_w\right)\mathbf{b}_j\;\overline{\mathcal{E}'}. \quad (\text{ by } (34))\end{aligned}$$

Here,

$$\mathcal{E}' = \mathbf{a}_h\left(\textstyle\prod_{w=h}^{i-1}\mathbf{c}_w\right)\left[\prod_{h'=1}^{h-1}\overline{\mathcal{E}^*}_{hh'}\overline{\mathcal{E}^*}_{h'h}\right]\left(\prod_{w=h}^{i-1}\overline{\mathbf{b}}_w\right).$$

Furthermore, by the above formulas, we have

$$\begin{aligned}(35)\left(\textstyle\prod_{w=i}^{j-1}\mathbf{c}_w\right)\mathbf{b}_j\left[\prod_{g=i}^{j}\overline{\mathcal{E}}_{hg}\right] &= \left(\textstyle\prod_{w=i}^{j-1}\mathbf{c}_w\right)\mathbf{b}_j\overline{\mathbf{a}_h\left(\prod_{w=h}^{i-1}\mathbf{c}_w\right)\left[\prod_{h'=1}^{h-1}\overline{\mathcal{E}^*}_{hh'}\overline{\mathcal{E}^*}_{h'h}\right]\left(\prod_{w=h}^{i-1}\overline{\mathbf{b}}_w\right)}\\ &= \left(\textstyle\prod_{w=i}^{j-1}\mathbf{c}_w\right)\mathbf{b}_j\overline{\mathcal{E}^*}_{hi} \qquad (\text{ by } (28)).\end{aligned}$$

Moreover, by (33),(35),

$$(36)\qquad \mathcal{E}_{ij} = \mathbf{a}_i\left(\textstyle\prod_{w=i}^{j-1}\mathbf{c}_w\right)\mathbf{b}_j\left[\prod_{h=1}^{i-1}\overline{\mathcal{E}^*}_{hi}\right]\left[\prod_{h=1}^{i-1}\prod_{g=i}^{n}\overline{\mathcal{E}}_{gh}\right]\left[\prod_{w=i}^{j-1}\overline{\mathcal{E}}_{iw}\right].$$

For any h, g, where $i+1 \le g \le n$ and $1 \le h \le i-1$, we have

$$(37)\qquad \textstyle\prod_{w=h}^{g-1}\overline{\mathbf{a}}_w = \overline{\mathbf{a}}_h\cdots\overline{\mathbf{a}}_i\cdots\overline{\mathbf{a}}_{g-1}\;.$$

Thus, since $\pi_{gh} \in Q_{f^\bullet,\pi_{ij}}$, we have

$$\begin{aligned}(38)\qquad &\mathbf{a}_i\left[\textstyle\prod_{g=i+1}^{n}\overline{\mathcal{E}^*}_{gh}\right]\\ =\;&\mathbf{a}_i\left[\textstyle\prod_{g=i+1}^{n}\overline{\mathbf{a}_g\left(\prod_{w=h}^{g-1}\mathbf{c}_w\right)\mathbf{b}_h\left[\prod_{h'=1}^{h-1}\overline{\mathcal{E}^*}_{hh'}\overline{\mathcal{E}^*}_{h'h}\right]\left(\prod_{w=h}^{g-1}\overline{\mathbf{a}}_w\right)}\right] \quad (\text{ by } (27))\\ =\;&\mathbf{a}_i \qquad (\text{ by } (37),(7)).\end{aligned}$$

Moreover, by (36),(38),

$$(39)\qquad \mathcal{E}_{ij} = \mathbf{a}_i\left(\textstyle\prod_{w=i}^{j-1}\mathbf{c}_w\right)\mathbf{b}_j\left[\prod_{h=1}^{i-1}\overline{\mathcal{E}^*}_{hi}\right]\left[\prod_{h=1}^{i-1}\overline{\mathcal{E}^*}_{ih}\right]\left[\prod_{w=i}^{j-1}\overline{\mathcal{E}}_{iw}\right].$$

