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WORST-CASE ANALYSIS OF INDEXING RULES FOR SINGLE MACHINE SEQUENCING

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Abstract The single machine sequencing problem is considered in which each job has a release date, a processing time, and a delivery time. The objective is to find a sequence of jobs which minimizes the time by which all jobs are delivered. We study priority sequencing rules which use an index to prioritize jobs. In particular, we establish worst-case bounds for families of weighted linear and quotient indexing rules. We also analyze an $O(n \log n)$ dynamic indexing rule A. Two subregions of the admissible input space are identified in which heuristic A has better worst-case performance ratios.

1. Introduction

We consider a scheduling problem (SH) with n jobs to be processed without interruption on a single machine. Associated with each job i are a release date $r_i \geq 0$, a processing time $p_i > 0$ on the machine and a delivery time $q_i \geq 0$, where q_i is the postprocessing time after leaving the machine. The objective is to minimize the makespan, i.e., to finish the n jobs as soon as possible. The problem can be viewed as a three-stage flow-shop problem, where an unlimited number of machines are available in the first and third stage with a processing time on the first and third stage r_i and q_i, respectively. Problem SH can also be viewed as equivalent to a scheduling problem with a due date d_i instead of a delivery time q_i associated with each job i and the objective of minimizing maximum lateness with respect to due dates. The equivalence can be established by letting $d_i = K - q_i$, where K is a constant.

We note that problem SH is strongly NP-hard [11]. As both an 'easy' NP-hard problem and a fundamental problem arising in the theoretical context of computing lower bounds for flow shop and job shop problems [2], problem SH has drawn a lot of attention from researchers. Research interests follow two lines: worst-case analysis of heuristics and the development of enumerative methods. Kise, Ibaraki and Mine [10] analyzed the performance of several heuristics and showed that each heuristic can deviate by an amount arbitrarily close to 100% from the optimum. Based on the extended Jackson's rule [9], also known as Schrage's heuristic [14], Potts [13] presented an $O(n^2 \log n)$ heuristic which ensures that a solution within 50% of the optimum is always produced. Hall and Shmoys [7] presented an $O(n^2 \log n)$ heuristic which ensures that a solution within 33% of the optimum is always produced for the problem when there are precedence constraints among the jobs, and two polynomial approximation schemes for the problem without precedence constraints. Hall and Rhee [8] considered fifteen heuristics for a related problem, thirteen of which belong to the family of weighted linear rules (F_l) considered in this paper. They empirically studied the average case performance of these heuristics by randomly generating the testing set. They failed to derive the worst-case performance ratios for these heuristics. Instead they used linear programming to estimate the worst-case bounds. Another line of research on pursuing the enumerative methods for solving problem SH includes Baker and Su [1], McMahon and Florian [12], Carlier [3] and Grabowski et al. [6].

In this research we follow the line of worst-case analysis of heuristics. In particular, we analyze the worst-case performance of one family (F_l) of weighted linear rules and one family

(F_q) of weighted quotient rules. Worst-case performance ratios are derived for these two families. We also analyze an $O(n \log n)$ dynamic linear rule (A). We identify one special case for which heuristic A has a worst-case performance ratio of 4/3, and one case for which heuristic A has a worst-case performance ratio of 5/4. The worst-case results on F_l can be viewed as an extension of Kise, Ibaraki and Mine [10] as well as Hall and Rhee [8]. In using a priority index for sequencing the jobs, intuitively one would expect that the smaller the release date, the higher should be the priority; the larger the delivery time, the higher should be the priority. It is unclear a priori how processing time may be related to a good job sequence. Our investigations on F_l and F_q, in some sense, answer how we aggregate the single job measures r_i, p_i, q_i in forming a priority index for sequencing the jobs.

The paper is organized as follows. Section 2 is on the analysis of the linear and quotient indexing rules. Section 3 is on the analysis of the dynamic linear rule A. We conclude with section 4. For a survey and discussion of worst-case analysis of heuristics, see Fisher [4] and Garey et al. [5].

2. Analysis of Two Families of Sequencing Rules

For a processing order of the n jobs on the machine, we define a 'busy schedule' as one with no forced idle time. In what follows, we will only consider 'busy schedule'. Given a problem instance I, let $T^H(I)$ represent the objective value (the makespan) obtained by using heuristic H and $T^*(I)$ represent the optimal solution, where heuristic H can be any algorithm producing a processing order of the n jobs on the machine. Also let $E^H(I) = T^H(I) - T^*(I)$. Let η_H denote the worst-case performance ratio associated with heuristic H, then $\eta_H = \sup_I\{T^H(I)/T^*(I)\}$, where I is a problem instance. In what follows, when no confusion arises, we will use T^H, T^* and E^H instead of $T^H(I)$, $T^*(I)$ and $E^H(I)$.

