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A TWO-ARMED BANDIT PROBLEM WITH ONE ARM KNOWN UNDER SOME CONSTRAINTS

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Abstract A finite-time horizon two-armed bandit problem with one arm known, in which there are constraints on the number of times each arm may be pulled, is considered. The loss incurred at the stage when there are n stages remaining is multiplied by the factor β_n, and the objective is to minimize the total expected weighted loss over n stages. This problem is formulated by dynamic programming and the critical value function which specifies the optimal strategy is derived along with its monotonicity properties. The recursive equation is solved explicitly in the case of the exponential distribution. Tables of the critical values are obtained for special cases of $\{\beta_n\}$.

1. Introduction

Suppose that there are n sequential stages remaining and there are two arms a_0 and a_1 which can be pulled at most k and l times, respectively, where $1 \leq k \leq n$, $1 \leq l \leq n$ and $k + l \geq n + 1$. In each stage one of two arms a_0 or a_1 is pulled. By pulling a_0, an observation z is obtained from the distribution with density $f(z|u)$ and a cost z is incurred as the result, where u is unknown. The expected cost incurred by pulling a_1 is 1. There is the prior knowledge that u has the conjugate prior distribution with density $g(u|x, y)$. The cost obtained when there are n stages remaining is multiplied by a positive constant β_n, that is, $\beta_n, \beta_{n-1}, \cdots, \beta_1$ have the role of discount factors which depend upon the number of remaining stages. The objective is to minimize the discounted total expected cost.

This problem is an extension of the finite-horizontal classical two-armed bandit problem with one arm known, where $n = k = l$. In the real world, the constraints $1 \leq k \leq n$ and/or $1 \leq l \leq n$ are sometimes important. For example, in the clinical trial, n patients will arrive one by one sequentially and for each patient, one of two treatments a_0 and a_1 is selected, where the treatments a_0 (or a_1) can be used at most k (or l) times, because of the amount of medicine available. Another example is to minimize the total expected flowtime of jobs in the single machine production system, where the same kind of products are produced by attaching one of two kinds of tools, a_0 and a_1, to the machine. Tool a_0 is available for at most k products and tool a_1 is available for at most l products. Tool a_1 has been used for a long time and therefore its mean processing time is known. Tool a_0 is new and one must decide whether it is to be used in place of a_1 or not. The processing time when a_0 is used has the density $f(z|u)$. Although u is unknown, we have a prior knowledge that u has the conjugate prior distribution with density $g(u|x, y)$. In this case, $\beta_i = i$ for $1 \leq i \leq n$.

The special case, $k + l = n$, is the sequencing problem and minimization of the expected weighted sum of flowtimes is discussed in Hamada and Glazebrook [4]. Since the critical values derived in [4] depend only on k and l and not on n, the index policy is useful in that case and the extended problem with more than two kinds of jobs is solved. In this paper, we derive the critical value which depends not only on (k, l) but also on n, and therefore the index policy is not useful in this problem.

In Section 2, the problem is formulated by the principle of optimality of dynamic programming and a stopping property is derived. In Section 3, the optimal strategy is specified by using the critical value function. Properties of this function are derived. In Section 4, the optimal strategy for the case of the exponential distribution is specified by solving the recursive equations explicitly.

2. Formulation by dynamic programming.

The state vector is denoted by $(n; k, l; x, y)$, where n denotes how many stages are remaining, k denotes how many times a_0 is available, l denotes how many times a_1 is available, and (x, y) is the parameters of the prior distribution. As the matter of convenience, $(n; k, l; x, y)$ also denotes the problem with state vector $(n; k, l; x, y)$. Let $C_n = \{(k, l) | 1 \leq k \leq n, 1 \leq l \leq n,\ k + l \geq n + 1\}$ and $S = S_x \times S_y$, where S is the state space of (x, y) and also S_x and S_y are the spaces of x and y, respectively. Let $(\varphi(x, y; Z), \psi(x, y; Z))$ be the parameters of the posterior distribution of u after obtaining Z by pulling a_0 when the parameters of the prior distribution of u is (x, y). Let

$$h(z|x, y) = \int f(z|u) g(u|x, y) du$$

and

$$R(x, y) = E[Z|x, y] = \int z h(z|x, y) dz.$$

Then, it is easily derived that

$$E[R(\varphi(x, y; Z), \psi(x, y; Z)) | x, y] = R(x, y) \tag{1}$$

for $(x, y) \in S$.

Now, we make the following assumptions as in Hamada [3]:

- A_1 : $\varphi(x, y; z)$ is continuous in x, nondecreasing in x and z and nonincreasing in y and $\psi(x, y; z)$ is continuous in y, nonincreasing in x and z and nondecreasing in y.
- A_2 : $h(z|x', y)/h(z|x, y) \leq h(z'|x', y) / h(z'|x, y)$ for any $z < z'$ when $x' > x$ and $h(z|x, y)/h(z|x, y') \leq h(z'|x, y)/h(z'|x, y')$ for any $z < z'$ when $y' > y$ (This property is called the likelihood ratio ordering: See, for example, Ross [6]).
- A_3 : $R(x, y) > 0$ for $(x, y) \in S$.
- A_4 : For $c > 0$ and $y > 0$, there exists $x_1 \in S_x$ such that $R(x_1, y) > c$ and also there exists $x_2 \in S_x$ such that $0 < R(x_2, y) < c$.

The monotonicity properties that $R(x, y)$ is continuous and strictly increasing in x and continuous and strictly decreasing in y are derived from A_2 (See, for example, the proof of Proposition 5.4 of Ross [6] or Appendix of Rosenfield and Shapiro [5]). Let $G_n(k, l; x, y)$ be the minimum expected total cost incurred in the remaining n stages when the current prior knowledge of u is (x, y). Then, the problem is formulated by dynamic programming as follows:

$$G_n(k, l; x, y) = \min\{G_n^0(k, l; x, y), G_n^1(k, l; x, y)\} \tag{2}$$

for $n = 1, 2, 3, \cdots$ and $G_0(k, l; x, y) = 0$, where

$$G_n^0(k, l; x, y) = R(x, y)\beta_n + E[G_{n-1}(k - 1, \min\{l, n - 1\}; \varphi(x, y; Z), \psi(x, y; Z)) | x, y] \tag{3}$$

