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AN OPTIMAL SEARCH FOR A DISAPPEARING TARGET WITH A RANDOM LIFETIME

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Abstract We consider an optimal search for a disappearing target with a random lifetime. A stationary target hides in some boxes at the beginning of the search and disappears at some time which is a random variable. A searcher can not detect the target after it disappears. The searcher is given some reward by the detection of the target. He is permitted to use some searching effort with cost for the detection. This paper deals with an optimal distribution of searching effort and an optimal stopping time minimizing the expected risk (the expected cost minus the expected reward) and the conditions of the optimality are obtained. Numerical examples are examined and the relations between this paper and previous studies are discussed, too.

1. Introduction

In this paper, we deal with a search for a disappearing target with a random lifetime. Let us consider a target such as a hostile submarine in the military search operation. The target does not leave any sign in the target space when he goes away from the space and then the searcher cannot obtain any information about the target after the disappearing time of the target (called the lifetime). Some searching effort are available for the searcher with the expenditure of cost. The detection of the target gives the searcher some reward. In this paper, the optimal search plan minimizing the expected risk of the search, which is defined as the expected searching cost minus the expected reward, is investigated.

There are a few studies on the optimal search plan for other type of the target, that is a mortal target such as a lost party in a mountain. The mortal target usually leaves some sign, e.g. his dead body, in the target space. Therefore, the searcher can detect the sign of the target after his lifetime but could never obtain the reward by the detection. Stone[4] gave the necessary and sufficient conditions for a survivor search plan to maximize the probability of detecting the target alive. Nakai[3] investigated an optimal search plan maximizing the expected return for the mortal target. Assuming the homogeneity of the searching cost in both the target space and the time space, he derived conditions for the optimal allocation of the searching effort and the optimal stopping time of the search. Iida[1] studied a whereabouts search for the mortal target and derived conditions for the optimal whereabouts search plan which minimizes the expected risk. However, no study on the optimal search for the disappearing target has been published until now.

In the next section, we describe assumptions of our model and define system parameters. The problem of the searching for the disappearing target is formulated as a variational problem and the optimal search plan is derived in Section 3. Numerical examples are presented in Section 4 and the properties of the optimal search plan and relations between our model and the previous studies are discussed in Section 5.

2. Assumptions of the Model

The model dealt with in this paper is described in detail as follows.

(i) A target is known to appear in one of n boxes with a probability distribution $\{p_i, i = 1, 2, \cdots, n\}$, where p_i is the probability of the target's appearing in box i and $p_i > 0, i = 1, 2, \cdots, n, \sum_i p_i = 1$.

(ii) The target is assumed to be a disappearing target and he does not leave any sign in the target space after he disappears, and once the target has left, he never comes back to the target space. Therefore, the searcher cannot get any information about the target by searching after the lifetime of the target.

(iii) The lifetime of the target in box i is assumed to be a random variable with a defective probability distribution function $F_i(t)$ which is differentiable with respect to t in $(0, \infty)$ and has a probability mass β_i at $t = 0$. β_i is the probability that the target in box i has disappeared before the start of the search. Let γ_i be the probability that the target stays in box i forever. The derivative of $F_i(t)$ is denoted by $f_i(t)$. Hence, we have

$$F_i(t) = \beta_i + \int_0^t f_i(s)ds,$$

$$\beta_i + \int_0^\infty f_i(s)ds + \gamma_i = 1.$$

In the following discussions, we sometimes assume the concavity of $F_i(t)$, which means the disappearing probability rate is decreasing with time.

(iv) The searcher obtains a reward R_i when he detects the target in box i.

(v) The conditional detection probability $b(i, \psi)$, given that the target is in box i and searching effort ψ is allocated in the box, is assumed to be an exponential detection function,

$$b(i, \psi) = 1 - \exp(-\alpha_i \psi).$$

This assumption of the detection function implies that the random search[1] is conducted in box i.

(vi) The search in box i costs $c_i > 0$ per unit density of searching effort.

(vii) The search is carried out continuously in time within a total searching cost rate $C > 0$ which is available to the searcher per unit time and is presented in advance of the search. Here, we assume that C can be divided arbitrarily and be allocated to any boxes.

(viii) The density of searching effort allocated to box i at t is denoted by $\phi(i, t)$ and the cumulative searching effort from time 0 to t is by $\psi(i, t)$. Then obviously, $\phi(i, t) = \partial\psi(i, t)/\partial t$ holds for almost everywhere t and the derivative at the external points, that is, the starting time or the stopping time of the search, is defined by the right or the left derivative, respectively. If the search is continued without detection, the searcher is assumed to be able to stop his search whenever he wants. The stopping time of the search is denoted by $T \geq 0$. A set $\phi_T = \{\phi(i, t), i = 1, 2, \cdots, n, \ t \in [0, T]\}$ is called the search plan, which specifies both the distribution of searching effort in $[0, T]$ and the stopping time of the search.

(ix) The system parameters of the search, $c_i, C, f_i(t), p_i, \alpha_i, \beta_i$ and γ_i mentioned above, are assumed to be known to the searcher precisely before the start of the search.

(x) We assume that the searcher wishes to minimize the expected risk of the search. The definition of the expected risk is the expected searching cost imposed by searching until the detection or the stopping, whichever comes first, minus the expected reward obtained by the detection of the target. The search plan which minimizes the expected risk is called the optimal search plan and is denoted by

$$\phi^*_{T^*} = \{\phi^*(i,t),\ i = 1,2,\cdots,n,\ t \in [0,T^*]\}.$$

3. Optimal Search Plan

3.1 Formulation of the model

Suppose a search plan ϕ_T is employed. The expected risk $G(\phi_T, T)$ for the search plan ϕ_T is obtained as follows.

$$(3.1)\quad G(\phi_T,T) = \sum_i p_i \left[\int_0^T \alpha_i\phi(i,t)(1-F_i(t))\exp(-\alpha_i\psi(i,t))\left\{\sum_i c_i\psi(i,t) - R_i\right\}dt + \left\{1 - \int_0^T \alpha_i\phi(i,t)(1-F_i(t))\exp(-\alpha_i\psi(i,t))dt\right\}\sum_i c_i\psi(i,T)\right].$$

The first term in the bracket of the right-hand side of Eq.(3.1) is the conditional expected risk given that the target is in box i and is detected at some time in $[0,T]$. The second term is the conditional expected risk when the target is not detected by T. Since the sum of these two terms is the conditional expected risk given the target being in box i, $G(\phi_T,T)$ is obtained by the sum of the conditional expected risk weighted by p_i.