It is well-known that

$$T^H = \max_{1\le i\le j\le n}\{r_{\sigma(i)} + \textstyle\sum_{h=i}^{j} p_{\sigma(h)} + q_{\sigma(j)}\} = r_{\sigma(u)} + \textstyle\sum_{h=u}^{v} p_{\sigma(h)} + q_{\sigma(v)},$$

where $(\sigma(1)\ \ldots\ \sigma(n))$ is the sequence generated by H and $1 \le u \le v \le n$. If there is a choice, it is assumed that u and v are both as small as possible. As u is as small as possible, either job $\sigma(u)$ is the first job or the machine will be idle immediately prior to processing job $\sigma(u)$. Furthermore, note that the machine is continuously busy processing jobs $\sigma(u)$ through $\sigma(v)$. We define the set of jobs between $\sigma(u)$ and $\sigma(v)$ as the *critical group*. We refer to this group as the critical group because the makespan is determined by this group. Let $r_{min} = \min_{u\le h\le v}\{r_{\sigma(h)}\}$ and $q_{min} = \min_{u\le h\le v}\{q_{\sigma(h)}\}$.

We first develop some properties of E^H, η_H and T^*.

Lemma 1.

(1) $T^* \ge r_{\sigma(h)} + p_{\sigma(h)} + q_{\sigma(h)}$, $1 \le h \le n$.

(2) $T^* \ge r_{min} + \sum_{h=u}^{v} p_{\sigma(h)} + q_{min}$.

(3) $E^H \le r_{\sigma(u)} - r_{min} + q_{\sigma(v)} - q_{min} \le r_{\sigma(u)} + q_{\sigma(v)}$.

(4) $\eta_H \le 2$ if any of the following conditions holds:

(*i*) $r_{\sigma(u)} \le r_{\sigma(v)}$,

(*ii*) $q_{\sigma(u)} \ge q_{\sigma(v)}$,

(*iii*) $r_{\sigma(u)} + p_{\sigma(u)} \le r_{\sigma(v)} + p_{\sigma(v)}$,

(*iv*) $q_{\sigma(u)} + p_{\sigma(u)} \ge q_{\sigma(v)} + p_{\sigma(v)}$.

Proof. (1) The completion time of any single job is obviously a lower bound for the optimal objective value.

(2) The optimal objective value can be no smaller than the minimum possible completion time, considering only jobs $\sigma(u)$, ..., $\sigma(v)$.

(3) The first inequality can be obtained by subtracting T^H from the inequality in (2). The second inequality holds since both release dates and delivery times are nonnegative.

(4) By (3) and then (1), for (*i*) or (*ii*), we have $E^H \le r_{\sigma(u)} + q_{\sigma(v)} \le \max\{r_{\sigma(u)} + q_{\sigma(u)}, r_{\sigma(v)} + q_{\sigma(v)}\} \le T^*$; while, for (*iii*) or (*iv*), we have $E^H \le r_{\sigma(u)} + q_{\sigma(v)} \le \max\{r_{\sigma(u)} + p_{\sigma(u)} + q_{\sigma(u)}, r_{\sigma(v)} + p_{\sigma(v)} + q_{\sigma(v)}\} \le T^*$. Hence, $\eta_H \le 2$. ◇

We now examine indexing rules for sequencing jobs. An index I_i of job i is a function of the attributes (e.g., r_i, p_i and q_i) of the job. Indexing rules use I_i to prioritize jobs, where we assume that a high value of I_i represents high priority. We consider one family (F_l) of weighted linear indexing rules and one family (F_q) of weighted quotient indexing rules for sequencing jobs, where $F_l = \{I_i : I_i = xq_i - yr_i + zp_i,\ x > 0,\ y > 0\}$ and $F_q = \{I_i : I_i = (xq_i + p_i)/(yr_i + p_i),\ x > 0,\ y \ge 1\}$. Let $F_l^1 = \{I_i : I_i = xq_i - yr_i + p_i,\ x \ge 1,\ y > 0\}$, $F_l^2 = \{I_i : I_i = xq_i - yr_i + p_i,\ 0 < x < 1,\ y > 0\}$, $F_l^3 = \{I_i : I_i = xq_i - yr_i - p_i,\ y \ge 1,\ x > 0\}$, $F_l^4 = \{I_i : I_i = xq_i - yr_i - p_i,\ 0 < y < 1,\ x > 0\}$, and $F_l^5 = \{I_i : I_i = q_i - yr_i,\ y > 0\}$. We note that $F_l = \cup F_l^i$, $i = 1, 2, 3, 4, 5$. This is true since $F_l = F_l^1 \cup F_l^2$ if $z > 0$; $F_l = F_l^3 \cup F_l^4$ if $z < 0$; and $F_l = F_l^5$ if $z = 0$.

Let σ^H denote the sequence generated by heuristic H and σ^* denote the optimal sequence. The worst-case performance ratios for these families of heuristics are derived in theorem 1.

Theorem 1.
(1) $\eta_H = 2$ if $H \in F_l^1$.
(2) $\eta_H = 3 - \frac{x+y}{1+y}$ if $H \in F_l^2$.
(3) $\eta_H = 2$ if $H \in F_l^3$.
(4) $\eta_H = 3 - \frac{y+x}{1+x}$ if $H \in F_l^4$.
(5) $\eta_H = 2$ if $H \in F_l^5$.
(6) $\eta_H = 2$ if $H \in F_q$.