and

$$G_n^1(k,l;x,y) = \beta_n + G_{n-1}(\min\{k, n-1\}, l-1; x,y).$$

In these equations, $G_n^i(k,l;x,y)(i = 0, 1)$ denotes the minimum expected total cost incurred when the optimal strategy is followed after the pull of a_i at the first stage. Now, consider the stopping property described as follows: Once a_1 is optimal at any stage, then it is also optimal in the subsequent $l-1$ stages. By Berry and Fristedt [1], whether or not this property holds for the classical bandit problem with one arm known depends upon the sequence $(\beta_n, \beta_{n-1}, \cdots, \beta_1)$. Now we define $\gamma_j = \beta_j + \beta_{j-1} + \cdots + \beta_1$. In [1], the sequence $(\beta_n, \beta_{n-1}, \cdots, \beta_1)$ is called regular if it satisfies the relation $\gamma_1/\gamma_2 \le \gamma_2/\gamma_3 \le \cdots \le \gamma_{n-1}/\gamma_n$. For example, $(\alpha^n, \alpha^{n-1}, \cdots, \alpha)$ is regular for $\alpha > 1$ since it satisfies this relation. By slightly modifying the proof of Theorem 5.2.2 of [1], the following theorem is obtained.

THEOREM 1. *Suppose that $\beta_n \ge \beta_{n-1} \ge \cdots \ge \beta_1 > 0$ and $(\beta_n, \beta_{n-1}, \cdots, \beta_1)$ is regular. Then, if a_1 is optimal in the first stage of $(n; k, l; x, y)$, a_1 will also be optimal in the subsequent $l-1$ stages.*

Now, let the set of the sequences $(\beta_n, \beta_{n-1}, \cdots, \beta_1)$ which satisfies both the stopping property and the relation $\beta_n \ge \beta_{n-1} \ge \cdots \ge \beta_1 > 0$ be denoted by B_n. If $(\beta_n, \beta_{n-1}, \cdots, \beta_1) \in B_n$, then

$$G_n^1(k,l;x,y) = \sum_{j=n-l+1}^{n} \beta_j + R(x,y)\sum_{j=1}^{n-l} \beta_j$$

whenever $G_n^0(k,l;x,y) \ge G_n^1(k,l;x,y)$ and

$$G_n^1(k,l;x,y) \le \sum_{j=n-l+1}^{n} \beta_j + R(x,y)\sum_{j=1}^{n-l} \beta_j$$

whenever $G_n^0(k,l;x,y) \le G_n^1(k,l;x,y)$. Therefore (2) is rewritten as follows:

$$G_n(k,l;x,y) = \min\{G_n^0(k,l;x,y), \sum_{j=n-l+1}^{n} \beta_j + R(x,y)\sum_{j=1}^{n-l} \beta_j\} \tag{4}$$

where $\sum_{j=m}^{n} \beta_j = 0$ if $m > n$.

3. Optimal strategy.

Let

$$H_n(k,l;x,y) = G_n^0(k,l;x,y) - \sum_{j=n-l+1}^{n} \beta_j - R(x,y)\sum_{j=1}^{n-l} \beta_j$$

for $n \ge 1$, $(k,l) \in C_n$ and $(x,y) \in S$. Then, a_0 is optimal if $H_n(k,l;x,y) \le 0$ and a_1 is optimal if $H_n(k,l;x,y) \ge 0$. We define $H_0(0,0;x,y) = 0$ for $(x,y) \in S$ and also $H_n(0,n;x,y) = 0$ and $H_n(n,0;x,y) = 0$ for $n \ge 1$ and $(x,y) \in S$. Furthermore, let $H_n^-(k,l;x,y) = \min\{H_n(k,l;x,y), 0\}$ for $k \ge 0$, $l \ge 0$, $n \ge 0$ and $(x,y) \in S$. If we define $\beta_0 = 0$, the following lemma is derived.

LEMMA 1. *For $n \ge 1$, $(k,l) \in C_n$, $(x,y) \in S$ and $(\beta_n, \beta_{n-1}, \cdots, \beta_1) \in B_n$,*

$$\begin{aligned} H_n(k,l;x,y) =& \{R(x,y) - 1\}(\beta_n - \beta_{n-l}) \\ &+ E[H_{n-1}^-(k-1, \min\{l, n-1\}; \varphi(x,y;Z), \psi(x,y;Z))|x,y]. \end{aligned} \tag{5}$$

PROOF. Since $C_1 = \{(1,1)\}$, the assertion is trivial for $n = 1$. Suppose that (5) holds for some $n \geq 1$. Then, for $(k,l) \in C_{n+1}$,

$$H_{n+1}(k,l;x,y) = G^0_{n+1}(k,l;x,y) - \sum_{j=n-l+2}^{n+1} \beta_j - R(x,y) \sum_{j=1}^{n+1-l} \beta_j$$

from which

$$\begin{aligned} H_{n+1}(k,l;x,y) =& R(x,y)\beta_{n+1} + E[G_n(k-1,\min\{l,n\};\varphi(x,y;Z),\psi(x,y;Z))|x,y] \\ & - \sum_{j=n-l+2}^{n+1} \beta_j - R(x,y) \sum_{j=1}^{n+1-l} \beta_j. \end{aligned}$$

Since

$$\begin{aligned} & G_n(k-1,\min\{l,n\};\varphi(x,y;Z),\psi(x,y;Z)) \\ &\quad = H^-_n(k-1,\min\{l,n\};\varphi(x,y;Z),\psi(x,y;Z)) \\ &\quad + \sum_{j=n-min\{l,n\}+1}^{n} \beta_j + R(\varphi(x,y;Z),\psi(x,y;Z)) \sum_{j=1}^{n-min\{l,n\}} \beta_j, \end{aligned}$$

the following equation is obtained by using (1),

$$\begin{aligned} H_{n+1}(k,l;x,y) =& \{R(x,y)-1\}(\beta_{n+1}-\beta_{n+1-l}) \\ & + E[H^-_n(k-1,\min\{l,n\};\varphi(x,y;Z),\psi(x,y;Z))|x,y] \end{aligned}$$

which completes the proof. □

In the case of $\beta_n = \beta_{n-1} = \cdots = \beta_1 = 1$, it is derived by induction on n that

$$G^0_n(k,l;x,y) = G^0_{n-1}(k,l-1;x,y) + 1$$

for $n \geq 2$, $(k,l) \in C_n$ with $k \leq n-1$ and $(x,y) \in S_n$. From this equation,