The total searching cost rate is limited within C, $\sum_i c_i\phi(i,t) \le C$, for $t \in [0,T]$ as mentioned in Assumption (vii). A set of non-negative $\phi(i,t)$ satisfying this equality is called a feasible allocation of searching effort and denoted by Ψ. However in our model, the target is assumed to disappear with the passing of time and to be stationary when he stays in the target space. Furthermore, the reward and the searching cost are assumed to be constant. In this case, the available searching cost C should be used up exhaustively in the optimal search plan. Then, we can omit the inequality sign in this restriction without any change on the optimal allocation and we have $\sum_i c_i\phi(i,t) = C$. Substituting this relation into Eq.(3.1) and integrating by parts, we obtain the next simplified expression of the expected risk.

$$\begin{aligned}(3.2)\quad G(\phi_T,T) &= CT - \sum_i p_i\left[\int_0^T \alpha_i\phi(i,t)(1-F_i(t))\exp(-\alpha_i\psi(i,t))\{R_i + C(T-t)\}dt\right] \\ &= \sum_i p_i\left[R_i(1-F_i(T))\exp(-\alpha_i\psi(i,T)) - R_i(1-\beta_i) + CT\beta_i \right.\\ &\quad \left. + \int_0^T \{C(1-F_i(t)) + f_i(t)(R_i + C(T-t))\}\exp(-\alpha_i\psi(i,t))dt\right].\end{aligned}$$

The following restrictions on the allocation of the searching effort $\phi(i,t)$ are imposed.

$$(3.3)\quad \begin{array}{l}\sum_i c_i\phi(i,t) = C \ \ for\ \ t \in [0,T], \\ \phi(i,t) \ge 0 \ \ for\ \ i = 1,\cdots,n,\ \ t \in [0,T].\end{array}$$

A set of the search plan $\phi_T = \{\phi(i,t),\ i = 1, 2, \cdots, n,\ t \in [0,T]\}$ which satisfies the above restrictions is denoted as Ψ_0. Our problem is formulated as a variational problem to find an optimal ϕ_T which minimizes the functional $G(\phi_T, T)$ defined by Eq.(3.2) subject to the restrictions given by Eqs.(3.3).

We derive conditions for ϕ_T to be optimal by two steps in the following section. First, by conditioning T, we obtain the effort allocation ϕ_T which minimizes the expected risk $G(\phi_T, T)$. This allocation is called the conditionally optimal search plan given T and denoted by $\phi_T^* = \{\phi^*(i,t),\ t \in [0,T]\}$.

$$G(\phi_T^*, T) = \min_{\phi_T} G(\phi_T, T).$$

In the second stage of our optimization, T is considered as a variable in $[0,\infty)$, and applying the conditionally optimal search plan by T, we choose the stopping time T so as to minimize $G(\phi_T^*, T)$. The stopping time T which minimizes $G(\phi_T^*, T)$ with respect to T is defined as the optimal stopping time of the search and is denoted by T^*.

$$G(\phi_{T^*}^*, T^*) = \min_{T} G(\phi_T^*, T).$$

Then the pair $(\phi_{T^*}^*, T^*)$ is the optimal plan.

3.2 Optimal allocation of searching effort

In this section, we derive conditions for the searching effort allocation ϕ_T to be conditionally optimal by applying the theorems given by Stromquist and Stone[5]. Their model is as follows.

The target space $\boldsymbol{Y}$ and the time space $\boldsymbol{T}$ are assumed to be spaces with δ-finite measure ν and τ, respectively. Let μ be the product measure on $\boldsymbol{Y} \times \boldsymbol{T}$ and $\boldsymbol{Z}$ be a μ-measurable subset of $\boldsymbol{Y} \times \boldsymbol{T}$. A t-section of $\boldsymbol{Z}$ is denoted by $\boldsymbol{Z}_t = \{y \in \boldsymbol{Y} \mid (y,t) \in \boldsymbol{Z}\}$. The cost rate $c:\ \boldsymbol{Z} \to (0,\infty)$ is τ-measurable, and Ψ is a set of τ-measurable functions $\phi : \boldsymbol{Z} \to [0,\infty)$ such that

$$\int_{Z_t} c(y,t)\phi(y,t)d\nu(y) \le m(t)$$

for a.e. $t \in \boldsymbol{T}$. Stromquist and Stone considered a maximization problem of a real-valued functional P defined on ϕ. Here, the Gateaux differential $\delta P(\phi, h)$ and its kernel $d(\phi, y, t)$ are defined as

$$(3.4)\qquad \begin{aligned} \delta P(\phi,h) &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\{P(\phi+\varepsilon h) - P(\phi)\} \\ &= \int_Z d(\phi,y,t)h(y,t)d\mu(y,t) \end{aligned}$$

for every h and $\phi \in \Psi$, where h is a μ-measurable function $h : \boldsymbol{Z} \to (-\infty,\infty)$ such that $\phi + \varepsilon h \in \Psi$ for all sufficiently small and positive ε.

The necessary and sufficient conditions for ϕ^* to be optimal are given by Theorems 1 and 2 in Stromquist and Stone[5] as follows.

Lemma 1. *Assume that $P(\phi)$ has a Gateaux differential at $\phi^* \in \Psi$ with a kernel $d(\phi^*, y, t)$. Then a necessary condition for ϕ^* to maximize $P(\phi)$ is that there exists a measurable function $\lambda : \boldsymbol{T} \to (-\infty,\infty)$ such that for a.e. (y,t),*

$$(3.5)\qquad \begin{aligned} &d(\phi^*,y,t) = \lambda(t)c(y,t), \quad \text{if } \phi^*(y,t) > 0, \\ &d(\phi^*,y,t) \le \lambda(t)c(y,t), \quad \text{if } \phi^*(y,t) = 0. \end{aligned}$$

In addition to the hypotheses of the necessary condition, if $P(\phi)$ is concave (in their maximization problem), then the necessary conditions Eq.(3.5) are also sufficient for ϕ^* to be optimal.

Our problem mentioned before is a special case of the Stromquist-Stone model. In our model, ν is a counting measure on $\boldsymbol{Y} = \{1, 2, \cdots, n\}$ and τ is a Lebesgue measure. Then if we set $P(\phi) = -G(\phi, T)$ given by Eq.(3.2), $c(y,t) = c_y$ and $m(t) = C$, we can apply Lemma 1 to our problem. However, we need the next lemma to derive theorems.

Lemma 2. *The expected risk $G(\phi, T)$ given by Eq.(3.2) is strictly convex with respect to ϕ.*

Proof: Let ϕ^1 and ϕ^2 be arbitrary effort allocations which satisfy the restrictions Eq.(3.3) and let $\phi^0 = (1-\theta)\phi^1 + \theta\phi^2$, $0 \le \theta \le 1$. Then, substituting ϕ^0 into Eq.(3.2) and using the property of the convexity of the negative exponential function, we can derive easily the next relations.

$$\phi^0 \in \Psi_0,$$
$$G(\phi^0, T) \le (1-\theta)G(\phi^1, T) + \theta G(\phi^2, T). \quad \square$$

The following theorem is obtained by applying Lemma 1 to our model.