Proof. If $r_{\sigma(u)} \le r_{\sigma(v)}$ or $q_{\sigma(u)} \ge q_{\sigma(v)}$, then, by lemma 1(4), $\eta_H \le 2$. Therefore, we need only to consider the case: $r_{\sigma(u)} > r_{\sigma(v)}$ and $q_{\sigma(u)} < q_{\sigma(v)}$. Assume that $r_{\sigma(u)} - r_{\sigma(v)} = \beta T^*$ and $q_{\sigma(v)} - q_{\sigma(u)} = \alpha T^*$. Also, assume that $p_{\sigma(u)} = \theta T^*$ and $p_{\sigma(v)} = \theta' T^*$.

For $H \in F_l^1$ or F_l^2, we have, by definition of H, $xq_{\sigma(u)} - yr_{\sigma(u)} + p_{\sigma(u)} \ge xq_{\sigma(v)} - yr_{\sigma(v)} + p_{\sigma(v)}$. Thus we have $q_{\sigma(v)} \le q_{\sigma(u)} + \frac{1}{x}p_{\sigma(u)} - \frac{y}{x}(r_{\sigma(u)} - r_{\sigma(v)})$ and $r_{\sigma(u)} \le r_{\sigma(v)} + \frac{1}{y}p_{\sigma(u)} - \frac{x}{y}(q_{\sigma(v)} - q_{\sigma(u)})$.

Similarly, for $H \in F_l^3$ or F_l^4, we have, by definition of H, $xq_{\sigma(u)} - yr_{\sigma(u)} - p_{\sigma(u)} \ge xq_{\sigma(v)} - yr_{\sigma(v)} - p_{\sigma(v)}$. Thus, we have $q_{\sigma(v)} \le q_{\sigma(u)} + \frac{1}{x}p_{\sigma(v)} - \frac{y}{x}(r_{\sigma(u)} - r_{\sigma(v)})$ and $r_{\sigma(u)} \le r_{\sigma(v)} + \frac{1}{y}p_{\sigma(v)} - \frac{x}{y}(q_{\sigma(v)} - q_{\sigma(u)})$.

(1) Since $x \ge 1$, we have $q_{\sigma(v)} \le q_{\sigma(u)} + p_{\sigma(u)}$.
Hence, $E^H \le r_{\sigma(u)} + q_{\sigma(v)} \le r_{\sigma(u)} + p_{\sigma(u)} + q_{\sigma(u)} \le T^*$, i.e., $\eta_H \le 2$.
To show that the bound is tight, consider the example with $n = (K/x) + (K/y) + 2$ and $0 < \epsilon < 1$ as shown in table 1.

Table 1

i	1	2	$\cdots$	$n-1$	n
r_i	$\frac{K}{y}$	0	$\cdots$	0	$\frac{K}{y}$
p_i	$1+\epsilon$	1	$\cdots$	1	$1-\epsilon$
q_i	$\frac{K}{x}$	0	$\cdots$	0	$\frac{K}{x}$

Then $\sigma^H = (1\ 2\ \ldots\ n)$ with $T^H = \frac{2K}{y} + \frac{2K}{x} + 2$, while $\sigma^* = \sigma_1\sigma_2$ with $T^* = \frac{K}{y} + \frac{K}{x} + 2$, where $\sigma_1 = (2\ 3\ \ldots\ \frac{K}{y} + 1)$ and $\sigma_2 = (1\ n\ n-1\ \ldots\ \frac{K}{y} + 2)$. Hence, $T^H/T^* \to 2$ as $K \to \infty$.

(2) Since, by lemma 1(1), $T^* \ge r_{\sigma(u)} + p_{\sigma(u)} + q_{\sigma(u)}$, we have $E^H \le \sum_{h=u}^{v} p_{\sigma(h)} - p_{\sigma(u)} + q_{\sigma(v)} \le$

$(2-\theta)T^*$. By lemma 1(3), we also have

$$E^H \le r_{\sigma(u)} + q_{\sigma(v)} \le r_{\sigma(v)} + r_{\sigma(u)} - r_{\sigma(v)} + q_{\sigma(v)} \le (1+\beta)T^*,$$

$$E^H \le r_{\sigma(u)} + q_{\sigma(v)} \le r_{\sigma(u)} + p_{\sigma(u)} + q_{\sigma(u)} + (\frac{1}{x} - 1)p_{\sigma(u)} - \frac{y}{x}(r_{\sigma(u)} - r_{\sigma(v)})$$
$$\le (1 + \frac{1-x}{x}\theta - \frac{y}{x}\beta)T^*,$$

$$E^H \le r_{\sigma(u)} + q_{\sigma(v)} \le r_{\sigma(u)} + p_{\sigma(u)} + q_{\sigma(u)} + q_{\sigma(v)} - q_{\sigma(u)} - p_{\sigma(u)}$$
$$\le (1 + \alpha - \theta)T^*$$

and

$$E^H \le r_{\sigma(u)} + q_{\sigma(v)} \le r_{\sigma(v)} + q_{\sigma(v)} + \frac{1}{y}p_{\sigma(u)} - \frac{x}{y}(q_{\sigma(v)} - q_{\sigma(u)})$$
$$\le (1 + \frac{1}{y}\theta - \frac{x}{y}\alpha)T^*.$$

