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PERISHABLE INVENTORY PROBLEM WITH TWO TYPES OF CUSTOMERS AND DIFFERENT SELLING PRICES

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Abstract This paper discusses an inventory control model for a single perishable product with two types of customers and different selling prices. This model is a one period horizon model and a generalization of Nahmias model [2] and Nose, et al. model [6]. Especially the model takes sensitive customers to freshness of commodities into consideration and treats two types of customers. Moreover, in the model there exist different selling prices according to the remaining lifetime of commodity to be sold. In the situation, an optimal ordering policy to maximize the expected profit is derived.

1. Introduction

This paper considers an inventory problem for perishable products with a fixed lifetime and optimal ordering policy. Different from models so far considered, our model has the following prominent features:

(1) There exist two types of customers, one buys only the newest commodity while the other buys either one, whether the newest or not. That is, the first is very sensitive to the freshness of the commodity and so is assigned high priority with respect to the purchase of the newest one, while the second is not so sensitive to the freshness of the commodity and buys old one if its price is cheap enough compared to the newest one.

(2) Different selling prices are set reflecting upon the remaining age of commodity.

Until now, models with (1) and (2) are few except our earlier model [6] and Nahmias [5] where the former sets two selling prices according to whether the remaining lifetime of the commodity to be sold is only one period or not and the latter assumes two types of customers but treats nonperishable product.

Section 2 formulates the model under (1) and (2) together with other necessary assumptions. Section 3 calculates expected quantities of outdating, inventory holding, shortage and sales corresponding to remaining life time of the commodity. Shortage costs are also different between high priority demand (type 1 customer) and low priority one (type 2 customer). Based on these quantities, Section 3 identifies the profit function to be maximized. Section 4 derives an optimal ordering policy to maximize the profit function. Finally Section 5 discusses further research problems.

2. Problem Formulation

The followings are assumed throughout this paper.

(1) A periodic review inventory model is considered for one planning period and single perishable product. The period length is arbitrary but fixed.

(2) Ordering takes places at the beginning of a period and unit purchasing cost c is charged.
(3) The maximum lifetime of the perishable commodity treated is m- periods. There exist two types of customers. One has a high priority and buys only the newest (remaining lifetime m) commodity, while the other has a low priority and buys not only the newest but also old ones. That is, at the start of each period high priority demand (type 1 customer) is first satisfied from the newest commodity in stock after ordering and next low priority demand from remaining stock according to FIFO issuing policy.
(4) If the commodity has not been depleted by the low priority demand until the period it reaches age m, then it perishes and must be discarded at a specified unit cost θ.
(5) Stock is depleted by demands at the beginning of each period and deterioration proceeds by one stage at each period after commodities are placed into stock.
(6) Low priority demands (type 2 customer) D_j^L in successive periods $j = 1, 2, \cdots$ are independent nonnegative random variables with known distributions $F_j^L(\cdot)$ and densities $f_j^L(\cdot)$. It is assumed that $F_j^L(0) = f_j^L(0) = 0$, $j = 1, 2, \cdots$ and they are continuous except zero. High priority demands (type 1 customer) D_j^H in successive periods also satisfy the same condition as low priority demands($F_j^H(\cdot)$ and $f_j^H(\cdot)$ are distributions and density respectively) and each high priority demand and each low priority demand are independent to each other.
(7) Shortage cost of high priority demand is p_H for each unit short and that of low priority demand p_L and $p_H \geq p_L$. On the other hand, a holding cost h is charged for unit carried over.
(8) The commodity whose remaining lifetime is k periods is sold at unit price $R_k, k = 1, 2, \cdots, m$. Further it is assumed that $R_{k+1} \geq R_k$, $k = 1, 2, \cdots, m-1$.

In stocking perishable commodities, it is often necessary to keep track of the amount of inventory on hand at each remaining lifetime level. For this or other purposes, we use the following notations.

x_i; the amount of the commodity on hand with remaining lifetime of i periods. $i = 1, 2, \cdots, m-1$.

$x \equiv \sum_{i=1}^{m-1} x_i$; total amount of inventory on hand at the beginning of each period.

$X_p \equiv (x_1, x_2, \cdots, x_p), p = 1, 2, \cdots, m-1$ and $X_0 \equiv \phi$ (null vector).

