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AN ε-FREE DEA AND A NEW MEASURE OF EFFICIENCY

Kaoru Tone
Saitama University

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Abstract This study presents a DEA model without the conventional non-Archimedian infinitesimal ε. This article also introduces a new DEA efficiency measure. Incorporating slacks in inputs and shortages in outputs, the new DEA measure expresses the relative efficiency of decision making units more properly than traditional one.

1 Introduction and Historical Background

In their ingenious paper [10], Charnes, Cooper and Rhodes introduced a fractional programming method to measure the relative efficiency of a Decision Making Unit (DMU), which was solved by transforming the fractional programming into a linear programming problem via the Charnes-Cooper scheme [6]. The method was referred to as DEA (Data Envelopment Analysis). The DEA model proposed in [10] maintained an assumption; all the weighting values to inputs and outputs were assumed to be *nonnegative*. In the subsequent "short communication " [11], Charnes *et. al.* changed their DEA problems and required that the weights be strictly *positive*. Thus, the introduction of the non-Archimedian infinitesimal ε was anticipated to distinguish between nonnegative and positive values. (This problem was already discussed in [10] implicitly.) Although the subsequent discussions can be found in [5], [7], and [12], and the role of ε has become unclear and weakened, it is still frequently used in the literature (e.g., [4],[8]) and in particular, in some cases of computational situations, values such as $\varepsilon = 10^{-5}, 10^{-6}$ (single precision) or $\varepsilon = 10^{-12}$ (double precision) are conveniently employed to substitute for the non-Archimedian infinitesimal ε. However, the approach may produce a theoretically contradicting issue. That is, we cannot uniquely determine what is the best ε. Different ε values yield different DEA results. Therefore, we need a completely ε-free development of DEA from both theoretical and computational points of view.

This article is organized as follows. Section 2 defines an input oriented DEA model based on the production possibility set. Its dual corresponds to the Charnes-Cooper-Rhodes (CCR) model with the weights to inputs and outputs as variables. Then, we define a DMU as *slackless* if, for every optimal solution to the DEA model, it has no slack in inputs and no shortages in outputs. By a theorem of the alternative or the strong theorem of complemantary slackness, it will be proved that for a slackless DMU there is a strictly positive weight solution in the corresponding CCR model. Subsequently, for a DMU with non-zero slacks in an optimal solution to the DEA model, there exist no positive weight solutions in the CCR model. Section 3 defines the *max-slack* solution and shows a procedure to find it. The max-slack solution can be used for deciding whether the DMU is slackless or not. Then, we propose a method for finding positive weights for slackless DMUs. Thus, all jobs of the CCR model can be successfully achieved with no recourse to ε. Section 4 introduces a new measure of relative efficiency, based on the max-slack solution, which takes account of slacks in the inputs and shortages in the outputs of the objective DMU. As a consequence, the new DEA measure expresses the relative efficiency of DMUs more properly than the traditional one.

2 Input Oriented DEA and Slackless DMU

We consider n decision making units (DMUs), each of which uses an input matrix $X = [x_{ik}] \in R^{m\times n}$ to produce an output matrix $Y = [y_{jk}] \in R^{s\times n}$, where x_{ik} is the amount of input i consumed by DMU k and y_{jk} is the amount of output j produced by DMU k. The unit k has the input vector $\boldsymbol{x}_k = (x_{ik}) \in R^m$, representing m input components and the output vector $\boldsymbol{y}_k = (y_{jk}) \in R^s$ representing s output components. We assume $X > O$ and $Y > O$. The production possibility set P (e.g., [2], [3], [9]) is defined by

$$(1) \qquad P = \{(\boldsymbol{x}, \boldsymbol{y}) \mid \boldsymbol{x} \geq X\boldsymbol{\lambda},\ \boldsymbol{y} \leq Y\boldsymbol{\lambda},\ \boldsymbol{\lambda} \geq \mathbf{0}\},$$

where $\boldsymbol{x}, \boldsymbol{y}$ and $\boldsymbol{\lambda}$ are m-, s- and n- vectors, respectively.