$$H_n(k,l;x,y) = H_{n-1}(k,l-1;x,y).$$

By using this relation subsequently,

$$H_n(k,l;x,y) = H_{n-1}(k,l-1;x,y) = \cdots = H_k(k,l-n+k;x,y)$$

This means that it is optimal for $(n;k,l;x,y)$ to assign a_1 at the last $n-k$ stages. Therefore, we have to consider only the case of $(k;k,l-n+k;x,y)$ in place of $(n;k,l;x,y)$ at the outset. Also, from Lemma 1,

$$\begin{aligned} H_k(k,l-n+k;x,y) &= H_k(k,k;x,y) \\ &= \{R(x,y)-1\} + E[H^-_{k-1}(k-1,k-1;\varphi(x,y;Z),\psi(x,y;Z))|x,y] \end{aligned}$$

if $l = n$ and

$$H_k(k,l-n+k;x,y) = E[H^-_{k-1}(k-1,l-n+k;\varphi(x,y;Z),\psi(x,y;Z))|x,y]$$

if $1 \leq l-n+k \leq k-1$. Therefore, if $l = n$, then whether a_0 is optimal or not in the first stage depends upon (x,y). But, if $1 \leq l \leq n-1$, $H_k(k,l-n+k;x,y) \leq 0$ for $n \geq 1$, $n \geq k \geq n-l+1$ and $(x,y) \in S$, that is, it is optimal to pull a_0 in the first subsequent

$n-l$ stages with neglecting the existence of a_1. In both cases, the problem reduces to the classical two-armed bandit problem with finite horizon.

In the case of $\beta_n > \beta_{n-1} > \cdots > \beta_1 > 0$, let B_n^* be the set of $(\beta_n, \beta_{n-1}, \cdots, \beta_1) \in B_n$ such that $\beta_n > \beta_{n-1} > \cdots > \beta_1 > 0$. Then, the following lemma gives the monotonicity of $H_n(k,l;x,y)$ in x and y.

LEMMA 2. *For $n \geq 1$, $(k,l) \in C_n$ and $(\beta_n, \beta_{n-1}, \cdots, \beta_1) \in B_n^*$,*

(i) *$H_n(k,l;x,y)$ is continuous and strictly increasing in x.*
(ii) *$H_n(k,l;x,y)$ is continuous and strictly decreasing in y.*

PROOF. Since $H_1(1,1;x,y) = \{R(x,y)-1\}\beta_1$, both (i) and (ii) are true for $n=k=l=1$. Suppose that both (i) and (ii) are true for $n \geq 1$, $(k,l) \in C_n$ *and* $(\beta_n, \beta_{n-1}, \cdots, \beta_1) \in B_n^*$. Then,

$$\begin{aligned} H_{n+1}(k,l;x,y) =& \{R(x,y)-1\}(\beta_{n+1}-\beta_{n+1-l}) \\ &+ \int H_n^-(k-1, \min\{l,n\}; \varphi(x,y;Z), \psi(x,y;Z))h(z|x,y)dz \end{aligned}$$

For $x \in S_x$ and $x' \in S_x$ such that $x' > x$, $h(z|x',y)/h(z|x,y) \leq h(z'|x',y)/h(z'|x,y)$ for any $z < z'$, there exists at least one z^* such that $h(z|x',y)/h(z|x,y) \leq 1$ if $z \leq z^*$ and $h(z|x',y)/h(z|x,y) > 1$ if $z > z^*$. Then,

$$\begin{aligned} H_{n+1}(k,l;x,y) =& \{R(x,y)-1\}(\beta_{n+1}-\beta_{n+1-l}) \\ &+ \int_{\{z|z\leq z^*\}} H_n^-(k-1, \min\{l,n\}; \varphi(x,y;z), \psi(x,y;z))h(z|x,y)dz \\ &+ \int_{\{z|z> z^*\}} H_n^-(k-1, \min\{l,n\}; \varphi(x,y;z), \psi(x,y;z))h(z|x,y)dz. \end{aligned}$$

Since $(\beta_{n+1}, \beta_n, \cdots, \beta_1) \in B_n^*$, the first term of the right hand side is strictly increasing in x. The nonincreasing property of the sum of the second and third terms can be proved in the same way as in the proof of Lemma 1 of Hamada [3]. □

The sequence $(\beta_n, \beta_{n-1}, \cdots, \beta_1)$ has a useful property which is given in Lemma 3.

LEMMA 3. *For $n \geq 1$, $(k,l) \in C_n$ and $(\beta_n, \beta_{n-1}, \cdots, \beta_1) \in B_n^*$,*

$$\sum_{j=n-l+1}^{n} \beta_j - \sum_{j=1}^{n-k} \beta_j > 0.$$

PROOF. For $n \geq 1$, since $0 \leq n-k \leq l-1$ holds for $(k,l) \in C_n$, $\sum_{j=n-l+1}^{n} \beta_j - \sum_{j=1}^{n-k} \beta_j \geq \sum_{j=n-l+1}^{n} \beta_j - \sum_{j=1}^{l-1} \beta_j$, the right hand side of this inequality is positive because of the fact that $(\beta_n, \beta_{n-1}, \cdots, \beta_1) \in B_n^*$. This completes the proof. □

LEMMA 4. *For $n \geq 1$, $(k,l) \in C_n$ and $(\beta_n, \beta_{n-1}, \cdots, \beta_1) \in B_n^*$,*

$$R(x,y)(\beta_n - \beta_{n-l}) - \sum_{j=n-l+1}^{n} \beta_j \leq H_n(k,l;x,y) \leq \{R(x,y)-1\}\left(\sum_{j=n-l+1}^{n} \beta_j - \sum_{j=1}^{n-k} \beta_j\right). \tag{6}$$

PROOF. From (5) and $C_1 = \{(1,1)\}$, (6) is trivial for $n = 1$. Suppose that (6) holds for some $n \geq 1$. Since

$$H_{n+1}(1, n+1; x, y) = \{R(x,y) - 1\}(\beta_{n+1} - \beta_0)$$

and

$$\begin{aligned} H_{n+1}(k, l; x, y) =& \{R(x,y) - 1\}(\beta_{n+1} - \beta_{n+1-l}) \\ &+ E[H_n^-(k-1, \min\{l, n\}; \varphi(x,y;Z), \psi(x,y;Z))|x,y] \end{aligned}$$

for $(k,l) \in C_{n+1}$ with $k \geq 2$, it is easily derived by using the inductive hypothesis and Lemma 3 that

$$R(x,y)(\beta_{n+1} - \beta_{n+1-l}) - \sum_{j=n+1-l+1}^{n+1} \beta_j \leq H_{n+1}(k,l;x,y)$$

$$\leq \{R(x,y) - 1\}\left(\sum_{j=n+1-l+1}^{n+1} \beta_j - \sum_{j=1}^{n+1-k} \beta_j\right),$$

which completes the proof. □

The next theorem gives the existence and the uniqueness of the critical value function $s_n(k,l;y)$ which specifies the optimal strategy and also it shows that $s_1(1,1;y)$ is the lower bound of the critical value function.