B_j; after depleting all of the commodities $x_j, x_{j-1}, \cdots, x_1$, the total unsatisfied low priority demand until period j which should be satisfied by the commodity in the inventory whose remaining lifetime is greater than j, i.e.,

$$B_j = [D_j^L + B_{j-1} - x_j]^+, \ j = 1, 2, \cdots, m-1$$

where

$$B_0 \equiv 0 \text{ and } [b]^+ = max(b, 0).$$

$Q_n(u : X_{n-1}) \equiv p_r\{D_n^L + B_{n-1} \leq u\} \quad n = 1, 2, \cdots, m$; the probability that the sum of D_n^L and B_{n-1} is not greater than u, where $Q_n(u : \cdot) \equiv 0$ for $u \leq 0$.

$q_n(u : X_{n-1})$; density of $Q_n(u : X_{n-1})$.(note that from [2], Q_n and q_n are continuous except possibility of jump at zero.)

$J(y : X_{m-1})$; the expected sales function when the quantity y is ordered under the current

stock level X_{m-1}.
$L(y : X_{m-1})$; the expected cost function when the quantity y is ordered under the current stock level X_{m-1}, including costs of ordering,holding shortage and outdating.
$K(y : X_{m-1}) \equiv J(y : X_{m-1}) - L(y : X_{m-1})$; the expected profit function to be maximized.

3. Identification of the Expected Profit Function $K(y : X_{m-1})$.

This section identifies the expected profit function $K(y : X_{m-1})$. For the purpose, we first identifiy the expected cost function by calculating the expected quantities of outdating for the ordered quantity y, shortage for both demands and holding, and next the expected sales function by calculating each expected sales quantity whose remaining life time is k, k=1,...,m.

[Expected outdating quantity]

Let R denote the number of unit of y scheduled to outdate after m-periods. First note that remaining quantity after high priority demand is satisfied by y.

$$\begin{cases} y - D_1^H & (\text{if } y > D_1^H) \\ 0 & (\text{otherwise}) \end{cases}$$

Thus the expected quantity of outdating $E(R)$ is as follows ([2]).

$$E(R) = \int_0^y f_1^H(v) \int_0^{y-v} Q_m(u : X_{m-1}) du dv \ = \ \int_0^y F_1^H(v) Q_m(y - v : X_{m-1}) dv \qquad (1)$$

[Expected shortage quantity]

(i) Expected shortage quantity of the high priority demand.

$$\int_y^\infty (v - y) f_1^H(v) dv = \int_y^\infty v f_1^H(v) dv - y\{1 - F_1^H(y)\} \qquad (2)$$

(ii) Expected shortage quantity of the low priority demand.

Shortage of the low priority demand occurs in two cases, i.e., one is that the high priority demand D_1^H is not short, but total sum of both demands exceeds the quantity of the inventory stock $x+y$. The other is that both demands are short. Thus expected shortage quantity of the low priority demand is

$$\int_0^y f_1^H(v)\{\int_{y+x-v}^\infty (u + v - x - y) f_1^L(u) du\} dv + \{1 - F_1^H(y)\} \int_x^\infty (u - x) f_1^L(u) du \qquad (3)$$

[Expected holding quantity]

We denote holding quantity H. Then

$$H = \begin{cases} [y + x - D_1^H - D_1^L]^+ & (D_1^H < y) \\ [x - D_1^L]^+ & (\text{otherwise}) \end{cases}$$

$$E(H) = \int_0^y \{\int_0^{x+y} (x + y - u - v) f_1^L(u) du\} f_1^H(v) dv + \{1 - F_1^H(y)\} \int_0^x (x - u) f_1^L(u) du \quad (4)$$

Thus

$L(y : X_{m-1}) = E$[ordering cost + holding cost +shortage cost + outdating cost]

$$= cy + h[\int_0^y \{\int_0^{x+y-v} (x+y-u-v) f_1^L(u)du\} f_1^H(v)dv$$
$$+\{1 - F_1^H(y)\} \int_0^x (x-u) f_1^L(u)du] + p_H[\int_y^\infty v f_1^H(v)dv - y\{1 - F_1^H(y)\}]$$
$$+p_L[\int_0^y f_1^H(v)\{\int_{y+x-v}^\infty (u+v-x-y) f_1^L(u)du\}dv + \{1 - F_1^H(y)\}$$
$$\times \int_x^\infty (u-x) f_1^L(u)du] + \theta \int_0^y F_1^H(v) Q_m(y-v : X_{m-1})dv \quad (5)$$

by using above calculations.