For a given DMU denoted by o with the input $\boldsymbol{x}_o$ and the output $\boldsymbol{y}_o$, we consider the following input oriented DEA model as expressed by the linear programming (LP_o).

$$(2) \qquad \begin{array}{rrcl} (LP_o) & \min\ \theta & & \\ \text{subject to} & \theta\boldsymbol{x}_o & \geq & X\boldsymbol{\lambda}, \\ & \boldsymbol{y}_o & \leq & Y\boldsymbol{\lambda}, \\ & \boldsymbol{\lambda} & \geq & \mathbf{0}, \end{array}$$

where $\theta \in R$ and $\boldsymbol{\lambda} \in R^n$ are variables.
This model contracts inputs as far as possible while controlling for outputs. The dual of (LP_o) is:

$$(3) \qquad \begin{array}{rrcl} (DP_o) & \max\ \boldsymbol{y}_o^T\boldsymbol{u} & & \\ \text{subject to} & \boldsymbol{x}_o^T\boldsymbol{v} & = & 1, \\ & \boldsymbol{v}^T X & \geq & \boldsymbol{u}^T Y, \\ & \boldsymbol{v} & \geq & \mathbf{0}, \quad \boldsymbol{u} \geq \mathbf{0}, \end{array}$$

where $\boldsymbol{v} \in R^m$ and $\boldsymbol{u} \in R^s$ are variables.

As is well known, (DP_o) is the linear programming version of the original CCR fractional programming problem. The dual variables $\boldsymbol{u}$ and $\boldsymbol{v}$ are the weights for outputs and inputs, respectively.

In (LP_o), the *slacks* $\boldsymbol{s}_x$ and $\boldsymbol{s}_y$ can be defined by

$$(4) \qquad \boldsymbol{s}_x = \theta\boldsymbol{x}_o - X\boldsymbol{\lambda} \quad \text{and} \quad \boldsymbol{s}_y = Y\boldsymbol{\lambda} - \boldsymbol{y}_o.$$

Here let optimal solutions for (LP_o) and (DP_o) be $(\theta^*, \boldsymbol{\lambda}^*, \boldsymbol{s}_x^*, \boldsymbol{s}_y^*)$ and $(\boldsymbol{u}^*, \boldsymbol{v}^*)$, respectively. These problems are often degenerated and the optimal solutions are not always unique.

Definition 1 (slackless DMU) *If the following condition for (LP_o):*

$$(5) \qquad \boldsymbol{s}_x^* = \mathbf{0} \quad \textit{and} \quad \boldsymbol{s}_y^* = \mathbf{0},$$

is observed at optimality, then we call the status of DMU_o as slackless.

Theorem 1 *DMU_o is slackless if and only if there exists an optimal dual solution $(\boldsymbol{u}^*, \boldsymbol{v}^*)$ for (DP_o) with $\boldsymbol{u}^* > \boldsymbol{0}$ and $\boldsymbol{v}^* > \boldsymbol{0}$.*

Proof. Although the theorem is a natural consequence of the strong theorem of complementary slackness, we will show another proof.

If DMU_o is slackless, there is no solution $(\boldsymbol{\lambda}, \boldsymbol{s}_x, \boldsymbol{s}_y)$ for the system:

$$\theta^* \boldsymbol{x}_o = X\boldsymbol{\lambda} + \boldsymbol{s}_x, \tag{6}$$
$$\boldsymbol{y}_o = Y\boldsymbol{\lambda} - \boldsymbol{s}_y, \tag{7}$$
$$\boldsymbol{\lambda} \geq \boldsymbol{0}, \tag{8}$$
$$(\boldsymbol{s}_x, \boldsymbol{s}_y) \geq \boldsymbol{0} \text{ and } (\boldsymbol{s}_x, \boldsymbol{s}_y) \neq \boldsymbol{0}. \tag{9}$$

By applying Slater's theorem of the alternative (see Appendix 1), modified for the nonhomogenous system, to the system (6)-(9), it is concluded that the system

$$\boldsymbol{v}^T X \geq \boldsymbol{u}^T Y, \tag{10}$$
$$\theta^*(\boldsymbol{v}^T \boldsymbol{x}_o) - \boldsymbol{u}^T \boldsymbol{y}_o + z = 0, \tag{11}$$

has a solution $\boldsymbol{v} \in R^m$, $\boldsymbol{u} \in R^s$ and $z \in R$ with

$$z > 0, \quad \boldsymbol{u} \geq \boldsymbol{0}, \quad \boldsymbol{v} \geq \boldsymbol{0} \tag{12}$$

or

$$z \geq 0, \quad \boldsymbol{u} > \boldsymbol{0}, \quad \boldsymbol{v} > \boldsymbol{0}. \tag{13}$$

Since $\boldsymbol{v}^T \boldsymbol{x}_o \geq 0$, there are two cases:

Case (i) $\boldsymbol{v}^T \boldsymbol{x}_o > 0$.

