ASYMPTOTIC THEORY OF SELECTION BY RELATIVE RANK WITH HIGH COST

Sigeiti Moriguti Professor Emeritus, University of Tokyo

(Received December 15, 1992; Revised March 16, 1993)

Abstract Selection from among n objects by relative rank with no recall — the "secretary problem" in the asymptotic case when $n \to \infty$ is considered, assuming that k, the cost ratio, becomes very large and $\kappa = k/n$ is kept to be a finite constant. It is known that in the "low cost" case where K = kn is kept to be a non-negative finite constant, the expected number of observations comes out to be O(n) and the expected absolute rank of the selected object comes out to be O(1), and that in the "medium cost" case where k is kept to be a positive finite constant, both expected values come out to be $O(\sqrt{n})$. Here the former comes out to be O(1) and the latter O(n). The graph of total "loss" vs. κ looks like a continuous broken line, while those of "expected cost of observations" and "expected absolute rank" have jumps at many points. As κ approaches 0, the curves approach smooth ones corresponding to the "medium cost" case.

1. Introduction.

Suppose there are *n* objects and they are ranked from 1 (the best) to *n* (the worst) without any tie. They are arranged in a row. We are allowed to observe them one by one starting at an end. Let the (absolute) rank of the *i*-th object be x_i . Then x_1, x_2, \dots, x_n is assumed to be a random permutation of $1, 2, \dots, n$. When we have observed first *i* objects, the relative rank of the *i*-th object is $y_i = (number of x_1, x_2, \dots, x_{i-1} less than x_i) + 1$.

The stopping criteria $s_1, s_2, \dots, s_n = n$ are pre-determined. For $i = 1, 2, \dots$, right after observing the first *i* objects, if the relative rank y_i of the *i*-th object is less than or equal to s_i , then we stop there and select the *i*-th object. Otherwise we continue the observation.

The "classical secretary problem" (cf. Chow et al. (1964)) assumes that the cost of observation is 0. Moriguti (1993a) discussed the basic theory of the "secretary problem" with cost, and Moriguti (1993b) dealt with asymptotic behavior as $n \to \infty$, keeping

(1.1)
$$k = \frac{\text{cost of observing one object}}{\text{loss of getting one lower expected rank}}$$

a finite constant. The gap between this "medium cost" case and "zero cost" case was successfully bridged by the "low cost" case discussed in Moriguti (1993c).

On the other hand, results of Moriguti (1992) suggest that "high cost" case, where

(1.2)
$$\kappa = k/n$$

is kept constant when $n \to \infty$, would be also interesting. That is what we discuss in this paper.

Among possible applications, the "marriage problem" would usually be a typical example of the "high cost" case.

2. Notations and fundamental formulae.

In Moriguti (1993a), besides the notations introduced in Section 1 above, the following notations were used:

 ω_i = the event that stopping does not occur at the *i*-th observation or earlier,

 $Q_i = \mathbf{P}(\omega_i),$

 $i_s = \min\{i \mid s_i \ge s\},$

 $I_s = \{i \mid i_s \le i \le i_{s+1} - 1\},\$

 $e_o =$ expected number of observations until stopping,

 e_i = expected number of further observations until stopping under the condition ω_i , $f(i,r) = P\{\text{stop right after the } i\text{-th observation, absolute rank } r\},$

f(r) = P{absolute rank of the selected object is r},

$$g(i,r) = f(i,r) - f(i,r+1),$$

 $c_o =$ expected absolute rank of the selected object,

 c_i = expected absolute rank of the selected object under the condition ω_i ,

 $t_o = c_o + k \cdot e_o =$ expected total "loss",

 $t_i = c_i + k \cdot e_i =$ expected total "loss" under the condition ω_i .

Optimal stopping criteria s_1, s_2, \cdots and resulting $t_i, c_i, e_i (i = 0, 1, \cdots)$ are given by backward recurrence formulae:

$$t_{i-1} = \frac{(n+1)s_i(s_i+1)}{2i(i+1)} + k + \left(1 - \frac{s_i}{i}\right)t_i,$$