Hence,

$$\frac{E^H}{T^*} \le \min\{2-\theta,\ 1+\beta,\ 1+\frac{1-x}{x}\theta - \frac{y}{x}\beta,\ 1+\alpha-\theta,\ 1+\frac{1}{y}\theta - \frac{x}{y}\alpha\} \le 2 - \frac{x+y}{1+y}.$$

Note that the last inequality becomes an equality when $\theta = \frac{x+y}{1+y}$, $\beta = \frac{1-x}{1+y}$ and $\alpha = 1$. Therefore, $\eta_H \le 3 - \frac{x+y}{1+y}$.

To show that the bound is tight, consider the example shown in table 2 with $n = 3$.

Table 2

i	1	2	3
r_i	$\frac{1-x}{1+y}K$	0	0
p_i	$\frac{x+y}{1+y}K+3$	$\frac{1-x}{1+y}K+2$	1
q_i	0	$\max\{0,\ \frac{2x+xy-1}{x(1+y)}K\}$	K

Then $\sigma^H = (1\ 2\ 3)$ with $T^H = (3 - \frac{x+y}{1+y})K + 6$, while $\sigma^* = (3\ 2\ 1)$ with $T^* = K+6$. Hence, $T^H/T^* \to 3 - \frac{x+y}{1+y}$ as $K \to \infty$.

(3) Since $y \ge 1$, we have $r_{\sigma(u)} \le r_{\sigma(v)} + p_{\sigma(v)}$.

Hence, $E^H \le r_{\sigma(u)} + q_{\sigma(v)} \le r_{\sigma(v)} + p_{\sigma(v)} + q_{\sigma(v)} \le T^*$, i.e., $\eta_H \le 2$.

To show that the bound is tight, see the example as shown in table 1.

(4) Similar to the proof in (2), we can show that

$$\frac{E^H}{T^*} \le \min\{2-\theta',\ 1+\alpha,\ 1+\frac{1-y}{y}\theta' - \frac{x}{y}\alpha,\ 1+\beta-\theta',\ 1+\frac{1}{x}\theta' - \frac{y}{x}\beta\} \le 2 - \frac{y+x}{1+x}.$$

Note that the last inequality achieves equality when $\theta' = \frac{y+x}{1+x}$, $\beta = 1$ and $\alpha = \frac{1-y}{1+x}$.

To show that the bound is tight, consider the example with $n = 3$ as shown in table 3.

Then $\sigma^H = (1\ 2\ 3)$ with $T^H = (3 - \frac{y+x}{1+x})K + 6$, while $\sigma^* = (3\ 2\ 1)$ with $T^* = K+6$. Hence, $T^H/T^* \to 3 - \frac{y+x}{1+x}$ as $K \to \infty$.

Table 3

i	1	2	3
r_i	K	$\max\{0, \frac{2y+xy-1}{y(1+x)}K\}$	0
p_i	1	$\frac{1-y}{1+x}K+2$	$\frac{x+y}{1+x}K+3$
q_i	0	0	$\frac{1-y}{1+x}K$

(5) Since $H \in F_l^5$, we have $q_{\sigma(u)} - yr_{\sigma(u)} \geq q_{\sigma(v)} - yr_{\sigma(v)}$, or alternatively, $q_{\sigma(u)} \geq q_{\sigma(v)} + y(r_{\sigma(u)} - r_{\sigma(v)})$. Since $y > 0$, we have either $q_{\sigma(u)} \geq q_{\sigma(v)}$ or $r_{\sigma(v)} > r_{\sigma(u)}$. In either case, we have, by lemma 1(4), $\eta_H \leq 2$.

To show that the bound is tight, see the example with $x = 1$ as shown in table 1.

(6) Since, by definition of H,

$$\frac{xq_{\sigma(u)} + p_{\sigma(u)}}{yr_{\sigma(u)} + p_{\sigma(u)}} \geq \frac{xq_{\sigma(v)} + p_{\sigma(v)}}{yr_{\sigma(v)} + p_{\sigma(v)}},$$

we have

$$\frac{xq_{\sigma(u)} - yr_{\sigma(u)}}{yr_{\sigma(u)} + p_{\sigma(u)}} \geq \frac{xq_{\sigma(v)} - yr_{\sigma(v)}}{yr_{\sigma(v)} + p_{\sigma(v)}}.$$

We have either $xq_{\sigma(u)} - yr_{\sigma(u)} \geq xq_{\sigma(v)} - yr_{\sigma(v)}$ or $yr_{\sigma(u)} + p_{\sigma(u)} \leq yr_{\sigma(v)} + p_{\sigma(v)}$. In the first case, we have, as in (5), $\eta_H \leq 2$. In the second case, we have $r_{\sigma(u)} \leq r_{\sigma(v)} + (1/y)p_{\sigma(v)} \leq r_{\sigma(v)} + p_{\sigma(v)}$ (since $y \geq 1$) and thus $E^H \leq r_{\sigma(u)} + q_{\sigma(v)} \leq r_{\sigma(v)} + p_{\sigma(v)} + q_{\sigma(v)} \leq T^*$, i.e., $\eta_H \leq 2$.