[Expected sales quantity]

We define U_k and $\bar{U}_k$, $k = 1, 2, \cdots, m$ as follows.

U_k; sales quantity of y whose remaining lifetime is k, $k = 1, 2, \cdots, m$.

$\bar{U}_k$; sales quantity of y whose remaining lifetime is not less than k, $k = 1, 2, \cdots, m$.

(i) Case $D_1^H \geq y$.

Then $U_m = y$ and $U_k = 0$, $k \neq m$. (6)

(ii) Case $D_1^H < y$.

Remaining quantity of y after satisfying the high priority demand D_1^H is $y - D_1^H$. Thus

$$U_m = \begin{cases} D_1^H & (D_1^L \leq x) \quad (7) \\ D_1^H + D_1^L - x & (x < D_1^L < x + y - D_1^H) \quad (8) \\ y & (D_1^H + D_1^L \geq x + y) \quad (9) \end{cases}$$

Since the newest commodity in the first period reaches remaining lifetime k after $m - k$ periods, $\bar{U}_k (k \neq m)$ is the sales quantity of y which is sold during period $1 \sim$ period $m - k + 1$. Thus

$$\bar{U}_k = \begin{cases} B_{m-k+1} - \sum_{j=m-k+2}^{m-1} x_j & (y - D_1^H + \sum_{j=m-k+2}^{m-1} x_j \geq B_{m-k+1} \geq \sum_{j=m-k+2}^{m-1} x_j) \\ y - D_1^H & (B_{m-k+1} > \sum_{j=m-k+2}^{m-1} x_j + y - D_1^H) \\ 0 & (\text{otherwise}) \end{cases}$$

since B_{m-k+1} is the overflow of low priority demands to the inventory part $x_{m-k+2} \sim x_{m-1}$. Accordingly,

$$E(\bar{U}_k) = \int_0^y [\int_{\sum_{j=m-k+2}^{m-1} x_j}^{y-v+\sum_{j=m-k+2}^{m-1} x_j} (u - \sum_{j=m-k+2}^{m-1} x_j) dQ_{m-k+1}(u + x_{m-k+1} : X_{m-k})$$
$$+(y-v)\{1 - Q_{m-k+1}(y - v + \sum_{j=m-k+1}^{m-1} x_j : X_{m-k})\}] f_1^H(v)dv$$
$$= \int_0^y \{(y-v) - \int_0^{y-v} Q_{m-k+1}(u + \sum_{j=m-k+1}^{m-1} x_j : X_{m-k})du\} f_1^H(v)dv \quad (10)$$

Since $U_k = \bar{U}_k - \bar{U}_{k+1}$ for $k \neq m$,

$$E(U_k) = E(\bar{U}_k) - E(\bar{U}_{k+1})$$
$$= \int_0^y f_1^H(v)[\int_0^{y-v} \{Q_{m-k}(u + \sum_{j=m-k}^{m-1} x_j : X_{m-k-1}) - Q_{m-k+1}(u + \sum_{j=m-k+1}^{m-1} x_j : X_{m-k})\}du]dv,$$

$k \neq m$.

By (i) and (7) $\sim$ (9),

$$E(U_m) = y\{1 - F_1^H(y)\} + \int_0^y v f_1^H(v)dv \times F_1^L(x)$$
$$+ \int_0^y \{\int_x^{x+y-v} (u+v-x) f_1^L(u)du\} f_1^H(v)dv$$
$$+y\int_0^y \{1 - F_1^L(x+y-v)\} f_1^H(v)dv = y - \int_0^y \int_x^{x+y-v} F_1^L(u)du f_1^H(v)dv \quad (11)$$