Let $\boldsymbol{v}^* = \boldsymbol{v}/\boldsymbol{v}^T \boldsymbol{x}_o$, $\boldsymbol{u}^* = \boldsymbol{u}/\boldsymbol{v}^T \boldsymbol{x}_o$ and $z^* = z/\boldsymbol{v}^T \boldsymbol{x}_0$. Then, we have:

$$(\boldsymbol{v}^*)^T \boldsymbol{x}_o = 1, \tag{14}$$
$$(\boldsymbol{v}^*)^T X \geq (\boldsymbol{u}^*)^T Y, \tag{15}$$
$$\theta^* - (\boldsymbol{u}^*)^T \boldsymbol{y}_o + z^* = 0, \tag{16}$$
$$z^* \geq 0, \quad \boldsymbol{u}^* \geq \boldsymbol{0}, \quad \boldsymbol{v}^* \geq \boldsymbol{0}. \tag{17}$$

Hence, $(\boldsymbol{u}^*, \boldsymbol{v}^*)$ is a feasible solution for (DP_o). By the duality relation, (16) becomes

$$\theta^* = (\boldsymbol{u}^*)^T \boldsymbol{y}_o \quad \text{and} \quad z^* = 0, \tag{18}$$

demonstrating the optimality of $(\boldsymbol{u}^*, \boldsymbol{v}^*)$ for (DP_o). By the result of (13), it is obtained that $\boldsymbol{u}^* > \boldsymbol{0}$ and $\boldsymbol{v}^* > \boldsymbol{0}$.

Case (ii) $\boldsymbol{v}^T \boldsymbol{x}_o = 0$.

From (10), we have $0 = \boldsymbol{v}^T \boldsymbol{x}_o \geq \boldsymbol{u}^T \boldsymbol{y}_o \geq 0$ and, hence $\boldsymbol{u}^T \boldsymbol{y}_o = 0$. Equation (11) results in $z = 0$ and, from (13), $\boldsymbol{u} > \boldsymbol{0}$ and $\boldsymbol{v} > \boldsymbol{0}$ are obtained. This contradicts with $\boldsymbol{v}^T \boldsymbol{x}_o = 0$, since $\boldsymbol{x}_o > \boldsymbol{0}$ by assumption. Thus, Case (ii) never occurs.

The reverse is also true by the nature of the theorem of the alternative. □

Evidently, we have:

Corollary 1 *(DP_o) has no strictly positive optimal solution $(\boldsymbol{u}^*, \boldsymbol{v}^*)$ if and only if (LP_o) has an optimal solution $(\theta^*, \boldsymbol{\lambda}^*, \boldsymbol{s}_x^*, \boldsymbol{s}_y^*)$ with $(\boldsymbol{s}_x^*, \boldsymbol{s}_y^*) \geq \boldsymbol{0}$ and $(\boldsymbol{s}_x^*, \boldsymbol{s}_y^*) \neq \boldsymbol{0}$.*

3 Max Slack Solution

The preceding Theorem and Corollary reveal the equivalence between the slackless solution in (LP_o) and the positive solution in (DP_o). Therefore, hereafter, we will mainly deal with (LP_o), since it can be more easily handled theoretically and computationally than the latter. Furthermore,(LP_o) does not need ε.

Definition 2 (max slack solution) *An optimal solution of* (LP_o) *is called as* max slack, *if it maximizes* $w = \boldsymbol{e}^T\boldsymbol{s}_x + \boldsymbol{e}^T\boldsymbol{s}_y$, *where the* $\boldsymbol{e}^T$ *is a vector of ones.*

The max slack solution can be obtained by a 2-phase process as follows:
The first phase minimizes θ of (LP_o). Then, the second phase maximizes $w = \boldsymbol{e}^T\boldsymbol{s}_x + \boldsymbol{e}^T\boldsymbol{s}_y$, while keeping $\theta = \theta^*$ (the optimal θ value). It hardly needs pointing out that DMU_o is slackless if and only if its max slack solution satisfies $w = \boldsymbol{e}^T\boldsymbol{s}_x + \boldsymbol{e}^T\boldsymbol{s}_y = 0$. From the definition of efficiency, it is a straight forward matter to state that a DMU is *efficient* if it has $\theta^* = 1$ and is slackless. Otherwise, it is *inefficient.* (The above procedure and the definition of efficiency are given in [12], as well.)