Theorem 1. *A necessary and sufficient condition for a conditionally optimal allocation ϕ_T^* of searching effort is that there exists a function $\lambda(t)$ such that for all i and a.e. t,*

$$\text{(3.6)} \qquad \begin{array}{ll} \text{if } \phi^*(i,t) > 0, & A_{it}^T(\phi_T^*) = \lambda(t), \\ \text{if } \phi^*(i,t) = 0, & A_{it}^T(\phi_T^*) \le \lambda(t), \end{array}$$

where

$$A_{it}^T(\phi_T^*) = \frac{\alpha_i p_i}{c_i}\left[(1 - F_i(T))R_i \exp(-\alpha_i \psi^*(i,T)) + H_i(t, T, \phi_T^*)\right],$$
$$H_i(t, T, \phi_T^*) = \int_t^T \{C(1 - F_i(x)) + f_i(x)(R_i + C(T - x))\} \exp(-\alpha_i \psi^*(i,x))dx.$$

Proof: The functional $G(\phi_T, T)$ defined by Eq.(3.2) has a Gateaux differential at ϕ_T to any direction h on $\phi_T + \varepsilon h \in \Psi$ for $\varepsilon \ll 1$. By the definition of the Gateaux differential Eq.(3.4), we obtain

$$\begin{aligned} \delta G(\phi_T, T) \;=\; & -\sum_i p_i \left[\int_0^T (1 - F_i(t))(R_i + C(T-t))\alpha_i \exp(-\alpha_i \psi(i,t)) \right. \\ & \left. \times \left\{h(i,t) - \alpha_i \phi(i,t) \int_0^t h(i,x)dx\right\} dt \right]. \end{aligned}$$

By interchanging the order of the double integrals and integrating by parts, we have

$$\delta G(\phi_T, T) = -\sum_i \int_0^T h(i,t)\alpha_i p_i \left[(1 - F_i(T))R_i \exp(-\alpha_i \psi(i,T)) + H_i(t, T, \phi_T)\right] dt.$$

Therefore, the kernel of the Gateaux differential $\delta G(\phi, T)$ defined by Eq.(3.4), is obtained as follows.

$$\text{(3.7)} \qquad d(\phi_T, i, t) = -\alpha_i p_i \left[(1 - F_i(T))R_i \exp(-\alpha_i \psi(i,T)) + H_i(t, T, \phi_T)\right].$$

Here, we set $P(\phi_T) = -G(\phi_T, T)$ in Lemma 1. The convexity and the minimization of $G(\phi_T, T)$ correspond to the concavity and the maximization of $P(\phi_T)$, respectively. By applying Lemma 1 to our model, we have Eq.(3.6) as the necessary and sufficient condition for ϕ_T^* to be optimal. □

From Theorem 1, several properties of the optimal allocation of searching effort are elucidated. The next corollary is derived directly by setting $t = T$ in Theorem 1.

Corollary 1.

$$\text{(3.8)} \quad \begin{aligned} &If \;\; \phi^*(i,T) > 0, \quad \frac{\alpha_i p_i}{c_i}(1 - F_i(T))R_i \exp(-\alpha_i \psi^*(i,T)) = \lambda(T), \\ &if \;\; \phi^*(i,T) = 0, \quad \frac{\alpha_i p_i}{c_i}(1 - F_i(T))R_i \exp(-\alpha_i \psi^*(i,T)) \le \lambda(T). \end{aligned}$$

Corollary 2. *If both $\phi^*(i,t) > 0$ and $\phi^*(j,t) > 0$ in $[t_1, t_2]$, then at any time t in $[t_1, t_2]$, we have*

$$\text{(3.9)} \quad \begin{aligned} &\frac{\alpha_i p_i}{c_i}[C(1 - F_i(t)) + f_i(t)(R_i + C(T - t))]\exp(-\alpha_i \psi^*(i,t)) \\ &= \frac{\alpha_j p_j}{c_j}[C(1 - F_j(t)) + f_j(t)(R_j + C(T - t))]\exp(-\alpha_j \psi^*(j,t)). \end{aligned}$$

Proof: From Theorem 1, we obtain

$$\begin{aligned} \lambda(t_1) - \lambda(t) &= \frac{\alpha_i p_i}{c_i}\int_{t_1}^{t}[C(1 - F_i(x)) + f_i(x)(R_i + C(T - x))]\exp(-\alpha_i \psi^*(i,x))dx \\ &= \frac{\alpha_j p_j}{c_j}\int_{t_1}^{t}[C(1 - F_j(x)) + f_j(x)(R_j + C(T - x))]\exp(-\alpha_j \psi^*(i,x))dx \end{aligned}$$

for $t_1 \le t \le t_2$. Since the above relation holds at any t in $[t_1, t_2]$, we can conclude Eq.(3.9). □

The value of the both sides of Eq.(3.9) means $-d\lambda(t)/dt$ as easily confirmed by differentiating Eq.(3.6) by t. As discussed later in Section 5, the physical meaning of $-\lambda(t)$ is the expected marginal risk versus cost ratio when a unit search effort is allocated to box i at t. Therefore, the value $-d\lambda(t)/dt$ means the increasing rate of $-\lambda(t)$, and in the optimal search, this rate is balanced each other among the boxes being searched at that time.

The next theorem states properties of $\lambda(t)$ in Theorem 1.

Theorem 2. *$\lambda(t)$ is a continuous and strictly decreasing function of t, and if $F_i(t)$ is concave, $\lambda(t)$ is a strictly convex function.*

Proof: Suppose $0 \le t_1 < t_2 \le T$. We can always find the boxes, say boxes i and j, such that $\phi^*(i, t_1) > 0$, and $\phi^*(j, t_2) > 0$. Applying Theorem 1 at t_1 and t_2, we obtain

$$\frac{\alpha_j p_j}{c_j}H_j(t_1, t_2, \phi^*) \le \lambda(t_1) - \lambda(t_2) \le \frac{\alpha_i p_i}{c_i}H_i(t_1, t_2, \phi^*).$$

Hence, we can conclude $\lambda(t_1) - \lambda(t_2) > 0$ for $t_1 < t_2$, and in the limit when $(t_2 - t_1)$ approaches zero, $\lambda(t_1) = \lambda(t_2)$. Therefore, $\lambda(t)$ is a strictly decreasing and continuous function of t.