 $s_i = \operatorname{int}\left[\frac{i+1}{n+1} \cdot t_i\right]$, where $\operatorname{int}[x]$ denotes the greatest integer $\leq x$,

$$c_{i-1} = \frac{(n+1)s_i(s_i+1)}{2i(i+1)} + \left(1 - \frac{s_i}{i}\right)c_i,$$
$$e_{i-1} = 1 + \left(1 - \frac{s_i}{i}\right)e_i,$$

starting with $s_n = n$, $t_{n-1} = (n+1)/2 + k$, $c_{n-1} = (n+1)/2$, and $e_n = 0$. Here let us introduce the following notations:

$$(2.1) T(i) = t_i/n,$$

$$(2.2) \qquad \qquad \rho = r/n$$

Then, recurrence formulae for e_i , C(i), T(i), and $s_i(i = 1, 2, \dots)$, become, in the limit $n \to \infty$:

(2.4)
$$e_{i-1} = (1 - s_i/i)e_i + 1,$$

(2.5)
$$C(i-1) = \frac{s_i(s_i+1)}{2i(i+1)} + \left(1 - \frac{s_i}{i}\right)C(i),$$

(2.6)
$$T(i-1) = \frac{s_i(s_i+1)}{2i(i+1)} + \kappa + \left(1 - \frac{s_i}{i}\right)T(i),$$

(2.7)
$$s_i = \inf[(i+1)T(i)].$$

Starting from a sufficiently large value of n, and the initial conditions:

(2.8)
$$e_n = 0, \ C(n) = 1/2, \ T(n) = 1/2, \ s_n = n,$$

and using the formulae (2.4) through (2.7) successively, we can obtain e_0 , C(0), T(0), and s_1, s_2, \cdots . The cost parameter κ has a decisive effect on all these quantities, of course. If we plot T(0), C(0) and $E(0) = \kappa \cdot e_o$ against κ , then we get a picture like Fig. 1.



Fig. 1. Expected "costs" against the cost parameter κ .

Besides the fine ripple due to the limited resolution of the plotter, we observe significant jumps in the graphs of both C(0) and E(0). But they cancel out each other when we add C(0) and E(0) to get T(0). The resulting T(0) curve looks almost like a broken line, consisting of line segments of slope $1, 2, 3, \cdots$ from right to left.

The dotted vertical lines in Fig. 1 indicate some of the positions of major jumps in C(0) and E(0) curves, and their tops shown by circles are the corners of T(0) curve. Their abscissae $\kappa_i (i = 1, 2, \dots, 5)$ are shown in Table 1, together with the corresponding amount of jumps

(2.9)
$$\delta C(0) = \{C(0)+\} - \{C(0)-\} = \{C(0) \text{ for } \kappa_i + h\} - \{C(0) \text{ for } \kappa_i - h\},\$$

where h was chosen to be 0.00001. With respect to the last column E(i)/i of Table 1, see Appendix 1.

Table 1. Abscissae and amounts of major jumps in C(0) curve.

i	κi	C(0)+	C(0)-	δC(0)	E(i)/i
1 2 3 4 5	.08949 .03529 .01874 .01162 .00790	.50000 .27048 .18907 .14313 .11667	.30208 .20763 .15860 .12418 .10418	.19792 .06285 .03047 .01896 .01250	.19792 .06285 .03046 .01895 .01250

The curves of $i_s(s = 1, 2, \dots, 10)$ vs. κ come out to be as shown in Fig. 2.



Fig. 2. Curves of $i_s(s = 1, 2, \dots, 10)$ vs. κ .

It is seen that $\kappa_i (i = 1, 2, \dots, 5)$ in Table 1 coincide with the abscissae in Fig. 2 where i_1 jumps from i to i + 1 (when we go from right to left).

3. Distribution of the number of observations.

The cumulative distribution function of i, the number of observations, is given by 1-Q(i), where Q(i) is obtained starting from Q(0) = 1 and using

(3.1)
$$Q(i) = Q(i-1) \cdot (1 - s_i/i)$$

successively (cf. Moriguti (1993a)).

Some examples are shown in Fig. 3,(a) through (e). In each of the five pictures, two curves are shown, one (solid line) for $\kappa = \kappa_i + 0$ and the other (dotted line) for $\kappa = \kappa_i - 0$. Corresponding expected values are shown by two vertical lines:— dashed line for $\kappa = \kappa_i + 0$ and dot-dash line for $\kappa = \kappa_i - 0$. Major jumps in the graph of $E(0) = \kappa \cdot e_0$ vs. κ in Fig. 1 are reflected in these graphs.

4. Distribution of the absolute rank.

The absolute rank of the selected object is O(n), so that we are to consider the limiting distribution of a continuous variate $\rho = r/n$ (cf.(2,2)).