THEOREM 2. *For $n \geq 1$, $(k,l) \in C_n$, $y \in S_y$ and $(\beta_n, \beta_{n-1}, \cdots, \beta_1) \in B_n^*$, the equation, $H_n(k,l;x,y) = 0$, of x has a unique root $s_n(k,l;y)$ which is strictly increasing in y and $s_n(k,l;y) \geq s_1(1,1;y)$.*

PROOF. From Lemmas 3 and 4 and A_4, there exists $x_1 \in S_x$ such that $R(x_1, y) > (\beta_n - \beta_{n-l})^{-1} \sum_{j=n-l+1}^{n} \beta_j$ and also there exists $x_2 \in S_x$ such that $R(x_2, y) < 1$. From these inequalities and (i) of Lemma 2, the existence and the uniqueness of $s_n(k,l;y)$ is trivial. From $H_n(k,l;s_n(k,l;y),y) = 0$ and (ii) of Lemma 2, $s_n(k,l;y)$ is strictly increasing in y. From the second inequality of (6), $H_n(k,l;s_1(1,1;y),y) \leq 0$, which means from (i) of Lemma 2 that $s_n(k,l;y) \geq s_1(1,1;y)$. □

From this theorem, for the state $(n;k,l;x,y)a_0$ is optimal if $x \leq s_n(k,l;y)$ and a_1 is optimal if $x \geq s_n(k,l;y)$.

LEMMA 5. *For $n \geq 1$, $(k,l) \in C_n$ with $k \leq n-1$, $(x,y) \in S$, and $(\beta_n, \beta_{n-1}, \cdots, \beta_1) \in B_n^*$,*

$$H_n(k,l;x,y) \geq H_n(k+1,l;x,y).$$

PROOF. Since $C_2 = \{(1,2),(2,2),(2,1)\}$,

$$H_2(1,2;x,y) = \{R(x,y) - 1\}\beta_2$$

and

$$H_2(2,2;x,y) = \{R(x,y) - 1\}\beta_2 + E[H_1^-(1,1;\varphi(x,y;Z),\psi(x,y;Z))|x,y],$$

the assertion is true for $n = 2$. The assertion for $n \geq 2$ is immediately derived from (5). □

The next theorem presents a monotonicity property of the critical value function $s_n(k, l; y)$ in n, k and l.

THEOREM 3. *For $n \geq 1$, $y \in S_y$ and $(\beta_n, \beta_{n-1}, \cdots, \beta_1) \in B_n^*$,*

(i) $s_1(1, 1; y) = s_n(1, n; y)$,
(ii) $s_n(n, n; y) \leq s_{n+1}(n + 1, n + 1; y)$,
(iii) $s_n(k, l; y) \leq s_n(k + 1, l; y)$ *for* $(k, l) \in C_n$ *with* $k \leq n - 1$,
(iv) $s_n(k, n; y) \leq s_n(k, n - 1; y)$ *for* $2 \leq k \leq n$, *and*
(v) $s_{n-1}(k, l - 1; y) \leq s_n(k, l; y)$ *for* $(k, l) \in C_n$ *with* $l \geq 2$.

PROOF. (i) is the immediate consequence of

$$\begin{aligned} H_n(1, n; x, y) &= \{R(x, y) - 1\}\beta_n \\ &= H_1(1, 1; x, y)\beta_n/\beta_1. \end{aligned}$$

(ii) is proved by the same way as Theorem 2.3 of Hamada [2]. From the definition of $s_n(k, n; y)$ and Lemma 5,

$$\begin{aligned} H_n(k + 1, l; s_n(k + 1, l; y), y) &= 0 \\ &= H_n(k, l; s_n(k, l; y), y) \\ &\geq H_n(k + 1, l; s_n(k, l; y), y), \end{aligned}$$

from which (iii) is easily obtained by using (i) of Lemma 2. Since

$$H_n(k, n; x, y) = \{R(x, y) - 1\}(\beta_n - \beta_0) + E[H_{n-1}^-(k - 1, n - 1; \varphi(x, y; Z), \psi(x, y; Z))|x, y]$$

and

$$H_n(k, n - 1; x, y) = \{R(x, y) - 1\}(\beta_n - \beta_1) + E[H_{n-1}^-(k - 1, n - 1; \varphi(x, y; Z), \psi(x, y; Z))|x, y],$$

it holds that

$$H_n(k, n; x, y) - H_n(k, n - 1; x, y) = \{R(x, y) - 1\}(\beta_1 - \beta_0).$$

This equality means that for $x > s_1(1, 1; y)$

$$H_n(k, n; x, y) \geq H_n(k, n - 1; x, y),$$

from which $s_n(k, n; y) \leq s_n(k, n - 1; y)$ is derived by (i) of Lemma 2 and Theorem 2. (v) is trivial from Theorem 1. □

This theorem describes several properties for the problem of the state $(n; k, l; x, y)$ with $\beta_n > \beta_{n-1} > \cdots > \beta_1$: (i) denotes that a_0 is optimal for $(n; 1, n; x, y)$ if and only if a_0 is optimal for $(1; 1, 1; x, y)$. (ii) denotes that if a_0 is optimal for the state $(n; n, n; x, y)$, then it is also optimal for the state $(n+1; n+1, n+1; x, y)$. (iii) denotes that if a_0 is optimal for the state $(n; k, l; x, y)$ with $1 \leq k \leq n - 1$, then it is also optimal for the state $(n; k + 1, l; x, y)$. (iv) denotes that if a_0 is optimal for the state $(n; k, n; x, y)$ with $2 \leq k \leq n$, it is also optimal for the state $(n; k, n - 1; x, y)$. (v) is the property denoted in Theorem 1.

Remark. The monotonicity of $s_n(k,l;y)$ in l is not true for $(k,l) \in C_n$ with $n-k+2 \leq l \leq n$.