It is important to note that in the original CCR model [10], the weights $\boldsymbol{u}$ and $\boldsymbol{v}$ were required to be *nonnegative* and then, in the subsequent paper [11], the problem was changed and required $\boldsymbol{u}$ and $\boldsymbol{v}$ to be *positive*, considering the slackness in (LP_o). Specifically, in [7], Charnes and Cooper introduced the non-Archimedian infinitesimal ε and replaced the condition $\boldsymbol{u} > \boldsymbol{0}$, $\boldsymbol{v} > \boldsymbol{0}$ by $\boldsymbol{u} \geq \varepsilon\boldsymbol{e}$, $\boldsymbol{v} \geq \varepsilon\boldsymbol{e}$. If the optimal solution $(\boldsymbol{u}^*, \boldsymbol{v}^*)$ for (DP_o) under the latter condition happened to be *Archimedian positive*, then DMU_o is slackless by Theorem 1. However, if some elements of $(\boldsymbol{u}^*, \boldsymbol{v}^*)$ are non-Archimedian infinitesimal, we cannot decide from $(\boldsymbol{u}^*, \boldsymbol{v}^*)$ whether DMU_o is slackless or not. We will be free from this kind of information gap so long as we deal with the max slack solution of (LP_o).

A practical procedure for finding positive weights $\boldsymbol{u}$ and $\boldsymbol{v}$ for a slackless DMU is as follows. The problem turns out to find a feasible solution for (DP_o) with $\boldsymbol{y}_o^T\boldsymbol{u} = \theta^*$ and $\boldsymbol{v} \geq t\boldsymbol{e}$ and $\boldsymbol{u} \geq t\boldsymbol{e}$, for some positive t. In order to solve it in the primal form, we will start from the optimal basis of the second phase above and change the objective function of (LP_o) to maximize $t(\boldsymbol{e}^T\boldsymbol{s}_x + \boldsymbol{e}^T\boldsymbol{s}_y)$, while keeping $\theta = \theta^*$, where t is a parameter. The parametric (LP_o) is solved to give a positive t whose existence is guaranteed by Theorem 1. Then, the corresponding simplex multipliers $\boldsymbol{v}$ and $\boldsymbol{u}$ give the positive weights. (See the forthcoming paper [16] for further details.)

4 New Measure of Efficiency

The traditional DEA considers θ^* as the efficiency measure. However, θ^* is indifferent to the level of slacks in inputs and outputs and hence is misleading as a practical means for relatively comparing DMUs. Now, we can define another type of efficiency by the following principles: (1) it should be the same as θ^* when the DMU is slackless, and (2) it should be decreasing in the relative value of slacks in inputs and outputs.

In an effort to achieve this purpose, we propose a new measure of efficiency, defined by

$$\eta^* = \left(\theta^* - \frac{\boldsymbol{e}^T\boldsymbol{s}_x^*}{\boldsymbol{e}^T\boldsymbol{x}_o}\right)\left(\frac{\boldsymbol{e}^T\boldsymbol{y}_o}{\boldsymbol{e}^T\boldsymbol{y}_o + \boldsymbol{e}^T\boldsymbol{s}_y^*}\right), \tag{19}$$

where $\boldsymbol{s}_x^*$ and $\boldsymbol{s}_y^*$ are slacks of the max slack solution, respectively. It is easily observed that η^* defined by (19) satisfies the above criteria. Furthermore, η^* can be rewritten as

$$\eta^* = \frac{\boldsymbol{e}^T X\boldsymbol{\lambda}^*}{\boldsymbol{e}^T\boldsymbol{x}_o}\frac{\boldsymbol{e}^T\boldsymbol{y}_o}{\boldsymbol{e}^T Y\boldsymbol{\lambda}^*}. \tag{20}$$