To show that the bound is tight, consider the example with $n = K^2 + 1$ as shown in table 4.

Table 4

i	1	2	$\cdots$	K^2	K^2+1
r_i	0	0	$\cdots$	0	0
p_i	$\frac{1}{K}$	$\frac{1}{K}$	$\cdots$	$\frac{1}{K}$	1
q_i	2	2	$\cdots$	2	K

Then $\sigma^H =(1\ 2 \ldots n)$ with $T^H = 2K + 1$, while $\sigma^* =(n \ldots 2\ 1)$ with $T^* = K + 3$. Hence, $T^H/T^* \to 2$ as $K \to \infty$. ◇

In the sense of worst-case performance, it is interesting to note that: (i) $I_i = xq_i + p_i$ is better than $I_i = yq_i - p_i$ (x, $y \geq 1$), (ii) $I_i = xq_i + p_i$ is better than $I_i = xq_i - p_i$ ($x > 0$), (iii) $I_i = -xr_i - p_i$ is better than $I_i = -yr_i + p_i$ (x, $y \geq 1$), (iv) $I_i = -xr_i - p_i$ is better than $I_i = -xr_i + p_i$ ($x, > 0$), (v) $I_i = q_i$ and $I_i = -r_i$ each are as good as the weighted index $I_i = q_i - xr_i$ ($x > 0$), and (vi) $I_i = \frac{q_i}{r_i}$ is as good as the weighted index $I_i = \frac{xq_i+p_i}{yr_i+p_i}$ ($x > 0$, $y \geq 1$).

3. Analysis of A Dynamic Linear Indexing Rule

In this section we analyze a dynamic linear rule, denoted by A, where we will use $I_i = q_i - r_i$ to prioritize jobs and allow the release date relative to the current time to be updated, so that jobs are dynamically sequenced. We note that heuristic A can be implemented in $O(n \log n)$ steps as follows. We define $(x)^+ = \max\{0, x\}$.

A : 1. Let S be an ordered set of $\{1, 2, \ldots, n\}$ arranged in nonincreasing order of the value of the index $I_i = q_i - r_i$. Let $N = 1$, $t_0 = 0$, $S_1 = \emptyset$. Let S' be an ordered set of $\{1, 2, \ldots, n\}$ arranged in nondecreasing order of r_i.

2. Let j_1 be the first job in S. If $S_N \neq \emptyset$, then go to step 3. Otherwise, set $j = j_1$ and go to step 4.

3. Let j_2 be the first job in S_N. If $q_{j_1} - (r_{j_1} - t_{N-1})^+ \geq q_{j_2}$, then $j = j_1$; otherwise, $j = j_2$. Go to step 4. (We note that $(r_{j_1} - t_{N-1})^+$ denotes the updated release date relative to time t_{N-1}, where t_{N-1} is the time at which the $(N-1)$-th job is ready for delivery. Also note that job j_2 is ready for processing and thus its updated release date is zero. Job j_1 has the hightest index value among those jobs which are not ready for processing at time t_{N-1} and job j_2 has the highest index value among those jobs which are ready for processing at time t_{N-1}. The *if-condition* tests to see which one job, j_1 or j_2, has the highest index value among the unsequenced jobs.)

4. Set $S = S - \{j\}$, $S' = S' - \{j\}$, $S_N = S_N - \{j\}$, $t_N = \max\{t_{N-1} + p_j,\ r_j + p_j\}$ and $\sigma(N) = j$. Go to step 5.

5. If $N = n$, then stop: $\sigma = (\sigma(1), \ldots, \sigma(n))$ is the generated sequence. Otherwise: set $N = N + 1$; set $S_N = S_{N-1} \cup \{h :\ h \in S',\ r_h \leq t_{N-1}\}$ and $S = S - S_N$, where S_N is an ordered set arranged in nonincreasing order of q_i; go to step 2. (We note that S_N contains the jobs which are ready for processing at time t_{N-1}.)

It is easy to verify that heuristic A runs in $O(n \log n)$ (e.g., Carlier [3], pp 45). As a preliminary to analyzing the worst-case behavior of heuristic A, we present some properties associated with heuristic A in the following three lemmas (2, 3 and 4) that are needed in our subsequent analysis. Let $T^A = r_{\sigma(u)} + \sum_{h=u}^{v} p_{\sigma(h)} + q_{\sigma(v)}$ and t_j represent the time at which job $\sigma(j)$ is ready for delivery, where $\sigma(\cdot)$ is the sequence generated by A. Note that, for $u \leq j \leq v$, $t_j = r_{\sigma(u)} + \sum_{h=u}^{j} p_{\sigma(h)}$. If $r_{\sigma(u)} < r_{\sigma(v)}$, let k be such that $r_{\sigma(v)} \leq t_k$ and $r_{\sigma(v)} > t_{k-1}$, and let $r_{\sigma(l)} = \min_{k+1 \leq h \leq v}\{r_{\sigma(h)}\}$.