4. Case of exponential distributions.

In this section, we consider the special case that both $f(z|u)$ and $f(z|1)$ are densities of the exponential distribution with parameters u and 1, respectively. Also, let $g(u|w,\alpha)$ is the density of gamma distribution with parameters w and α, that is, $g(u|w,\alpha) = \{\Gamma(\alpha)\}^{-1}w^{\alpha}u^{\alpha-1}e^{-wu}$. Since $\varphi(w,\alpha;z) = w+z$, $\psi(w,\alpha;z) = \alpha+1$, $h(z|w,\alpha) = \alpha w^{\alpha}(w+z)^{-\alpha-1}$ and $R(x,y) = w(\alpha-1)^{-1}$, assumptions A_1 through A_4 are satisfied in this case. Both equations (4) and (3) are replaced by the following equations (7) and (8), respectively.

$$G_n(k,l;w,\alpha) = \min\left\{G_n^0(k,l;w,\alpha)\,,\ \sum_{j=n-l+1}^{n}\beta_j + w(\alpha-1)^{-1}\sum_{j=1}^{n-l}\beta_j\right\} \tag{7}$$

and

$$G_n^0(k,l;w,\alpha) = w(\alpha-1)^{-1}\beta_n + \int_1^{\infty} G_{n-1}(k-1,\min\{l,n-1\};wu,\alpha+1)\alpha u^{-\alpha-1}du. \tag{8}$$

We introduce the function

$$G_n^{1*}(k,l;w,\alpha) = \sum_{j=n-l+1}^{n}\beta_j + w(\alpha-1)^{-1}\sum_{j=1}^{n-l}\beta_j.$$

and the notation $\binom{-a}{b}$ for positive integers a and b as follows:

$$\binom{-a}{b} = (-a)(-a-1)\cdots(-a-b+1)/(1\cdot 2\cdots b).$$

Then, the following theorem is derived.

THEOREM 4. *For $n \geq 1$, $(k,l) \in C_n$, $(w,\alpha) \in S$ and $(\beta_n,\beta_{n-1},\cdots,\beta_1) \in B_n^*$,*

(9)

$$G_n(k,l;w,\alpha) = \begin{cases} w(\alpha-1)^{-1}\displaystyle\sum_{j=n-k+1}^{n}\beta_j + \sum_{j=1}^{n-k}\beta_j \\ \quad + \displaystyle\sum_{j=1}^{k-1}\binom{-a}{j-1} D_{n-j+1}(k-j+1,\min\{l,n-j+1\};\alpha+j-1)w^{\alpha+j-1}, \\ \quad \text{if } 0 < w < s_n(k,l;\alpha) \\ \displaystyle\sum_{j=n-l+1}^{n}\beta_j + w(\alpha-1)^{-1}\sum_{j=1}^{n-l}\beta_j, \ \text{if} s_n(k,l;\alpha) \leq w, \end{cases}$$

where

$$\begin{aligned} D_n(k,l;\alpha) = {} & \left(\sum_{j=n-l}^{n-1}\beta_j - \sum_{j=1}^{n-k}\beta_j\right)\Big(-(\alpha-1)^{-1}\{s_{n-1}(k-1,\min\{l,n-1\};\alpha+1)\}^{-\alpha+1} \\ & + \{s_{n-1}(k-1,\min\{l,n-1\};\alpha+1)\}^{-\alpha}\Big) \\ & - \sum_{j=1}^{k-2}\binom{-\alpha}{j} D_{n-j}(k-j,\min\{l,n-j\};\alpha+j)\{s_{n-1}(k-1,\min\{l,n-1\};\alpha+1)\}^{j} \end{aligned}$$

and $s_n(k,l;\alpha)$ *is the unique root of the following equation of* w*:*

$$\sum_{j=1}^{k-1}\binom{-\alpha}{j-1}D_{n-j+1}(k-j+1,\min\{l,n-j+1\};\alpha+j-1)w^{\alpha+j-1}$$
$$+\left(\sum_{j=n-l+1}^{n}\beta_j-\sum_{j=1}^{n-k}\beta_j\right)\{w(\alpha-1)^{-1}-1\}=0.$$

PROOF. Two cases, $k=1$ and $k\geq 2$, have to be considered separately. Since

$$G_n^0(1,n;w,\alpha)=w(\alpha-1)^{-1}\beta_n+\sum_{j=1}^{n-1}\beta_j$$

and

$$G_n^{1*}(1,n;w,\alpha)=\sum_{j=1}^{n}\beta_j,$$

the assertion is true in case of $k=1$. For $k\geq 2$,

$$G_2^0(2,l;w,\alpha)=w(\alpha-1)^{-1}\beta_2+\int_1^\infty G_1(1,1;wu,\alpha+1)\alpha u^{-\alpha-1}du$$
$$=\begin{cases} w(\alpha-1)^{-1}\sum_{j=1}^{2}\beta_j+D_2(2,l;\alpha)w^\alpha, & \text{if } 0<w<s_1(1,1;\alpha+1)\\ w(\alpha-1)^{-1}\sum_{j=2}^{2}\beta_j+\sum_{j=1}^{1}\beta_j, & \text{if } s_1(1,1;\alpha+1)\leq w\end{cases}$$

and

$$G_2^{1*}(2,l;w,\alpha)=\sum_{j=2-l+1}^{2}\beta_j+w(\alpha-1)^{-1}\sum_{j=1}^{2-l}\beta_j,$$

for $1\leq l\leq 2$, where

$$D_2(2,l;\alpha)=\beta_1[-(\alpha-1)^{-1}\{s_1(1,1;\alpha+1)\}^{-\alpha+1}+\{s_1(1,1;\alpha+1)\}^{-\alpha}].$$

If $w>s_1(1,1;\alpha+1)$, then $G_2^0(2,l;w,\alpha)>G_2^{1*}(2,l;w,\alpha)$, which means $s_2(2,l;\alpha)\leq s_1(1,1;\alpha+1)$. Therefore, $s_2(2,l;\alpha)$ is the unique root of

$$D_2(2,l;\alpha)w^\alpha+\{w(\alpha-1)^{-1}-1\}\sum_{j=2-l+1}^{2}\beta_j=0$$

and the assertion is true for $n=2$ and $k=2$. Suppose that the assertion is true for $n\geq 2$ and $k\geq 2$. Then, for $k\geq 2$,

$$G_{n+1}^0(k,l;w,\alpha)=w(\alpha-1)^{-1}\beta_{n+1}+\int_1^\infty G_n(k-1,\min\{l,n\};wu,\alpha+1)\alpha u^{-\alpha-1}du. \tag{10}$$