The proof of the convexity of $\lambda(t)$. We consider t such that

$$t = (1 - \theta)t_1 + \theta t_2, \quad 0 \le \theta \le 1, \ 0 \le t_1 < t_2 \le T.$$

Then there always exists a box, say box i, such that $\phi^*(i,t) > 0$. Then from Theorem 1, we obtain

$$\begin{aligned}
\lambda(t) &= \frac{\alpha_i p_i}{c_i}[(1-F_i(T))R_i\exp(-\alpha_i\psi^*(i,T)) + H_i(t,T,\phi^*)] \\
&= (1-\theta)\frac{\alpha_i p_i}{c_i}[(1-F_i(T))R_i\exp(-\alpha_i\psi^*(i,T)) + H_i(t_1,T,\phi^*)] \\
&\quad +\theta\frac{\alpha_i p_i}{c_i}[(1-F_i(T))R_i\exp(-\alpha_i\psi^*(i,T)) + H_i(t_2,T,\phi^*)] \\
&\quad +\frac{\alpha_i p_i}{c_i}[\theta H_i(t,t_2,\phi^*) - (1-\theta)H_i(t_1,t,\phi^*)] \\
&\leq (1-\theta)\lambda(t_1) + \theta\lambda(t_2) \\
&\quad +\frac{\alpha_i p_i}{c_i}[\theta H_i(t,t_2,\phi^*) - (1-\theta)H_i(t_1,t,\phi^*)].
\end{aligned}$$

If $F_i(t)$ is concave, the integrand of the last term is a non-negative and decreasing function of t. Using this property, the last term of the above inequality is easily proved to be non-positive. Therefore, we have

$$\lambda(t) \leq (1-\theta)\lambda(t_1) + \theta\lambda(t_2),$$

where the equality sign holds only when $\theta = 0$ or 1. Thus $\lambda(t)$ is proved to be a strictly convex function of t if $F_i(t)$ is concave. □

Let T_i^s and T_i^e be the starting time and the stopping time of the optimal search in box i, respectively. Namely,

$$T_i^s = \inf\{T_i \mid \phi^*(i,T_i) > 0\},$$
$$T_i^e = \sup\{T_i \mid \phi^*(i,T_i) > 0\}.$$

Then the next theorem holds.

Theorem 3. *Suppose $F_i(t)$ is concave and both $\psi^*(i,T)$ and $\psi^*(j,T)$ are positive (namely both box i and j are boxes to be searched by T).*
1. If and only if the next inequality

$$\text{(3.10)}\quad \begin{aligned}&\frac{\alpha_i p_i}{c_i}[C(1-F_i(T_i^s)) + f_i(T_i^s)(R_i + C(T-T_i^s))] \\ &\geq \frac{\alpha_j p_j}{c_j}\left[C(1-F_j(T_j^s)) + f_j(T_j^s)(R_j + C(T-T_j^s))\right]\end{aligned}$$

holds, then $T_i^s \leq T_j^s$.
2. If and only if the next inequality

$$\text{(3.11)}\quad \begin{aligned}&\frac{\alpha_i p_i \exp(-\alpha_i\psi^*(i,T_i^e))}{c_i}[C(1-F_i(T_i^e)) + f_i(T_i^e)(R_i + C(T-T_i^e))] \\ &\leq \frac{\alpha_j p_j \exp(-\alpha_j\psi^*(j,T_j^e))}{c_j}\left[C(1-F_j(T_j^e)) + f_j(T_j^e)(R_j + C(T-T_j^e))\right]\end{aligned}$$

holds, then $T_i^e \geq T_j^e$.

Proof: 1. Applying Theorem 1 to box i and j at T_i^s and T_j^s, we obtain

$$\text{(3.12)}\quad \begin{aligned}&\frac{\alpha_i p_i}{c_i}\int_{T_j^s}^{T_i^s}[C(1-F_i(x)) + f_i(x)(R_i + C(T-x))]\exp(-\alpha_i\psi^*(i,x))dx \\ &\leq \frac{\alpha_j p_j}{c_j}\int_{T_j^s}^{T_i^s}[C(1-F_j(x)) + f_j(x)(R_j + C(T-x))]\exp(-\alpha_j\psi^*(j,x))dx.\end{aligned}$$

Here, we assume Inequality (3.10) and $T_i^s > T_j^s$. Noting $\psi^*(i,t) = 0$ and $\psi^*(j,t) > 0$ in any $t \in [T_j^s, T_i^s)$ and using the above relation, we obtain

$$(3.13)\quad \frac{\alpha_i p_i}{c_i}\int_{T_j^s}^{T_i^s} [C(1-F_i(x)) + f_i(x)(R_i + C(T-x))]\,dx$$
$$< \frac{\alpha_j p_j}{c_j}\int_{T_j^s}^{T_i^s} [C(1-F_j(x)) + f_j(x)(R_j + C(T-x))]\,dx.$$

Since the integrands in the both sides of Eq.(3.13) are decreasing functions by the characteristics of $F.(t)$, we obtain the next inequality.

$$\frac{\alpha_i p_i}{c_i}[C(1-F_i(T_i^s)) + f_i(T_i^s)(R_i + C(T-T_i^s))]$$
$$< \frac{\alpha_j p_j}{c_j}\left[C(1-F_j(T_j^s)) + f_j(T_j^s)(R_j + C(T-T_j^s))\right].$$

This relation contradicts to the assumption, and therefore, we can conclude $T_i^s \le T_j^s$.

The proof of sufficiency. If $T_i^s \le T_j^s$, then $\psi^*(i,t) > 0$ and $\psi^*(j,t) = 0$ in $t \in [T_i^s, T_j^s)$, and therefore, Inequality (3.10) is derived from Eq.(3.12).

2. We assume Inequality (3.11) and $T_i^e < T_j^e$. Then $\psi^*(i,t) = \psi^*(i,T_i^e)$ and $\psi^*(j,t) < \psi^*(j,T_j^e)$ in any $t \in [T_i^e, T_j^e)$, and we have

$$\frac{\alpha_i p_i \exp(-\alpha_i \psi^*(i,T_i^e))}{c_i}[C(1-F_i(T_i^e)) + f_i(T_i^e)(R_i + C(T-T_i^e))]$$
$$> \frac{\alpha_j p_j \exp(-\alpha_j \psi^*(j,T_j^e))}{c_j}\left[C(1-F_j(T_j^e)) + f_j(T_j^e)(R_j + C(T-T_j^e))\right].$$

This contradicts Inequality (3.11) .

The proof of sufficiency. Since both $\psi^*(i,T_i^e)$ and $\psi^*(j,T_j^e)$ are positive, we obtain the next relation from Theorem 1.