We define the set of jobs between $\sigma(u)$ and $\sigma(v)$ as the *critical group*. We refer to this group as the critical group because the makespan is determined by this group. We note that if $r_{\sigma(u)} < r_{\sigma(v)}$, then job $\sigma(k)$ is the first job in the critical group for which job $\sigma(v)$ is available after $\sigma(k)$ has been processed.

Lemma 2.

(1) If $r_{\sigma(u)} < r_{\sigma(v)}$, then:
$q_{\sigma(h)} \geq q_{\sigma(v)}$ for $k+1 \leq h \leq v$; $q_{\sigma(k)} \geq q_{\sigma(v)}$ if $r_{\sigma(l)} \leq t_{k-1}$ also.

(2) If $r_{\sigma(k)} \leq r_{\sigma(u)} < r_{\sigma(v)}$, then $q_{\sigma(h)} \geq q_{\sigma(k)}$, $u \leq h \leq k$.

(3) If $r_{\sigma(l)} \leq r_{\sigma(u)} < r_{\sigma(v)}$, then $q_{\sigma(v)} = q_{min}$.

(4) If $r_{\sigma(u)} \geq r_{\sigma(v)}$, then $q_{\sigma(v)} = q_{min}$.

Proof. (1) Since $r_{\sigma(u)} < r_{\sigma(v)}$, we have, by definition of k, $r_{\sigma(v)} \leq t_k$. Thus, by definition of heuristic A, we have $q_{\sigma(h)} - (r_{\sigma(h)} - t_k)^+ \geq q_{\sigma(v)}$, $k+1 \leq h \leq v$, i.e., $q_{\sigma(h)} \geq q_{\sigma(v)}$, $k+1 \leq h \leq v$. In particular, we have $q_{\sigma(l)} \geq q_{\sigma(v)}$. Similarly, if $r_{\sigma(l)} \leq t_{k-1}$, then we have by definition of heuristic A, $q_{\sigma(k)} \geq q_{\sigma(l)}$. Therefore, $q_{\sigma(k)} \geq q_{\sigma(v)}$.

(2) Since, by definition of heuristic A, $q_{\sigma(u)} - (r_{\sigma(u)} - t_{u-1})^+ \geq q_{\sigma(k)} - (r_{\sigma(k)} - t_{u-1})^+$ and since $r_{\sigma(u)} \geq r_{\sigma(k)}$, we have, $q_{\sigma(u)} \geq q_{\sigma(k)}$. Also, since $(r_{\sigma(k)} - t_u)^+ = 0$, by definition of heuristic A, we know that $q_{\sigma(h)} \geq q_{\sigma(k)}$, $u+1 \leq h \leq k$. Hence, $q_{\sigma(h)} \geq q_{\sigma(k)}$, $u \leq h \leq k$.

(3) Similar to the proof in (2), we can show that $q_{\sigma(h)} \geq q_{\sigma(l)}$, $u \leq h \leq l$.
By (1), $q_{\sigma(h)} \geq q_{\sigma(v)}$, $k+1 \leq h \leq v$, so $q_{\sigma(v)} = q_{min}$.

(4) Following the proof of (2) and using v instead of k, we can obtain $q_{\sigma(h)} \geq q_{\sigma(v)}$, $u \leq h \leq v$, i.e., $q_{\sigma(v)} = q_{min}$. $\diamond$

Lemma 3.

If $r_{\sigma(u)} < r_{\sigma(v)}$, then $E^A \leq \max\{r_{\sigma(u)},\ t_{k-1}\} - \min\{r_{\sigma(k)}, r_{\sigma(l)}\} \leq \max\{r_{\sigma(u)},\ t_{k-1}\}$.

Proof. We claim that $T^* \geq \min\{r_{\sigma(k)}, r_{\sigma(l)}\} + \sum_{h=k}^{v} p_{\sigma(h)} + q_{\sigma(v)}$. By rewriting $T^A = \max\{r_{\sigma(u)},\ t_{k-1}\} + \sum_{h=k}^{v} p_{\sigma(h)} + q_{\sigma(v)}$ and then subtracting the above inequality, the results follow.

Let T_k^* be the optimal objective value by only considering jobs $\sigma(k), \sigma(k+1), \ldots, \sigma(v)$. We will show that $T_k^* \geq \min\{r_{\sigma(k)}, r_{\sigma(l)}\} + \sum_{h=k}^{v} p_{\sigma(h)} + q_{\sigma(v)}$, which will yield the desired result, since $T^* \geq T_k^*$.