From (10) and $G_n(k-1,\min\{l,n\};wu,\alpha+1)$ obtained by replacing k, l w and α of (9) by $k-1$, $\min\{l,n\}$, wu and $\alpha+1$, respectively, if $0<w<s_n(k-1,\min\{l,n\};\alpha+1)$, then

$$G_{n+1}^0(k,l;w,\alpha)=w(\alpha-1)^{-1}\beta_{n+1}+\int_1^{s_n(k-1,\min\{l,n\};\alpha+1)/w}\left\{wu\alpha^{-1}\sum_{j=n-k+2}^{n}\beta_j\right. \tag{11}$$
$$\left.+\sum_{j=1}^{n-k+1}\beta_j+\sum_{j=1}^{k-2}\binom{-\alpha-1}{j-1}D_{n-j+1}(k-j,\min\{l,n-j+1\};\alpha+j)w^{\alpha+j}u^{\alpha+j}\right\}\alpha u^{-\alpha-1}du$$
$$+\int_{s_n(k-1,\min\{l,n\};\alpha+1)/w}^\infty\left\{\sum_{j=n-\min\{l,n\}+1}^{n}\beta_j+wu\alpha^{-1}\sum_{j=1}^{n-\min\{l,n\}}\beta_j\right\}\alpha u^{-\alpha-1}du.$$

After slight calculation by using the equality

$$\sum_{j=n-k+2}^{n} \beta_j - \sum_{j=1}^{n-min\{l,n\}} \beta_j = \sum_{j=n-\min\{l,n\}+1}^{n} \beta_j - \sum_{j=1}^{n-k+1} \beta_j$$

in (11),

$$G^0_{n+1}(k,l;w,\alpha) = w(\alpha-1)^{-1} \sum_{j=n-k+2}^{n+1} \beta_j + \sum_{j=1}^{n-k+1} \beta_j$$
$$+ \sum_{j=1}^{k-1} \binom{-a}{j-1} D_{n+1-j+1}(k-j+1, \min\{l, n+1-j+1\}; \alpha+j-1) w^{\alpha+j-1},$$

where

$$D_{n+1}(k,l;\alpha) = \left(\sum_{j=n-\min\{l,n\}+1}^{n} \beta_j - \sum_{j=1}^{n-k+1} \beta_j \right)$$
$$\times \left(-(\alpha-1)^{-1} \{s_n(k-1, \min\{l,n\}; \alpha+1)\}^{-\alpha+1} + \{s_n(k-1, \min\{l,n\}; \alpha+1)\}^{-\alpha} \right)$$
$$- \sum_{j=1}^{k-2} \binom{-\alpha}{j} D_{n+1-j}(k-j, \min\{l, n-j+1\}; \alpha+j) \{s_n(k-1, \min\{l,n\}; \alpha+1)\}^{j}.$$

Also, if $s_n(k-1, \min\{l,n\}; \alpha+1) \leq w$, then

$$G^0_{n+1}(k,l;w,\alpha) = w(\alpha-1)^{-1}\beta_{n+1}$$
$$+ \int_1^\infty \left\{ \sum_{j=n-\min\{l,n\}+1}^{n} \beta_j + wu\alpha^{-1} \sum_{j=1}^{n-min\{l,n\}} \beta_j \right\} \alpha u^{-\alpha-1} du.$$
$$= w(\alpha-1)^{-1}\beta_{n+1} + \sum_{j=n-\min\{l,n\}+1}^{n} \beta_j + w(\alpha-1)^{-1} \sum_{j=1}^{n-min\{l,n\}} \beta_j.$$

On the other hand,

$$G^{1*}_{n+1}(k,l;w,\alpha) = \sum_{j=n-l+2}^{n+1} \beta_j + w(\alpha-1)^{-1} \sum_{j=1}^{n+1-l} \beta_j.$$

Since $\beta_{n+1} > \beta_n > \cdots > \beta_1 > 0$,

$$w(\alpha-1)^{-1}\beta_{n+1} + \sum_{j=n-\min\{l,n\}+1}^{n} \beta_j + w(\alpha-1)^{-1} \sum_{j=1}^{n-min\{l,n\}} \beta_j$$
$$> \sum_{j=n-l+2}^{n+1} \beta_j + w(\alpha-1)^{-1} \sum_{j=1}^{n+1-l} \beta_j$$

if and only if $w > s_1(1,1;\alpha)$. From the inequalities $s_n(k-1, \min\{l,n\}; \alpha+1) > s_n(k-1, \min\{l,n\}; \alpha) \geq s_1(1,1;\alpha)$ obtained in Theorem 2, $G^0_{n+1}(k,l;w,\alpha) > G^{1*}_{n+1}(k,l;w,\alpha)$ if $w > s_n(k-1, \min\{l,n\}; \alpha+1)$. Therefore, $s_{n+1}(k,l;\alpha)$ is the unique root of the following equation of w :

$$\sum_{j=1}^{k-1}\binom{-\alpha}{j-1}D_{n+1-j+1}(k-j+1,\min\{l,n+1-j+1\};\alpha+j-1)w^{\alpha+j-1}$$
$$+\left(\sum_{j=n+1-l+2}^{n+1}\beta_j+\sum_{j=1}^{n+1-k}\beta_j\right)\{w(\alpha-1)^{-1}-1\}=0.$$

This completes the proof. □

The special case that $\beta_j = j$ for $1 \leq j \leq n$ is important in applications. In this case, $t_n(k,l;\alpha)$ is used in place of $s_n(k,l;\alpha)$. Then, for $n \geq 1$, $(k,l) \in C_n$ and $(w,\alpha) \in S$,

$$D_n(k,l;\alpha)=\{l(2n-l-1)-(n-k)(n-k+1)\}/2$$
$$\times\left(-(\alpha-1)^{-1}\{t_{n-1}(k-1,\min\{l,n-1\};\alpha+1)\}^{-\alpha+1}\right.$$
$$\left.+\{t_{n-1}(k-1,\min\{l,n-1\};\alpha+1)\}^{-\alpha}\right)$$
$$-\sum_{j=1}^{k-2}\binom{-\alpha}{j}D_{n-j}(k-j,\min\{l,n-j\};\alpha+j)\{t_{n-1}(k-1,\min\{l,n-1\};\alpha+1)\}^{j}$$

and $t_n(k,l;\alpha)$ is the unique root of the following equation of w :

$$\sum_{j=1}^{k-1}\binom{-\alpha}{j-1}D_{n-j+1}(k-j+1,\min\{l,n-j+1\};\alpha+j-1)w^{\alpha+j-1}$$
$$+\{w(\alpha-1)^{-1}-1\}\{l(2n-l+1)-(n-k)(n-k+1)\}/2=0.$$