As discussed in Section 4 of Moriguti (1993a), f(i, r), the probability of stopping right after the *i*-th observation and the absolute rank of the selected object being r, is given by



Fig. 3. Graphs of Q(i) vs. *i* for $\kappa = \kappa_i \pm 0$ ($i = 1, 2, \dots, 5$).

(4.1)
$$f(i,r) = Q(i-1) \cdot \sum_{j=1}^{s_i} \frac{1}{i} \binom{r-1}{j-1} \binom{n-r}{i-j} / \binom{n}{i}$$

(See (4.9) there.)

Using the notation

(4.2)
$$x^{(p)} = x(x-1)\cdots(x-p+1),$$

we can transform (4.1) into

(4.3)
$$f(i,r) = Q(i-1) \cdot \sum_{j=1}^{s_i} \frac{1}{i} \frac{(r-1)^{(j-1)}}{(j-1)!} \frac{(n-r)^{(i-j)}}{(i-j)!} \frac{i!}{n^{(i)}}$$

Copyright © by ORSJ. Unauthorized reproduction of this article is prohibited.

247

S. Moriguti

$$= Q(i-1) \cdot \sum_{j=1}^{s_i} {\binom{i-1}{j-1}} \frac{(r-1)^{(j-1)}(n-r)^{(i-j)}}{n^{(i)}}.$$

Summing up (4.3) for $i = i_1, i_1 + 1, \dots$, substituting $r = \rho n$, and taking the limit $n \to \infty$, we have the limiting probability density function of ρ :

(4.4)
$$f(\rho) = \sum_{i=i_1}^{\infty} Q(i-1) \sum_{j=1}^{s_i} {i-1 \choose j-1} \rho^{j-1} (1-\rho)^{i-j}$$

Values of Q(i) herein are obtained with (3.1) as discussed in Section 3. So after obtaining $s_i(i = i_1, i_1 + 1, \cdots)$, there is no difficulty in the numerical computation of (4.4) for $\rho = 0, 0.01, 0.02, \cdots, 1.00$, say. Then, using Simpson's rule for numerical integration, we can get the values of the cumulative distribution function

(4.5)
$$F(\rho) = \int_0^{\rho} f(\rho) d\rho$$



Fig. 4. Cumulative distribution function $F(\rho)$ for $\kappa = \kappa_i \pm 0$ $(i = 1, 2, \dots, 5)$.

Copyright © *by ORSJ. Unauthorized reproduction of this article is prohibited.*

248

for $\rho = 0, 0.02, \dots, 1.00$. Plotting the results for $\kappa = \kappa_i \pm 0$ $(i = 1, 2, \dots, 5)$, we get graphs (a) through (e) of Fig. 4. It is interesting to note here that the graph for $\kappa = \kappa_i - 0$ is closer to that for $\kappa = \kappa_{i+1} + 0$ than to that for $\kappa = \kappa_i + 0$. This feature is in line with the nearly horizontal lines of C(0) in each interval $(\kappa_{i+1}, \kappa_i)(i = 1, 2, \dots)$.

It is possible to express (4.5) in an analytically more beautiful form using the incomplete beta function. But the above mentioned procedure will be preferable for numerical computation.

5. Final remarks.

Transition to the "medium cost" case (Moriguti (1993b)) is observed to be smooth in many respects. (See Appendix 2 for some examples.)

Thus, with "medium cost" case in the middle, and "low cost" case and "high cost" case on its two sides, the three asymptotic theories cover all situations completely. To see how these theories are reflected in the case of finite n, Fig. 5 plots t_0 , c_0 , and $k \cdot e_0$ against k (all on logarithmic scale) for the population size n = 1000.



Fig. 5. $t_0, c_0, k \cdot e_0$ vs. k for n = 1000.

Fig. 5 suggests that, in the case of n = 1000, the case of cost-ratio k between 0.03 and 1.0 can be treated as the "medium cost" case, whereas the case of k below 0.03 belongs to the "low cost" case, and the case of k above 1.0 should be treated as the "high cost" case.

Case of the astronomer Johannes Kepler (1571-1630) as described in Ferguson (1989), seems to belong to the "high cost" case, because he decided on the fifth he interviewed. The population size was at first n = 11, although he could have gone on beyond that number. It would not be unreasonable to assume that his potential population size n was very large. In that case, his estimate of interviewing one candidate might have been around $\kappa_2 = 0.035$ times n (see Fig. 3(b)). Then the expected absolute rank would have been somewhere between 20% and 27% of the whole potential population (see Fig. 4(b)). This explains why he was happy with his choice (Ferguson (1989), p. 285).