Summarizing above results,

$$J(y : X_{m-1}) = \sum_{k=1}^{m-1} R_k \int_0^y f_1^H(v)[\int_0^{y-v} \{Q_{m-k}(u + \sum_{j=m-k}^{m-1} x_j : X_{m-k-1})$$
$$-Q_{m-k+1}(u + \sum_{j=m-k+1}^{m-1} x_j : X_{m-k})\}du]dv + R_m\{y - \int_0^y \int_x^{x+y-v} F_1^L(u)du f_1^H(v)dv\}$$

and

$$K(y : X_{m-1}) = \sum_{k=1}^{m-1} R_k \int_0^y f_1^H(v)[\int_0^{y-v} \{Q_{m-k}(u + \sum_{j=m-k}^{m-1} x_j : X_{m-k-1})$$
$$-Q_{m-k+1}(u + \sum_{j=m-k+1}^{m-1} x_j : X_{m-k})\}du]dv + R_m\{y - \int_0^y \int_x^{x+y-v} F_1^L(u)du f_1^H(v)dv\}$$
$$-\langle\, cy + h[\int_0^y \{\int_0^{x+y-v} (x+y-u-v) f_1^L(u)du\} f_1^H(v)dv$$
$$+\{1 - F_1^H(y)\}\int_0^x (x-u) f_1^L(u)du] + p_H[\int_y^\infty v f_1^H(v)dv - y\{1 - F_1^H(y)\}]$$
$$+p_L[\int_0^y f_1^H(v)\{\int_{y+x-v}^\infty (u+v-x-y) f_1^L(u)du\}dv + \{1 - F_1^H(y)\}$$
$$\times \int_x^\infty (u-x) f_1^L(u)du] + \theta \int_0^y F_1^H(v) Q_m(y-v : X_{m-1})dv \rangle \quad (12)$$

4. An Optimal Ordering Policy

This section investigates the optimal ordering policy with respect to $K(y : X_{m-1})$ obtained in section 3. First we show the concavity of $K(y : X_{m-1})$.

Theorem 1. The expected profit function $K(y : X_{m-1})$ is concave with respect to y.

Proof : It is sufficient to show $\partial^2 K/\partial y^2 \le 0$.

$$\partial K/\partial y = \sum_{k=1}^{m-1} R_k \int_0^y f_1^H(v)\{Q_{m-k}(y - v + \sum_{j=m-k}^{m-1} x_j : X_{m-k-1})$$
$$-Q_{m-k+1}(y - v + \sum_{j=m-k+1}^{m-1} x_j : X_{m-k})\}dv - R_m\{1 - \int_0^y F_1^L(x+y-v) f_1^H(v)dv\}$$
$$-\theta \int_0^y F_1^H(v) q_m(y-v : X_{m-1})dv - c - h\int_0^y \int_0^{x+y-v} f_1^L(u) f_1^H(v)dudv$$
$$-p_H\{F_1^H(y) - 1\} - p_L \int_0^y f_1^H(v)\{F_1^L(x+y-v) - 1\}dv \quad (13)$$

Since $\theta \int_0^y F_1^H(v) q_m(y-v : X_{m-1})dv = \theta \int_0^y f_1^H(v) Q_m(y-v : X_{m-1})dv$ from the integration by parts,

$$\partial^2 K/\partial y^2 = \sum_{k=1}^{m-1} R_k f_1^H(y)\{Q_{m-k}(\sum_{j=m-k}^{m-1} x_j : X_{m-k-1}) - Q_{m-k+1}(\sum_{j=m-k+1}^{m-1} x_j : X_{m-k})\}$$
$$+ \int_0^y f_1^H(v)dv\{q_{m-k}(y - v + \sum_{j=m-k}^{m-1} x_j : X_{m-k-1}) - q_{m-k+1}(y - v + \sum_{j=m-k+1}^{m-1} x_j : X_{m-k})\}$$

$$-R_m\{F_1^L(x)f_1^H(y)+\int_0^y f_1^L(x+y-v)f_1^H(v)dv\}-\theta\int_0^y f_1^H(v)q_m(y-v:X_{m-1})dv$$
$$-h\{F_1^L(x)f_1^H(y)+\int_0^y f_1^H(v)f_1^L(x+y-v)dv\}-p_H f_1^H(y)$$
$$-p_L[\int_0^y f_1^H(v)f_1^L(x+y-v)dv+f_1^H(y)\{F_1^L(x)-1\}] \quad (14)$$

By the definition of $Q_1(u:\phi)$ and $q_1(u:\phi)$, $Q_1(x:\phi)=p_r(D_1^L+B_0\le x)=\int_0^x f_1^L(u)du=F_1^L(x)$

and $q_1(x+y-v:\phi)=dQ_1(u:\phi)/du\mid_{u=x+y-v}=f_1^L(x+y-v)$ (15)