By the following formula

$$\begin{aligned}(40)\qquad \textstyle\prod_{w=i}^{j-1}\overline{\mathbf{b}}_w &= \overline{\mathbf{b}}_i\,\overline{\mathbf{b}}_{i+1}\cdots\overline{\mathbf{b}}_{j-1}\\ &= \overline{\mathbf{b}}_i\,\overline{\mathbf{b}_{i+1}\overline{\mathbf{b}}_i}\cdots\overline{\mathbf{b}_{j-1}\overline{\mathbf{b}}_i\,\overline{\mathbf{b}}_{i+1}\cdots\overline{\mathbf{b}}_{j-2}} \qquad (\text{ by } (6))\\ &= \overline{\mathbf{b}}_i\left[\textstyle\prod_{w=i+1}^{j-1}\overline{\mathbf{b}_w\prod_{w'=i}^{w-1}\overline{\mathbf{b}}_{w'}}\;\right],\end{aligned}$$

we have

$$
\begin{aligned}
(41) \qquad & \mathbf{a}_i\left(\prod_{w=i}^{j-1}\mathbf{c}_w\right)\left[\prod_{h=1}^{i-1}\overline{\mathcal{E}^*}_{hi}\ \overline{\mathcal{E}^*}_{ih}\right]\left[\prod_{w=i}^{j-1}\overline{\mathcal{E}}_{iw}\right] \\
= \ & \mathbf{a}_i\left(\prod_{w=i}^{j-1}\mathbf{c}_w\right)\left[\prod_{h=1}^{i-1}\overline{\mathcal{E}^*}_{hi}\ \overline{\mathcal{E}^*}_{ih}\right]\overline{\mathbf{a}_i\mathbf{b}_i\left[\prod_{h=1}^{i-1}\overline{\mathcal{E}^*}_{hi}\ \overline{\mathcal{E}^*}_{ih}\right]} \\
& \left[\prod_{w=i+1}^{j-1}\overline{\mathbf{a}_i\left(\prod_{w'=i}^{w-1}\mathbf{c}_{w'}\right)\mathbf{b}_w\left[\prod_{h=1}^{i-1}\overline{\mathcal{E}^*}_{hi}\ \overline{\mathcal{E}^*}_{ih}\right]\left(\prod_{w'=i}^{w-1}\overline{\mathbf{b}}_{w'}\right)}\right] \quad (\text{ by } (27)\ ) \\
= \ & \mathbf{a}_i\left(\prod_{w=i}^{j-1}\mathbf{c}_w\right)\left[\prod_{h=1}^{i-1}\overline{\mathcal{E}^*}_{hi}\ \overline{\mathcal{E}^*}_{ih}\right]\overline{\mathbf{b}}_i\left[\prod_{w=i+1}^{j-1}\overline{\mathbf{b}_w\left(\prod_{w'=i}^{w-1}\overline{\mathbf{b}}_{w'}\right)}\right] \quad (\text{ by } (6)\ ) \\
= \ & \mathbf{a}_i\left(\prod_{w=i}^{j-1}\mathbf{c}_w\right)\left[\prod_{h=1}^{i-1}\overline{\mathcal{E}^*}_{hi}\ \overline{\mathcal{E}^*}_{ih}\right]\left(\prod_{w=i}^{j-1}\overline{\mathbf{b}}_w\right). \qquad (\text{ by } (40)\ )
\end{aligned}
$$

Moreover, by (39),(41), we have

$$\mathcal{E}_{ij} \ = \ \mathbf{a}_i\left(\prod_{w=i}^{j-1}\mathbf{c}_w\right)\mathbf{b}_j\left[\prod_{h=1}^{i-1}\overline{\mathcal{E}^*}_{hi}\overline{\mathcal{E}^*}_{ih}\right]\left(\prod_{w=i}^{j-1}\overline{\mathbf{b}}_w\right)$$

and

$$\mathcal{E}^*_{ij} \ = \ \mathbf{a}_i\left(\prod_{w=i}^{j-1}\mathbf{c}_w\right)\left[\prod_{h=1}^{i-1}\overline{\mathcal{E}^*}_{hi}\overline{\mathcal{E}^*}_{ih}\right]\left(\prod_{w=i}^{j-1}\overline{\mathbf{b}}_w\right).$$

Namely, (23),(24) hold for $f^*(\pi_{ij}) = k$ where $i < j$.

Similarly, by assuming that $i > j$ or $i = j$, we can prove (23),(24) for $f^*(\pi_{ij}) = k$ by the same method. Thus we have proved Lemma 11 by induction. □

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