In concluding the last paper of the series starting with Moriguti (1992), the author would like to express sincere gratitude to Professor Herbert Robbins, who first introduced him to this intriguing set of problems in 1964, and who encouraged him on occasions of his visits to New York, sometimes through telephone conversations.

Thanks are also due to Mr. Isao Watanabe who made substantial contributions in his graduation thesis (Watanabe (1965)) under the guidance of the present author.

REFERENCES

- Chow, Y. S., Moriguti, S., Robbins, H., and Samuels, S. M. (1964) "Optimal selection based on relative rank (the 'secretary problem')" *Israel Journal of Mathematics*, Vol. 2, pp. 81-90.
- [2] Ferguson, T. S. (1989) "Who solved the secretary problem?" Statistical Sciences 4, pp. 282-296.
- [3] Moriguti, S. (1992) "A selection problem with cost—'secretary problem' when unlimited recall is allowed" Journal of Operations Research Society of Japan, Vol. 35, pp. 373-382.
- [4] Moriguti, S. (1993a) "Basic theory of selection by relative rank with cost" Journal of Operations Research Society of Japan, Vol. 36, No. 1, pp. 46-61.
- [5] Moriguti, S. (1993b) "Asymptotic theory of selection by relative rank with medium cost" Journal of Operations Research Society of Japan, Vol. 36, No. 2, pp. 102-117.
- [6] Moriguti, S. (1993c) "Asymptotic theory of selection by relative rank with low cost" Journal of Operations Research Society of Japan, Vol. 36, No. 3, pp. 175-195.
- [7] Watanabe, I. (1965) "Optimal selection rule" Graduation Thesis, Departmeng of Mathematical Engineering and Instrumentation Physics, Faculty of Engineering, University of Tokyo. 44pp. [in Japanese]

Appendex 1. About the Major Jumps in Fig. 1.

First, let us establish that, as κ changes, T(0) changes continuously. If any discontinuity occurred, it would be at a poing where s_i changes by 1. Let κ_{si} denote the value of κ for which s_i in (2.7) changes from s + 1 to s. Namely,

(A1.1) For
$$\kappa_{si} + 0$$
, $s \le (i+1)T(i) \le s+1$,

and

(A1.2) For
$$\kappa_{si} = 0$$
, $s + 1 \le (i+1)T(i) \le s + 2$.

For the sake of proof by induction, let us assume that T(i) is continuous at κ_{si} . Then (A1.1), (A1.2) imply

(A1.3) For
$$\kappa = \kappa_{si}$$
, $(i+1)T(i) = s+1$.

Hence (2.6) gives us

(A1.4) For
$$\kappa = \kappa_{si} + 0$$
, $T(i-1) = \frac{s(s+1)}{2i(i+1)} + \kappa + \left(1 - \frac{s}{i}\right)\frac{s+1}{i+1} = \kappa + \frac{s+1}{i+1} - \frac{s(s+1)}{2i(i+1)}$,

For
$$\kappa = \kappa_{si} - 0$$
, $T(i-1) = \frac{(s+1)(s+2)}{2i(i+1)} + \kappa + \left(1 - \frac{s+1}{i}\right)\frac{s+1}{i+1} = \kappa + \frac{s+1}{i+1} - \frac{s(s+1)}{2i(i+1)}$.

Thus, if T(i) is continuous at κ_{si} , then T(i-1) is also continuous there. At the starting point i = n, T(n) is 1/2 and so continuous. Hence, backward induction establishes the continuity of T(i) for any i, including i = 0. This completes the continuity of T(0) with respect to κ .

Next, let us examine the jump of C(0) at $\kappa = \kappa_{si}$. Since Fig. 1 and Fig. 2 show that major jumps occur at $\kappa = \kappa_{0i}$, let us denote κ_{0i} simply by κ_i hereafter. Then, $s_i = 1$ for $\kappa = \kappa_i + 0$ and $s_i = 0$ for $\kappa = \kappa_i - 0$. Now (2.5) gives us

(A1.6) For
$$\kappa = \kappa_i + 0$$
, $C(i-1) = \frac{1}{i(i+1)} + \left(1 - \frac{1}{i}\right)C(i)$,

and

(A1.7) For
$$\kappa = \kappa_i - 0$$
, $C(i-1) = C(i)$.