The values of $t_n(k,l;\alpha)$ for $1 \leq n \leq 5$, $(k,l) \in C_n$ and $\alpha = 2,3,\cdots,10$ are calculated and Table 1 is obtained. As a numerical example, consider the state (4,3,2,1.1,2), that is, $n = 4$, $k = 3$, $l = 2$, $w = 1.1$ and $\alpha = 2$. Since $w < t_4(3,3;2) = 1.2967$, a_0 is optimal at the first stage. Suppose that $z = 0.8$ is observed at the first stage. Then, the new state is (3,2,2,1.9,3) and $t_3(2,2;3) = 2.1962$, a_0 is optimal at the second stage. Let $z = 1.17$ is observed at the second stage. Since the new state is (2,1,2,3.07,4) and $t_2(1,2;4) = 3.0000$, a_1 is optimal at the third stage and therefore a_1 is also optimal at the last stage.

Another special case that $\beta_j = \gamma^{j-1}$ for $1 \leq j \leq n$ is also important in application. In this case, $r_n(k,l;\alpha)$ is used in place of $s_n(k,l;\alpha)$. For $n \geq 1$, $(k,l) \in C_n$ and $(w,\alpha) \in S$,

$$D_n(k,l;\alpha)=\{\gamma^{n-l-1}(1-\gamma^l)-(1-\gamma^{n-k})\}(1-\gamma)^{-1}$$
$$\times\left(-(\alpha-1)^{-1}\{r_{n-1}(k-1,\min\{l,n-1\};\alpha+1)\}^{-\alpha+1}\right.$$
$$\left.+\{r_{n-1}(k-1,\min\{l,n-1\};\alpha+1)\}^{-\alpha}\right)$$
$$-\sum_{j=1}^{n-k}\binom{-\alpha}{j}D_{n-j}(k-j,\min\{l,n-j\};\alpha+j)\{r_{n-1}(k-1,\min\{l,n-1\};\alpha+1)\}^{j}$$

and $r_n(k,l;\alpha)$ is the unique root of the following equation of w :

$$\sum_{j=1}^{k-1}\binom{-a}{j-1}D_{n-j+1}(k-j+1,\min\{l,n-j+1\};\alpha+j-1)w^{\alpha+j-1}$$
$$+\{w(\alpha-1)^{-1}-1\}(\gamma^{n-l}-\gamma^{n}-1+\gamma^{n-k})(1-\gamma)^{-1}=0.$$

Table 1. Values of $t_n(k, l; \alpha)$ for $1 \leq n \leq 6$, $(k, l) \in C_n$ and $\alpha = 2, 3, \cdots, 10$

$(n, k, l) \backslash \alpha$	2	3	4	5	6	7	8	9	10
(1, 1, 1)	1.0000	2.0000	3.0000	4.0000	5.0000	6.0000	7.0000	8.0000	9.0000
(2, 2, 1)	1.1716	2.1962	3.2065	4.2121	5.2158	6.2183	7.2201	8.2215	9.2226
(2, 2, 2)	1.1010	2.1172	3.1240	4.1278	5.1303	6.1319	7.1332	8.1341	9.1349
(2, 1, 2)	1.0000	2.0000	3.0000	4.0000	5.0000	6.0000	7.0000	8.0000	9.0000
(3, 3, 1)	1.2967	2.3425	3.3632	4.3751	5.3829	6.3884	7.3925	8.3957	9.3982
(3, 3, 2)	1.2399	2.2766	3.2929	4.3022	5.3082	6.3124	7.3155	8.3179	9.3199
(3, 3, 3)	1.1822	2.2125	3.2260	4.2337	5.2388	6.2423	7.2449	8.2470	9.2486
(3, 2, 2)	1.1716	2.1962	3.2065	4.2121	5.2158	6.2183	7.2201	8.2215	9.2226
(3, 2, 3)	1.1270	2.1465	3.1548	4.1593	5.1623	6.1643	7.1657	8.1669	9.1678
(3, 1, 3)	1.0000	2.0000	3.0000	4.0000	5.0000	6.0000	7.0000	8.0000	9.0000
(4, 4, 1)	1.3963	2.4605	3.4909	4.5089	5.5210	6.5297	7.5362	8.5413	9.5454
(4, 4, 2)	1.3496	2.4055	3.4314	4.4466	5.4568	6.4640	7.4694	8.4736	9.4770
(4, 4, 3)	1.2997	2.3477	3.3697	4.3826	5.3911	6.3972	7.4017	8.4053	9.4081
(4, 4, 4)	1.2508	2.2935	3.3133	4.3249	5.3326	6.3380	7.3421	8.3453	9.3479
(4, 3, 2)	1.2967	2.3425	3.3632	4.3751	5.3829	6.3884	7.3925	8.3957	9.3982
(4, 3, 3)	1.2599	2.3000	3.3178	4.3281	5.3347	6.3394	7.3429	8.3456	9.3478
(4, 3, 4)	1.2156	2.2509	3.2668	4.2759	5.2818	6.2860	7.2891	8.2915	9.2935
(4, 2, 3)	1.1716	2.1962	3.2065	4.2121	5.2158	6.2183	7.2201	8.2215	9.2226
(4, 2, 4)	1.1390	2.1600	3.1688	4.1737	5.1768	6.1790	7.1805	8.1817	9.1827
(4, 1, 4)	1.0000	2.0000	3.0000	4.0000	5.0000	6.0000	7.0000	8.0000	9.0000
(5, 5, 1)	1.4798	2.5600	3.5992	4.6232	5.6395	6.6513	7.6603	8.6674	9.6732
(5, 5, 2)	1.4401	2.5127	3.5478	4.5690	5.5833	6.5936	7.6015	8.6077	9.6127
(5, 5, 3)	1.3976	2.4628	3.4938	4.5124	5.5249	6.5339	7.5407	8.5461	9.5504
(5, 5, 4)	1.3531	2.4113	3.4389	4.4554	5.4664	6.4743	7.4803	8.4850	9.4888
(5, 5, 5)	1.3104	2.3643	3.3900	4.4053	5.4156	6.4230	7.4286	8.4330	9.4366
(5, 4, 2)	1.3963	2.4605	3.4909	4.5089	5.5210	6.5297	7.5362	8.5413	9.5454
(5, 4, 3)	1.3658	2.4246	3.4521	4.4683	5.4791	6.4869	7.4927	8.4972	9.5009
(5, 4, 4)	1.3264	2.3791	3.4035	4.4179	5.4275	6.4343	7.4394	8.4434	9.4466
(5, 4, 5)	1.2858	2.3344	3.3571	4.3704	5.3793	6.3857	7.3904	8.3942	9.3971
(5, 3, 3)	1.2967	2.3425	3.3632	4.3751	5.3829	6.3884	7.3925	8.3957	9.3982
(5, 3, 4)	1.2695	2.3111	3.3297	4.3404	5.3474	6.3523	7.3559	8.3587	9.3610
(5, 3, 5)	1.2339	2.2718	3.2888	4.2986	5.3050	6.3095	7.3129	8.3155	9.3176
(5, 2, 4)	1.1716	2.1962	3.2065	4.2121	5.2158	6.2183	7.2201	8.2215	9.2226
(5, 2, 5)	1.1459	2.1677	3.1768	4.1819	5.1851	6.1874	7.1890	8.1902	9.1912
(5, 1, 5)	1.0000	2.0000	3.0000	4.0000	5.0000	6.0000	7.0000	8.0000	9.0000