If $T_i^e \ge T_j^e$, $\psi^*(i,t) \le \psi^*(i,T_i^e)$ and $\psi^*(j,t) = \psi^*(j,T_j^e)$ in any $t \in [T_j^e, T_i^e)$,

$$(3.14)\quad \frac{\alpha_i p_i}{c_i}\int_{T_j^e}^{T_i^e} [C(1-F_i(x)) + f_i(x)(R_i + C(T-x))]\exp(-\alpha_i\psi^*(i,x))dx$$
$$\le \frac{\alpha_j p_j}{c_j}\int_{T_j^e}^{T_i^e} [C(1-F_j(x)) + f_j(x)(R_j + C(T-x))]\exp(-\alpha_j\psi^*(j,x))dx.$$

Since the integrands in the both sides of Eq.(3.14) are decreasing functions with respect to x, we can derive easily the relation (3.11). □

3.3 Optimal stopping time of the search

In this section, employing the conditionally optimal search plan ϕ_T^* derived in the previous section, we investigate the optimal stopping time T^* which minimizes the expected risk $G(\phi_T^*, T)$ with respect to T. From now on, we deal with the stopping time T as a variable. Let $p_i(T)$ be the (defective) posterior probability of the target in box i given that the conditionally optimal search plan $\phi_T^* = \{\phi^*(i,t),\ i = 1,\cdots,n,\ t \in [0,T]\}$ fails to detect the target by T.

$$(3.15)\quad \{p_i(T)\} = \left\{\frac{(1-F_i(T))p_i\exp(-\alpha_i\psi^*(i,T))}{Q(T)}\right\},$$

where $Q(T)$ is the non-detection probability of the target given by

(3.16) $Q(T) = 1 - \sum_i p_i \int_0^T (1 - F_i(t))\alpha_i\phi^*(i,t)\exp(-\alpha_i\psi^*(i,t))dt.$

Theorem 4. *A necessary condition for the optimal stopping time of the search, say T^*, is*

(3.17) $\lambda(T^*) = Q(T^*),$

where $\lambda(T^*)$ is given by Corollary 1.

Proof: Since the distribution function $F_i(t)$ of the lifetime of the target and the cumulative searching effort ψ_T^* is differentiable with respect to T by the assumptions of the model, the risk function $G(\phi_T^*, T)$ given by Eq.(3.2) is also differentiable. Let $\delta G(\phi_T^*, T)$ and $\{\delta\phi^*(i,t)\}$ be variations of the expected risk $G(\phi_T^*, T)$ and the searching effort $\{\phi^*(i,t)\}$, respectively, by prolonging the stopping time T to $T + \delta T$. From Eq.(3.2), we have

$$(3.18)\quad \delta G(\phi_T^*, T) = CQ(T)\delta T - \sum_i \alpha_i p_i (1 - F_i(T))R_i\phi^*(i,T)\exp(-\alpha_i\psi^*(i,T))\delta T - \sum_i c_i \int_0^T A_{it}^T(\phi_T^*)\delta\phi^*(i,t)dt.$$

where $A_{it}^T(\phi_T^*)$ is given by the definition in Theorem 1. According to Corollary 1, the value $\alpha_i p_i(1 - F_i(T))R_i \exp(-\alpha_i\psi^*(i,T))/c_i$ equals to the constant $\lambda(T)$ for all boxes such that $\phi^*(i,T) > 0$. Therefore, the second term of the right-hand side of Eq.(3.18) is rewritten as follows.

$$-\sum_i \alpha_i p_i (1 - F_i(T))R_i\phi^*(i,T)\exp(-\alpha_i\psi^*(i,T))\delta T$$
$$= -\delta T \frac{\alpha_i p_i (1 - F_i(T))R_i \exp(-\alpha_i\psi^*(i,T))}{c_i} \sum_i c_i\phi^*(i,T) = -\lambda(T)C\delta T.$$

The third term results in zero after several calculations by using Theorem 1. Consequently, from Eq.(3.18), we have

$$\frac{dG(\phi_T^*, T)}{dT} = C(Q(T) - \lambda(T)),$$

and $\lambda(T^*) = Q(T^*)$ is obtained from $dG(\phi_T^*, T)/dT = 0$. □

We obtain the next corollaries by Theorem 4.

Corollary 3. *If $\lambda(T) > Q(T)$, the search should not be stopped at T.*

Corollary 4. *If the limit of the non-detection probability $Q(T)$ given by Eq.(3.16) draws closer to some positive value as T tends to infinity, the search should be stopped at some finite time, namely $T^* < \infty$.*

Proof: We assume that the search should be continued until the detection when $Q(\infty) > 0$. Since $\lambda(T)$ given by Eq.(3.8) approaches zero as T tends to infinity, we have $\lambda(T) < Q(T)$ and $dG(\phi_T^*, T)/dT > 0$ for large enough T. Therefore, the search should be stopped at a finite time. □

Theorem 4 and Corollary 3 are rewritten as follows.

Corollary 5. *1. A necessary condition for the optimal stopping time T^* is, for any i such that $\phi_T^*(i,T) > 0$,*

$$\frac{\alpha_i p_i(T) R_i}{c_i} = 1 \quad if\ T = T^*. \tag{3.19}$$

2. The search should not be stopped at T, if

$$\frac{\alpha_i p_i(T) R_i}{c_i} > 1. \tag{3.20}$$

Proof: Substituting $\lambda(T^*)$ given by Eq.(3.8) into Eq.(3.17) and dividing both sides of Eq.(3.17) by $Q(T^*)$, then we obtain easily the conditions Eq.(3.19) by using Eq.(3.15). By the same way, Eq.(3.20) is derived from Corollary 3. □

The following corollaries are derived directly from the above theorem by setting $T = 0$.

Corollary 6. *The search should be started, if*

$$\max_i \frac{\alpha_i p_i(1-\beta_i) R_i}{c_i} > 1.$$

4. Numerical Examples

In this section, we examine the optimal search plans for several examples. In the following examples, the defective distribution function of the lifetime of the target is assumed to be an exponential distribution;

$$F_i(t) = \beta_i + (1 - \beta_i - \gamma_i)(1 - \exp(-\mu_i t)).$$

A standard case and its variants are analyzed to show the sensitivity of the optimal search plan on the various parameters involved in the model. The standard case, named Case 1, has parameter values such that $n = 2, p_1 = p_2 = 0.5, \alpha_1 = \alpha_2 = 1, c_1 = c_2 = 1, \beta_1 = \beta_2 = 0, \gamma_1 = \gamma_2 = 0, R_1 = 5, R_2 = 10, C = 1, \mu_1 = 0.2, \mu_2 = 0$. In the standard case, two boxes are identical with respect to the target detectability, the appearing probability of the target and the searching cost. The lifetime distribution of the target appeared in box 1 is exponential with mean time 5 ($\mu_1 = 0.2$), whereas the target in box 2 is a non-disappearing target and the reward in box 2 is twice as large as that in box 1. Therefore, searching in box 2 is more attractive for the searcher. We examine the optimal search plan here.