Therefore, the jump is

(A1.8)
$$\delta C(i-1) = \frac{1}{i(i+1)} - \frac{C(i)}{i} \quad \text{at } \kappa = \kappa_i.$$

Similarly, we can get from (2.4)

(A1.9) For
$$\kappa = \kappa_i + 0$$
, $e_{i-1} = (1 - 1/i)e_i + 1$,

and

(A1.10) For
$$\kappa = \kappa_i - 0$$
, $e_{i-1} = e_i + 1$.

Therefore, the jump is

(A1.11)
$$\delta e_{i-1} = -e_i/i \quad \text{at } \kappa = \kappa_i.$$

If we introduce the notation

(A1.12)
$$E(i) = \kappa \cdot e_i$$

then the jump of E(i-1) is

(A1.13)
$$\delta E(i-1) = -E(i)/i \quad \text{at } \kappa = \kappa_i$$

Since there cannot be any jump in the interval [0, i - 1], the jumps (A1.8) and (A1.13) will be reflected in the jumps of C(0) and E(0), so that their jumps are

(A1.14)
$$\delta C(0) = \frac{1}{i(i+1)} - \frac{C(i)}{i} \quad \text{at } \kappa = \kappa_i,$$

(A1.15)
$$\delta E(0) = -\frac{E(i)}{i} \quad \text{at } \kappa = \kappa_i.$$

Adding up (A1.14) and (A1.15), and using (A1.3) for s = 0, we can reassure that T(0) does not have any jump at $\kappa = \kappa_i$.

The last column in Table 1 shows the value of E(i)/i for $\kappa = \kappa_i (i = 1, 2, \dots, 5)$. Comparison of $\delta C(0)$ and E(i)/i should be the numerical verification of the fact that the jumps of C(0) and E(0) at $\kappa = \kappa_i$ have the same magnitude and the opposite signs, thus cancelling each other when added up to get the jump of T(0).

Appendix 2. Transition to the Medium Cost Case.

It is natural to expect that, as $\kappa \to 0$, the curves in "high cost" case will show smooth transition to the corresponding curves in "medium cost" case, if we choose proper coordinates. In this Appendix, we will show two representative examples of that kind of transition.

The first example is Q(i), the probability that stopping does not occur at the *i*-th observation or before, as shown in Fig. 3, (a) through (e). Changing the abscissae into

(A2.1)
$$i\sqrt{(2\kappa)} = (i/\sqrt{n}) \cdot \sqrt{(2k)},$$

and superposing those curves, we get ten curves in Fig. 6. Of them, five solid curves (broken lines, in fact) with circles at some corners correspond to $\kappa = \kappa_i + 0$, and five dotted curves with crosses at some corners correspond to $\kappa = \kappa_i - 0$. The uppermost solid curve with solid squares at corners shows the corresponding curve of $\bar{Q}(u)$ vs. $u\sqrt{(2k)} = (i/\sqrt{n}) \cdot \sqrt{(2k)}$. (See Fig. 2 of Moriguti (1993b).)



Reflecting the "jumps" in the E(0) curve in Fig. 1, the transition is not completely smooth. But Fig. 6 suggests that the transition will eventually get smoother as κ tends to 0.

The second example is $F(\rho)$, the cumulative distribution function of the absolute rank r of the selected object, against $\rho = r/n$. The curves are shown in Fig. 4, (a) through (e) for $\kappa = \kappa_i \pm 0$ $(i = 1, 2, \dots, 5)$. Changing the abscissa into

(A2.2)
$$\rho/\sqrt{(2k)} = (r/\sqrt{n})/\sqrt{(2k)},$$

and superposing those curves (omitting (e)), we get eight curves in Fig. 7, solid ones corresponding to $\kappa_i + 0$ and dotted ones to $\kappa_i - 0$ (i = 1, 2, 3, 4). The limiting curve for the "medium cost" case is shown here by a dashed curve. (See Fig. 3 of Moriguti (1993b).)

General impression is that curves for $\kappa = \kappa_i + 0$ approach the limiting curve from below, and curves for $\kappa = \kappa_i - 0$ from above. Closer look, however, reveals a little more complex picture. Any way, in a region near the origin, all curves seem to approach the dashed curve from below.



Fig. 7. $F(\rho)$ vs. $\rho/\sqrt{(2\kappa)}$ for $\kappa = \kappa_i \pm 0$ $(i = 1, \dots, 4)$ and the limiting curve.

Sigeiti Moriguti Syoan 2-16-10 Suginami-ku Tokyo 167, JAPAN