Case 1: $r_{\sigma(l)} \leq \max\{r_{\sigma(u)},\ t_{k-1}\}$. The inequality holds since, by lemmas 2(1) and 2(3), $q_{\sigma(v)} = \min_{k \leq h \leq v}\{q_{\sigma(h)}\}$.

Case 2: $r_{\sigma(l)} > \max\{r_{\sigma(u)},\ t_{k-1}\}$. Since we know that

$T_k^* \geq \min\{r_{\sigma(k)} + \sum_{h=k}^{v} p_{\sigma(h)} + q_{\sigma(v)},\ r_{\sigma(l)} + \sum_{h=k}^{v} p_{\sigma(h)} + q_{\sigma(k)}\}$,

the inequality holds if we can show that $r_{\sigma(k)} + q_{\sigma(v)} \leq r_{\sigma(l)} + q_{\sigma(k)}$. We then consider two cases: $k = u$ (i.e., $r_{\sigma(k)} \geq t_{k-1}$) and $k > u$ (i.e., $r_{\sigma(k)} \leq t_{k-1} < r_{\sigma(l)}$). If $k = u$, we have, by definition of heuristic A at time t_{k-1},

$q_{\sigma(k)} - (r_{\sigma(k)} - t_{k-1})^+ \geq q_{\sigma(l)} - (r_{\sigma(l)} - t_{k-1})^+ \geq q_{\sigma(v)} - (r_{\sigma(l)} - t_{k-1})^+$

since $q_{\sigma(l)} \geq q_{\sigma(v)}$. Alternatively, we have $q_{\sigma(k)} + r_{\sigma(l)} \geq q_{\sigma(v)} + r_{\sigma(k)}$ since $r_{\sigma(l)} > r_{\sigma(k)}$ $(= r_{\sigma(u)}) > t_{k-1}$. If $k > u$, we have, by definition of heuristic A at time t_{k-1}, $q_{\sigma(k)} \geq q_{\sigma(l)} - (r_{\sigma(l)} - t_{k-1}) \geq q_{\sigma(v)} - (r_{\sigma(l)} - t_{k-1}) \geq q_{\sigma(v)} - (r_{\sigma(l)} - r_{\sigma(k)})$, and thus, $q_{\sigma(k)} + r_{\sigma(l)} \geq q_{\sigma(v)} + r_{\sigma(k)}$. $\diamond$

Lemma 4. If $r_{\sigma(u)} = r_{min}$, then $r_{\sigma(v)} \geq 2E^A$.

Proof. Since, $r_{\sigma(u)} = r_{min}$, we have, by lemma 1(3), $E^A \leq q_{\sigma(v)} - q_{min}$. We only need to consider the case: $r_{\sigma(u)} < \min\{r_{\sigma(l)},\ r_{\sigma(v)}\}$; otherwise, we have $q_{\sigma(v)} = q_{min}$ by lemmas 2(3) and 2(4). We consider two cases: $r_{\sigma(u)} < r_{\sigma(l)} \leq t_{k-1}$ and $r_{\sigma(l)} > t_{k-1}$.

Case 1: $r_{\sigma(u)} < r_{\sigma(l)} \leq t_{k-1}$.

Since in this case $r_{\sigma(l)} \leq t_{k-1}$, we have, by lemma 2(1), $q_{\sigma(k)} \geq q_{\sigma(v)}$. Letting $q_{min} = q_{\sigma(j)}$, $j \neq v$, we have $j < k$ by lemma 2(1) and using $q_{\sigma(k)} \geq q_{\sigma(v)}$. By definition of heuristic A at t_{j-1}, for $i = k,\ l$, we have:

$q_{min} \geq q_{\sigma(i)} - (r_{\sigma(i)} - t_{j-1})^+ \geq q_{\sigma(i)} - (r_{\sigma(i)} - r_{\sigma(u)})$ if $u < j < k$; and

$q_{min} - (r_{\sigma(u)} - t_{u-1})^+ \geq q_{\sigma(i)} - (r_{\sigma(i)} - t_{u-1})^+$ if $j = u$.

Alternatively, we have, for $i = k,\ l$, $r_{\sigma(i)} - r_{\sigma(u)} \geq q_{\sigma(i)} - q_{min} \geq q_{\sigma(v)} - q_{min}$.

Hence, by letting $E_2 = \min\{r_{\sigma(k)},\ r_{\sigma(l)}\} - r_{\sigma(u)}$, we have $E^A \leq q_{\sigma(v)} - q_{min} \leq E_2$.

Letting $E_1 = t_{k-1} - \min\{r_{\sigma(l)}, r_{\sigma(k)}\}$, by lemma 3, we have $E^A \leq E_1$.

Since $E_1 + E_2 \leq \max\{r_{\sigma(u)},\ t_{k-1}\} \leq r_{\sigma(v)}$, we have $r_{\sigma(v)} \geq 2E^A$.

Case 2: $r_{\sigma(l)} > t_{k-1}$. Letting $q_{min} = q_{\sigma(j)}$, $j \neq v$, we have, by lemma 2(1), $u \leq j \leq k$.