First, conditioning on the stopping time T of the search, we calculate the conditionally optimal ϕ_T^* and $G(\phi_T^*, T)$. Next, the optimal stopping time T^* minimizing $G(\phi_T^*, T)$ with respect to T is sought and then the optimal search plan $\phi_{T^*}^*$ is obtained. The calculation of the conditionally optimal allocation of the searching effort is carried out by FAB (Forward and Backward) algorithm proposed by Washburn[6]. From now, $G(\phi_T^*, T)$ and $G(\phi_{T^*}^*, T^*)$ may be abbreviated as G_T^* and G^*, respectively.

A part of the results of the conditionally optimal G_T^* is shown in Table 1. The expected risk given in Table 1 is visualized in Fig.1-A. As shown in this figure, the optimal stopping time of the disappearing target is $T^* = 4.0$ and the expected risk $G^* = -3.90$. It should be noted that the curve of G_T^* is not so sensitive to T in the neighborhood of the optimal T^*. The optimal searching effort $\phi_{T^*}^*(1,t)$ is shown in Fig.1-B. ($\phi_{T^*}^*(2,t)$ is obtained by $1 - \phi_{T^*}^*(1,t)$.) As shown in this figure, the searching effort is concentrated in box 1 in the first stage of the search. Although the reward when the target is detected in box 1 is low, it must be searched early since the target disappears in box 1 with the passing of time. On the other hand, since the target in box 2 does not disappear, the search in box 2 is not necessary to be hastened.

Table 1. The conditionally optimal G_T^* for Case 1

T	1	2	3	4	5	6
G_T^*	-2.32	-3.27	-3.75	-3.90	-3.82	-3.58

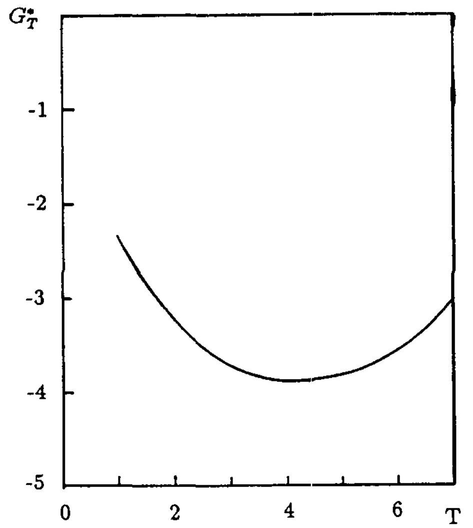

Fig.1-A G_T^* of Case 1

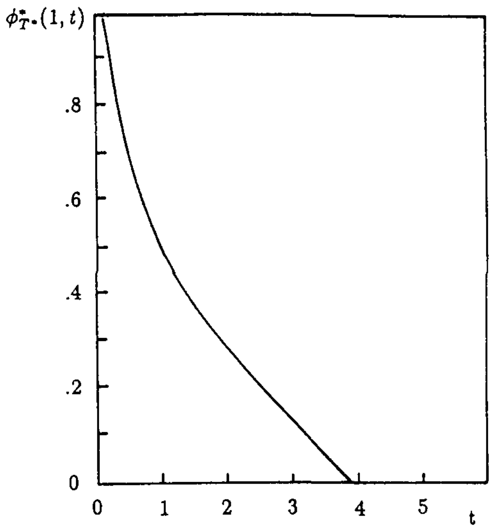

Fig.1-B $\phi_{T^*}^*(1,t)$ of Case 1

(1) Influence of the lifetime distribution of the target

To see the influence of the lifetime of the target to the optimal search plan, in Case 2, μ_1 is changed from 0.2 in Case 1 to 0.3 and 0.6, whereas the other parameters are kept the same as Case 1. T^* and G^* are shown in Table 2. Fig.2-A and Fig.2-B show the relation of $T^* \sim \mu_1$, $G^* \sim \mu_1$ and the conditionally optimal $\phi_T^*(1,t)$ with $T=3$, respectively. As seen in Table 2, T^* is a decreasing function of μ_1 and G^* is an increasing function. These properties are reasonable because the efficiency of the searching effort on the detection becomes lower for larger μ_1. As shown in Fig.2-B, the conditionally optimal allocation of searching effort in box 1, $\phi_T^*(1,t)$, $T=3$ is concentrated in the first stage of the search, and this concentration grows intense as μ_1 increases. These properties are observed for any stopping time T.

Table 2. T^* and G^* of Case 2

Case No.	1	2-1	2-2
μ_1	0.2	0.3	0.6
T^*	4.0	3.7	3.2
G^*	-3.90	-3.73	-3.45

Table 3. T^* and G^* of Case 3

Case No.	1	3-1	3-2
β_1	0	0.3	0.6
T^*	4.0	3.4	2.4
G^*	-3.90	-3.21	-2.76

Next, we examine Case 3 in which β_1 is changed from 0 in Case 1 to 0.3 and 0.6. The optimal stopping time T^* and the expected risk G^* are shown in Table 3. Fig.3-A and

Fig.3-B show T^*, G^* and the conditionally optimal allocation $\phi_T^*(1,t), T = 3$. As seen in Fig.3-B, $\phi_T^*(1,t)$ becomes smaller as β_1 increases. Since the risk of the search in box 1 becomes larger as β_1 increases, this property is natural.