Since

$$R_m\{F_1^L(x)f_1^H(y)+\int_0^y f_1^L(x+y-v)f_1^H(v)dv\}$$
$$=R_m\{Q_1(x)f_1^H(y)+\int_0^y q_1(x+y-v:\phi)f_1^H(v)dv\}$$

by using (15),

$$\partial^2K/\partial y^2=\sum_{k=1}^{m-1}(R_k-R_{k+1})f_1^H(y)\{Q_{m-k}(\sum_{j=m-k}^{m-1}x_j:X_{m-k-1})$$
$$+\int_0^y f_1^H(v)dv\{q_{m-k}(y-v+\sum_{j=m-k}^{m-1}x_j:X_{m-k-1})\}\}$$
$$-\theta\int_0^y f_1^H(v)q_m(y-v:X_{m-1})dv$$
$$-h\{F_1^L(x)f_1^H(y)+\int_0^y f_1^H(v)f_1^L(x+y-v)dv\}-f_1^H(y)[p_H-p_L+p_LF_1^L(x)]$$

$-p_L[\int_0^y f_1^H(v)f_1^L(x+y-v)dv]$ by including $R_m\{---\}$ into Σ and rearranging Σ in (14). (16)

Since $R_{k+1}\ge R_k$ and $p_H\ge p_L$ by the assumption, it holds that $\partial^2K/\partial y^2\le 0$ which means that K is concave with respect to y.

Theorem 2. An optimal ordering quantity y^* exists among $(0,\infty)$ as a function of X_{m-1}. Further an optimal ordering policy is

$$\begin{cases}\text{order} \quad y^*=\hat{y} \quad (R_m+p_H-c>0)\\ \text{do not order} \quad\quad (\text{otherwise})\end{cases}$$

where $\hat{y}$ is a solution of $\partial K/\partial y=0$.

Proof: From (13)

$$lim_{y\to+0}\partial K/\partial y=R_m+p_H-c=\begin{cases}>0 & (R_m+p_H-c>0)\\ \le 0 & (\text{otherwise})\end{cases} \quad (17)$$

Similarly,

$$lim_{y\to\infty}\partial K/\partial y=lim_{y\to\infty}\{-c-\theta\int_0^y F_1^H(v)q_m(y-v:X_{m-1})dv$$
$$-h\int_0^y\int_0^{x+y-v}f_1^L(u)f_1^H(v)dudv\}$$
$$=-c-h-\theta lim_{y\to\infty}\int_0^y f_1^H(v)Q_m(y-v:X_{m-1})dv$$
$$=-c-h-\theta\int_0^\infty f_1^H(v)dv=-c-h-\theta<0 \quad (18)$$

Thus from (17),(18) and Theorem 1, Theorem 2 is derived.

5. Conclusion

In this paper, we investigated the optimal ordering policies for a perishable commodity under the discriminating sales prices and two types of customers. Though our model is more general than [6],[1], in a point that it contains two types of customers reflecting actual behaviors of customers, followings are left as futher research problems.

(i) Analysis of multi-period versions of the model.

(ii) Sensitivity analysis on changes of inventory on hand ([6]).

(iii) Introduction of leadtime and/or inclusion of fixed charge cost into ordering cost.

(iv) Investigation of other realistic inventory depletion policies such as LIFO issuing, in order to cope with the increase of customer service.

(v)Construction of more concrete optimal policy for a suitable probability distribution of demands though it needs tedious numerical integration and numerical solution of nonlinear equations.

References

[1] Ishii,H., T.Nose, S.Shiode and T.Nishida:Perishable Inventory Management Subject to Stochastic Lead Time, *European Journal of Operational Research* 8 (1981) 76-85.

[2] Nahmias,S.: *Optimal and Approximate Ordering Policies for a Perishable Product subject to Stochastic Demand,* ph. D. Dissertation, North-Western University (1972).

[3] Nahmias,S.: The Fixed-Charge Perishable Inventory Problem, *Operations Research* 26 (1978) 464-481.

[4] Nahmias,S.: Perishable Inventory Theory: A Review, *Operations Research* 30 (1982) 680-708.

[5] Nahmias S. and W.S.Demmy: Operating Characteristics of an Inventory System with Rationing, *Management Science* 27 (1981) 1236-1245.

[6] Nose,T., H.Ishii and T.Nishida: Perishable Inventory Management with Stochastic Lead Time and Different Selling Prices, *European Journal of Operational Research* 18 (1984) 332-338.

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