By definition of heuristic A, we have

$$q_{min} \geq q_{\sigma(k)} - (r_{\sigma(k)} - t_{j-1})^+ \quad \text{and} \quad q_{\sigma(k)} \geq q_{\sigma(v)} - (r_{\sigma(v)} - t_{k-1})^+$$

and thus $r_{\sigma(v)} \geq q_{\sigma(v)} - q_{min} + (r_{\sigma(k)} - t_{j-1})^+ + t_{k-1}$. Since

$$(r_{\sigma(k)} - t_{j-1})^+ + t_{k-1} \geq \max\{r_{\sigma(u)},\ t_{k-1}\} \geq E^A \text{ (by lemma 3)},$$

we have $r_{\sigma(v)} \geq E^A + E^A = 2E^A$. $\diamond$

We now present our worst-case results for heuristic A.

Theorem 2. (1) $\eta_A = 2$.

(2) If $r_{\sigma(u)} = r_{min}$, then $\eta_A = 4/3$.

(3) If $r_{\sigma(u)} = r_{min}$ and $q_{\sigma(v)} \geq r_{\sigma(v)}$, then $\eta_A = 5/4$.

Proof. (1) If $r_{\sigma(u)} \leq r_{\sigma(v)}$, we have, by lemma 1(4), $\eta_A \leq 2$. Consider the case: $r_{\sigma(u)} > r_{\sigma(v)}$. In this case, we have $q_{\sigma(u)} > q_{\sigma(v)}$ and thus, by lemma 1(4), $\eta_A \leq 2$. To show that this bound is tight, consider the example with $n = K + 1$ and $K\epsilon < 1$ as shown in table 5.

Then $\sigma^A = (1\ 2 \ldots n)$ with $T^A = 2K+1+\epsilon$, while $\sigma^* = (n\ 1\ 2 \ldots n-1)$ with $T^* = K+2+K\epsilon$. Hence, $T^A/T^* \to 2$ as $K \to \infty$.

Table 5

i	1	2	$\cdots$	i	$\cdots$	K	$K+1$
r_i	1	2	$\cdots$	i	$\cdots$	K	0
p_i	ϵ	ϵ	$\cdots$	ϵ	$\cdots$	ϵ	K
q_i	2	2	$\cdots$	2	$\cdots$	2	0

(2) By only considering job $\sigma(v)$ and then by lemmas 4 and 1(3), we have $T^* \geq r_{\sigma(v)} + p_{\sigma(v)} + q_{\sigma(v)} > 2E^A + q_{\sigma(v)} \geq 3E^A$, i.e., $\eta_A \leq 4/3$. To show that the bound is tight, consider the example with $n = 3$ and $K > 2$ as shown in table 6.

Table 6

i	1	2	3
r_i	0	0	2K
p_i	K	$2K$	1
q_i	2	1	K

Then $\sigma^A = (1\ 2\ 3)$ with $T^A = 4K + 1$, while $\sigma^* = (2\ 3\ 1)$ with $T^* = 3K + 3$. Hence, $T^A/T^* \to \frac{4}{3}$ as $K \to \infty$.

(3) By only considering job $\sigma(v)$ and then by lemma 4, we have $T^* \geq r_{\sigma(v)} + p_{\sigma(v)} + q_{\sigma(v)} > 2r_{\sigma(v)} \geq 4E^A$, i.e., $\eta_A \leq 5/4$.
To show that the bound is tight, consider the example with $n = 3$ as shown in table 7.

Table 7

i	1	2	3
r_i	0	0	$2K$
p_i	K	$2K$	1
q_i	$K+2$	$K+1$	$2K$

Then $\sigma = (1\ 2\ 3)$ with $T^A = 5K + 1$, while $\sigma^* = (2\ 3\ 1)$ with $T^* = 4K + 3$. Hence, $T^A/T^* \to \frac{5}{4}$ as $K \to \infty$. $\diamond$

4. Conclusion

It seems appealing to use indexing rules for sequencing jobs. Therefore, one interesting question would be the optimal design of an indexing rule in the sense of worst-case performance or average-case performance. This paper partially addressed that issue in the sense of worst-case performance for the single machine sequencing problem with release dates and delivery times. One further research avenue would be to investigate the worst-case performance for more general indexing rules beyond linear and quotient forms.

We designed and analyzed one family of weighted linear rules and one family of weighted quotient rules. All the rules in F_l^1, F_l^3, F_l^5 and F_q have the same worst-case performance. Another research avenue would be to conduct probabilistic analysis and then identify the rule with the best average-case performance.

For heuristic A, our analysis revealed that the worst-case performance ratio is 2 for all the admissible input space, 4/3 in one subregion ($r_{\sigma(u)} = r_{min}$) and 5/4 in another subregion ($r_{\sigma(u)} = r_{min}$ and $q_{\sigma(v)} \geq r_{\sigma(v)}$). This suggests one way to improve the performance of a heuristic: if we can modify a heuristic such that it terminates in a subregion with a better worst-case performance ratio, then the modified heuristic will have a better performance.

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