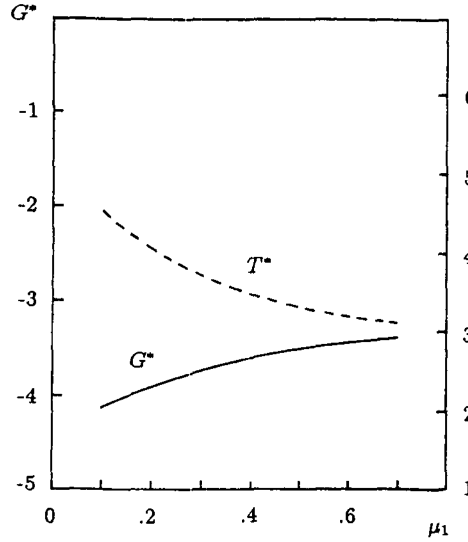

Fig.2-A T^* and G^* of Case 2

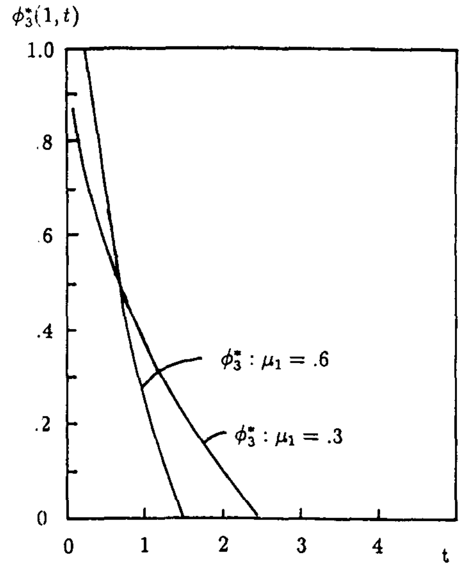

Fig.2-B $\phi_T^*(1,t)$ of Case 2

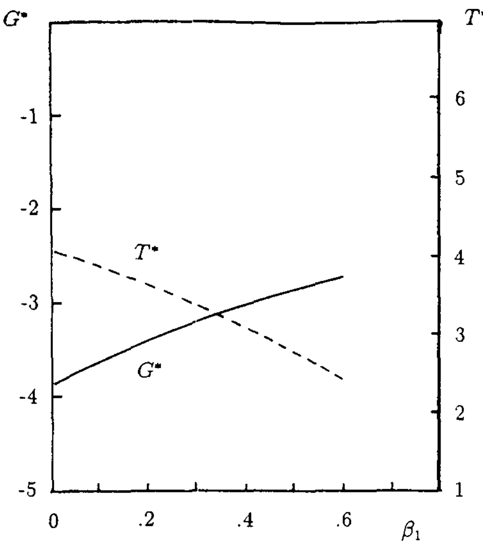

Fig.3-A T^* and G^* of Case 3

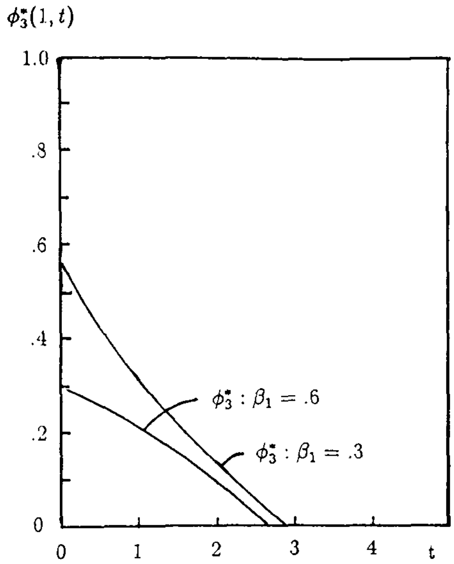

Fig.3-B $\phi_T^*(1,t)$ of Case 3

(2) Effect of the reward

Next, Case 4 in which R_1 varies from 5 in Case 1 to 2.5 and 7.5 is examined. Table 4 gives the optimal stopping time T^* and the expected risk G^* and Fig.4 shows the conditionally optimal allocation of searching effort $\phi_T^*(1,t)$ given $T = 5$. As seen in Table 4, the optimal stopping time is prolonged and the expected risk decreases as R_1 increases. In Fig.4, it should be noted that the conditionally optimal allocation of searching effort in box 1 becomes larger as the reward in box 1 increases. These results seem to be natural.

(3) Influence of the target distribution

Finally, we examine five cases varying p_1 from 0.1 to 0.5 and show the conditionally optimal allocation $\phi_T^*(1,t)$ given $T = 4$ in Fig.5. As seen in Fig.5, $\phi_T^*(1,t)$ has a unimodal shape for $p_1 = 0.1$ and 0.2. These p_1's are too low to allocate some searching effort in box 1 and to expect the detection there. It is optimal for the searcher to concentrate the searching effort in box 2 at the first stage and to divide them into box 1 at the later stage when the non-detection in box 2 makes the posterior probability of the target in box 1 higher. As p_1 becomes higher, $\phi_T^*(1,t)$ for small t increases rapidly and the curve of $\phi_T^*(1,t)$ turns to a decreasing curve of t.

Table 4. T^* and G^* of Case 4

Case No.	1	4-1	4-2
R_1	5.0	2.5	7.5
T^*	4.0	3.6	4.3
G^*	-3.90	-3.17	-4.69

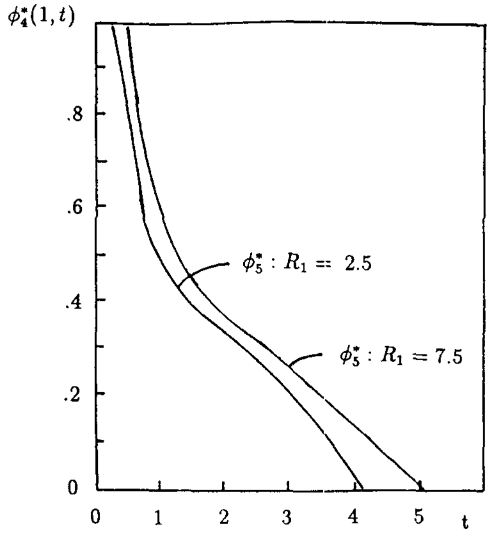

Fig.4 $\phi_T^*(1,t)$ of Case 4

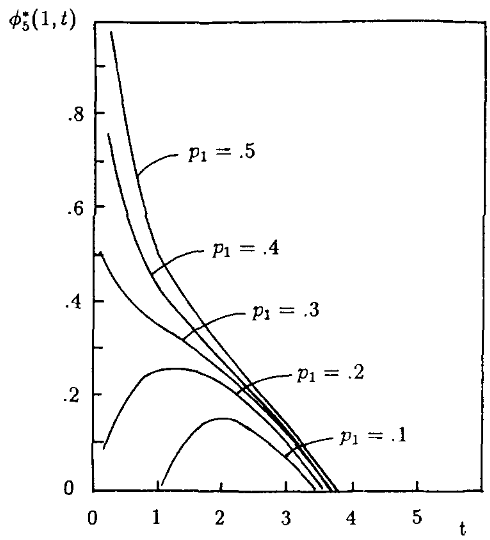

Fig.5 $\phi_T^*(1,t) \sim p_1$ $(T = 4)$

5. Discussions

In this section, we discuss the properties of the conditions of the optimal search plan and the relations between our model and the previous studies.