Table 2. Values of $r_n(k, l; \alpha)$ for $1 \le n \le 6$, $(k, l) \in C_n$ and $\alpha = 2, 3, \cdots, 10$.

$(n,k,l) \setminus \alpha$	2	3	4	5	6	7	8	9	10
(1, 1, 1)	1.0000	2.0000	3.0000	4.0000	5.0000	6.0000	7.0000	8.0000	9.0000
(2, 2, 1)	1.6418	2.6758	3.6889	4.6960	5.7004	6.7034	7.7055	8.7072	9.7085
(2, 2, 2)	1.6417	2.6757	3.6889	4.6959	5.7003	6.7033	7.7055	8.7072	9.7085
(2, 1, 2)	1.0000	2.0000	3.0000	4.0000	5.0000	6.0000	7.0000	8.0000	9.0000
(3, 3, 1)	1.8930	2.9844	4.0254	5.0490	6.0644	7.0753	8.0834	9.0897	10.0947
(3, 3, 2)	1.8931	2.9845	4.0254	5.0490	6.0644	7.0753	8.0834	9.0897	10.0947
(3, 3, 3)	1.7394	2.7913	3.8132	4.8255	5.8334	6.8390	7.8431	8.8462	9.8487
(3, 2, 2)	1.5406	2.5784	3.5933	4.6013	5.6063	6.6098	7.6123	8.6142	9.6157
(3, 2, 3)	1.5822	2.6188	3.6331	4.6408	5.6456	6.6488	7.6512	8.6530	9.6545
(3, 1, 3)	1.0000	2.0000	3.0000	4.0000	5.0000	6.0000	7.0000	8.0000	9.0000
(4, 4, 1)	2.0044	3.1378	4.2031	5.2429	6.2698	7.2893	8.3041	9.3158	10.3252
(4, 4, 2)	2.0043	3.1378	4.2031	5.2429	6.2698	7.2893	8.3041	9.3158	10.3252
(4, 4, 3)	1.9480	3.0585	4.1103	5.1410	6.1615	7.1762	8.1872	9.1958	10.2028
(4, 4, 4)	1.8026	2.8736	3.9060	4.9252	5.9379	6.9470	7.9539	8.9593	9.9637
(4, 3, 2)	1.7230	2.8101	3.8495	4.8724	5.8874	6.8980	7.9059	8.9120	9.9169
(4, 3, 3)	1.7898	2.8793	3.9196	4.9430	5.9583	6.9691	7.9771	8.9834	9.9883
(4, 3, 4)	1.6973	2.7540	3.7782	4.7918	5.8006	6.8068	7.8113	8.8148	9.8176
(4, 2, 3)	1.4775	2.5163	3.5317	4.5400	5.5452	6.5488	7.5514	8.5534	9.5549
(4, 2, 4)	1.5369	2.5748	3.5898	4.5978	5.6028	6.6063	7.6088	8.6107	9.6122
(4, 1, 4)	1.0000	2.0000	3.0000	4.0000	5.0000	6.0000	7.0000	8.0000	9.0000
(5, 5, 1)	2.0617	3.2228	4.3063	5.3590	6.3957	7.4228	8.4437	9.4604	10.4740
(5, 5, 2)	2.0617	3.2228	4.3063	5.3590	6.3957	7.4228	8.4437	9.4604	10.4740
(5, 5, 3)	2.0347	3.1821	4.2565	5.3026	6.3343	7.3576	8.3754	9.3895	10.4009
(5, 5, 4)	1.9799	3.1040	4.1644	5.2011	6.2260	7.2441	8.2579	9.2687	10.2775
(5, 5, 5)	1.8480	2.9340	3.9751	5.0000	6.0170	7.0294	8.0388	9.0463	10.0523
(5, 4, 2)	1.8046	2.9234	3.9818	5.0174	6.0415	7.0591	8.0724	9.0829	10.0913
(5, 4, 3)	1.8809	3.0061	4.0676	5.1051	6.1305	7.1489	8.1629	9.1740	10.1828
(5, 4, 4)	1.8705	2.9794	4.0308	5.0615	6.0819	7.0966	8.1077	9.1164	10.1233
(5, 4, 5)	1.7657	2.8398	3.8738	4.8938	5.9071	6.9167	7.9239	8.9295	9.9340
(5, 3, 3)	1.6295	2.7115	3.7489	4.7706	5.7848	6.7949	7.8024	8.8082	9.8129
(5, 3, 4)	1.7173	2.8041	3.8435	4.8663	5.8812	6.8918	7.8997	8.9058	9.9107
(5, 3, 5)	1.6608	2.7202	3.7458	4.7603	5.7696	6.7762	7.7810	8.7848	9.7877
(5, 2, 4)	1.4325	2.4712	3.4867	4.4951	5.5004	6.5040	7.5067	8.5087	9.5102
(5, 2, 5)	1.5004	2.5390	3.5543	4.5625	5.5677	6.5712	7.5738	8.5757	9.5773
(5, 1, 5)	1.0000	2.0000	3.0000	4.0000	5.0000	6.0000	7.0000	8.0000	9.0000

The values of $r_n(k,l;\alpha)$ for $1 \le n \le 6$, $(k,l) \in C_n$ and $\alpha = 2,3,\cdots,10$ are obtained in Table 2.

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