5.1 The meaning of the optimal conditions

The meaning of Theorem 1 is interpreted as follows. Let $\eta_T^*(i, t_1, t_2)$ be the searching effort allocated in box i from time t_1 to t_2;

$$\eta_T^*(i, t_1, t_2) = \int_{t_1}^{t_2} \phi^*(i, t) dt.$$

Equation (3.6) is transformed by the integration by parts and the relation $\psi^*(i, t_2) = \psi^*(i, t_1) + \eta_T^*(i, t_1, t_2)$ as follows.

if $\phi^*(i, t) > (=)0$,

$$(5.1) \quad \frac{\alpha_i p_i (1 - F_i(t)) \exp(-\alpha_i \psi^*(i, t))}{c_i} \times \left[R_i + \left\{C(T-t) - \int_t^T \frac{(1 - F_i(x))}{(1 - F_i(t))}(R_i + C(T-x))\alpha_i \phi^*(i, x) \exp(-\alpha_i \eta_T^*(i, t, x)) dx\right\}\right] = (\leq)\lambda(t).$$

As shown by the first equation of Eq.(3.2), the term in the braces of Eq.(5.1) is the conditional expected risk of the optimal search from t to T given that the target is in box i and is not detected by t. Here, we recall that the search is to be continued by T if the target is not detected. Therefore, if the target is detected at t in box i, the searcher earns the reward R_i and also saves the risk of the search which is to be incurred in $(t, T]$. The term in the brackets means this return which motivates the search in box i at t. The numerator in the first term of the left-hand side of Eq.(5.1) is the marginal detection probability of the target with the unit density of searching effort in box i at t, and the denominator is the cost of the unit density of searching effort in box i. Hence, the left-hand side of Eq.(5.1) is the expected marginal return versus cost ratio with the unit density of searching effort allocated to box i at t. Therefore, Theorem 1 is interpreted as follows; if the searching effort is to be allocated to box i at t, the density of the searching effort should be determined in such a way that the expected marginal return versus cost ratio mentioned above is balanced to $\lambda(t)$ among the boxes being searched at t, and if the searching effort should not be allocated to box i at t, the value of the ratio is smaller than or equal to $\lambda(t)$.

The conditions of the optimal stopping time for the disappearing target are intepreted as follows. The left-hand side of Eq.(3.19) mean the expected marginal return versus cost ratio at T. Therefore, Corollary 5 implies that the expected marginal return versus cost ratio decreases across unity when the stopping time T increases just across T^*. The search should not be stopped just at the time when the expected marginal return is larger than the searching cost.

5.2 Extension to an optimal search plan for the mortal-type target

As mentioned in Introduction, one of different types of the target with a random lifetime from the disappearing target is the mortal-type target. The mortal target could be dead but could be detected by the searcher even after his death. His death changes a value R_i of his body alive in box i to nothing. Therefore, the searcher gains no reward by the detection of the dead target. The optimal search plan with the criterion of the expected risk for the mortal target are easily to be sought analytically by the same way as procedures in Section 2 and 3. Nakai[3] investigated such a problem with the criterion of the expected return.

His model corresponds to our model with such system parameters as $c_i = c, C = c$. Nakai's model and the optimal search plan are generalized in terms of cost as follows. $F_i(t)$ of Assumption (iii) in Section 2 is interpreted as the distribution function of the lifetime of the mortal target. The expected risk function with a searching effort ϕ_T during $[0,T]$ is given as follows.

$$\begin{aligned} G(\phi_T, T) &= \sum_i p_i \left[\int_0^T \{Ct - R_i(1 - F_i(t))\} \alpha_i \phi(i,t) \exp(-\alpha_i \psi(i,t)) dt \right. \\ &\quad \left. + CT \exp(-\alpha_i \psi(i,T)) \right] \\ &= \sum_i p_i \left[R_i(1 - F_i(T)) \exp(-\alpha_i \psi(i,T)) - R_i(1 - \beta_i) \right. \\ &\quad \left. + \int_0^T \{C + R_i f_i(t)\} \exp(-\alpha_i \psi(i,t)) dt \right]. \end{aligned}$$

We obtain a necessary and sufficient condition for an optimal allocation ϕ^* of the searching effort, corresponding to Theorem 1, as follows.

if $\phi^*(i,t) > 0$,

$$\begin{aligned} &\frac{\alpha_i p_i}{c_i} \left[R_i(1 - F_i(T)) \exp(-\alpha_i \psi^*(i,T)) + \int_t^T (C + R_i f_i(x)) \exp(-\alpha_i \psi^*(i,x)) dx \right] \\ (5.2) \qquad &= \lambda(t), \end{aligned}$$

if $\phi^*(i,t) = 0$,

$$\begin{aligned} &\frac{\alpha_i p_i}{c_i} \left[R_i(1 - F_i(T)) \exp(-\alpha_i \psi^*(i,T)) + \int_t^T (C + R_i f_i(x)) \exp(-\alpha_i \psi^*(i,x)) dx \right] \\ &\leq \lambda(t). \end{aligned}$$

The non-detection probability $Q(T)$ for the mortal target and the posterior probability $p_i(T)$ of the target in box i with the searching effort ϕ_T^* are given by the next equations.

$$(5.3) \qquad Q(T) = \sum_i p_i \exp(-\alpha_i \psi^*(i,T)),$$

$$(5.4) \qquad p_i(T) = \frac{p_i \exp(-\alpha_i \psi^*(i,T))}{Q(T)}.$$

With the above equations, Theorem 4 is still valid for the mortal target, namely, a necessary condition for the optimal stopping time T^* is

$$\lambda(T^*) = Q(T^*).$$

5.3 Relations between our models and the previous studies

In this paper, we study the optimal search for the disappearing target. The relations between our study and the previous studies are as follows.

(1) If we set $F_i(t) = 0$ for all i and t in our models, the search situation considered in this paper becomes the search problem of the appeared target. In this special case, all results of the optimal search plan described in this paper are completely identical with the results obtained by Iida[1].

(2) In the whereabouts search model studied by Iida[2], if we set both the whereabouts search cost and the reward of the correct guess at zero, our results for the mortal target in the previous subsection are obtained.

(3) If we set $c_i = c, C = c$ in our model, the search plan for the mortal target in the previous subsection is identical with Nakai's model[3].

6. Concluding Remarks

In this paper, we deal with the optimization problem of the search for the disappearing target. We assume the followings; (1)the continuous time space, (2) the discrete target space, (3) the exponential detection function, (4) the continuous divisibility of the searching resource. These assumptions except for (4) are not so essential to deal with the problem. The model can be generalized to the discrete time system, the continuous target space and the regular detection function, and similar results as obtained in this paper will be derived without any difficulty. However, the continuous divisibility of searching resource is a very important assumption for our model to simplify the treatment of the problem.

Many problems may remain to be investigated in future. One of the important problems is the optimal search plan for a moving target with random lifetime (the disappearing or the mortal target). If the target is a moving target with a conditionally deterministic motion, the problem may be formulated by a similar pattern as described in this paper and Theorems will be obtained.

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