Let us define a DMU $(\boldsymbol{x}_e, \boldsymbol{y}_e)$ by

(21) $$\boldsymbol{x}_e = X\boldsymbol{\lambda}^* \quad \text{and} \quad \boldsymbol{y}_e = Y\boldsymbol{\lambda}^*.$$

Theorem 2 *The DMU $(\boldsymbol{x}_e, \boldsymbol{y}_e)$ is efficient.*

Proof. The efficiency of $(\boldsymbol{x}_e, \boldsymbol{y}_e)$ is estimated by solving the following problem:

$$\begin{array}{lrcl} (LP_e) & \min & \theta_e & \\ \text{subject to} & \theta_e \boldsymbol{x}_e & = & X\boldsymbol{\lambda} + \boldsymbol{s}_x, \\ & \boldsymbol{y}_e & = & Y\boldsymbol{\lambda} - \boldsymbol{s}_y, \\ & \boldsymbol{\lambda} & \geq & 0, \quad \boldsymbol{s}_x \geq \boldsymbol{0}, \quad \boldsymbol{s}_y \geq \boldsymbol{0}. \end{array}$$

Here let a max slack optimal solution of (LP_e) be $(\theta_e^*, \boldsymbol{\lambda}_e^*, \boldsymbol{s}_{x_e}^*, \boldsymbol{s}_{y_e}^*)$. From $\theta^* \boldsymbol{x}_o = \boldsymbol{x}_e + \boldsymbol{s}_x^*$ and $\boldsymbol{y}_o = \boldsymbol{y}_e - \boldsymbol{s}_y^*$, the following conditions are obtained:

$$\begin{aligned} \theta_e^* \theta^* \boldsymbol{x}_o &= X\boldsymbol{\lambda}_e^* + \boldsymbol{s}_{x_e}^* + \theta_e^* \boldsymbol{s}_x^*, \\ \boldsymbol{y}_o &= Y\boldsymbol{\lambda}_e^* - \boldsymbol{s}_{y_e}^* - \boldsymbol{s}_y^*. \end{aligned}$$

Since $(\theta^*, \boldsymbol{\lambda}^*, \boldsymbol{s}_x^*, \boldsymbol{s}_y^*)$ is a max slack optimal solution for $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ and $\theta_e^* \leq 1$, $\theta_e^* = 1$, $\boldsymbol{s}_{x_e}^* = \boldsymbol{0}$ and $\boldsymbol{s}_{y_e}^* = \boldsymbol{0}$ are obtained. Thus, $(\boldsymbol{x}_e, \boldsymbol{y}_e)$ is efficient. □

Thus, $(\boldsymbol{x}_e, \boldsymbol{y}_e)$ is a projection of $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ onto its efficiency facet. η^* can be interpreted as the product of the average input efficiency $\boldsymbol{e}^T\boldsymbol{x}_e/\boldsymbol{e}^T\boldsymbol{x}_o (\leq 1)$ with the average output efficiency $\boldsymbol{e}^T\boldsymbol{y}_o/\boldsymbol{e}^T\boldsymbol{y}_e (\leq 1)$.

Example

Table 1 shows the input X and the output Y for six DMUs along with the max slack solutions. DMU_6 has $\theta_6^* = 1$ which seems to be better than DMU_1 and DMU_2. Because of $s_{x_1}^* = 2$ is observed for DMU_6, its new efficiency becomes $\eta_6^* = 9/11 = 0.82$. Thus, DMU_6 drops to the lowest level.

Table 1: Efficiency: Old and New

DMU	1	2	3	4	5	6
X	4	6	8	4	2	10
	3	2	1	2	4	1
Y	1	1	1	1	1	1
$\boldsymbol{s}_{x_1}^*$	0	0	0	0	0	2
$\boldsymbol{s}_{x_2}^*$	0	0	0	0	0	0
$\boldsymbol{s}_y^*$	0	0	0	0	0	0
θ^*	.86	.86	1	1	1	1
η^*	.86	.86	1	1	1	.82

Note: The new efficiency measure is not invariant to the scaling of the data (X, Y) if the DMU has positive slacks. We can adjust the measure by considering some weights $\boldsymbol{w}_x \in R^m$ and $\boldsymbol{w}_y \in R^s$ corresponding to the relative importance of input resources and output products of DMU_o, respectively as follows:

The first phase minimizes θ of (LP_o). Then, the second phase maximizes $\gamma = \boldsymbol{w}_x^T \boldsymbol{s}_x$, while keeping $\theta = \theta^*$ (the optimal θ value of the first phase). Lastly, the third phase maximizes $\boldsymbol{w}_y^T \boldsymbol{s}_y$, while keeping $\theta = \theta^*$ and $\boldsymbol{w}_x^T \boldsymbol{s}_x = \gamma^*$ (the optimal γ value of the second phase). Let the optimal solution of the third phase be $(\boldsymbol{\lambda}^*,\ \boldsymbol{s}_x^*,\ \boldsymbol{s}_y^*)$. Then, a new measure of efficiency is defined by

$$\eta^* = \left(\theta^* - \frac{\boldsymbol{w}_x^T \boldsymbol{s}_x^*}{\boldsymbol{w}_x^T \boldsymbol{x}_o}\right)\left(\frac{\boldsymbol{w}_y^T \boldsymbol{y}_o}{\boldsymbol{w}_y^T \boldsymbol{y}_o + \boldsymbol{w}_y^T \boldsymbol{s}_y^*}\right). \tag{22}$$

The η^* defined above is uniquely determined for the given weights $\boldsymbol{w}_x$ and $\boldsymbol{w}_y$ and reflects the intention of the *input oriented* DEA.

5 Concluding Remarks

Considering the primal and the dual sides of the DEA model, this article pointed out the equivalence of the slackless solution in the primal and the existence of a positive weight in the dual. This study also proposed a new measure of efficiency. Although we have been mainly concerned with the input-oriented DEA, we can easily extend the results to the output-oriented DEA which is usually represented by:

$$\begin{array}{rrcl} & \max\ \xi & & \\ \text{subject to} & \boldsymbol{x}_o & \geq & X\boldsymbol{\lambda}, \\ & \xi \boldsymbol{y}_o & \leq & Y\boldsymbol{\lambda}, \\ & \boldsymbol{\lambda} & \geq & \mathbf{0}, \end{array}$$

where $\xi (\geq 1)$ is the expansion factor of the outputs for DMU_o.
Following an analogous rationale, we can define a new measure of efficiency τ^* for the output oriented DEA, using the max slack solution $(\xi^*, \boldsymbol{\lambda}^*, \boldsymbol{s}_x^*, \boldsymbol{s}_y^*)$ by the formula:

$$\tau^* = \left(\xi^* + \frac{\boldsymbol{e}^T \boldsymbol{s}_y^*}{\boldsymbol{e}^T \boldsymbol{y}_o}\right)\left(\frac{\boldsymbol{e}^T \boldsymbol{x}_o}{\boldsymbol{e}^T \boldsymbol{x}_o - \boldsymbol{e}^T \boldsymbol{s}_x^*}\right). \tag{23}$$

Several DEA models (Banker-Charnes-Cooper [3], increasing returns to scale, decreasing returns to scale, among others) are presented and extensively studied. (See e.g., [1],[4].) The new measure of efficiency proposed in this study can be easily incorporated within these models as long as the models are derived from some production possibility set.

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Appendix 1(Slaster [15])

Let A, B, C and D be given matrices with A and B being nonvacuous. Then the system (I) *or* (II) has a solution but never *both*.

(I)

$$A\boldsymbol{x} > \mathbf{0}, \quad B\boldsymbol{x} \geq \mathbf{0}, \quad B\boldsymbol{x} \neq \mathbf{0}, \quad C\boldsymbol{x} \geq \mathbf{0}, \quad D\boldsymbol{x} = \mathbf{0}$$

has a solution $\boldsymbol{x}$.

(II)

$$\boldsymbol{y}_1^T A + \boldsymbol{y}_2^T B + \boldsymbol{y}_3^T C + \boldsymbol{y}_4^T D = \mathbf{0}$$

$$\text{with } \boldsymbol{y}_1 \geq \mathbf{0}, \ \boldsymbol{y}_1 \neq \mathbf{0}, \ \boldsymbol{y}_2 \geq \mathbf{0}, \ \boldsymbol{y}_3 \geq \mathbf{0}$$

or

$$\boldsymbol{y}_1 \geq \mathbf{0}, \ \boldsymbol{y}_2 > \mathbf{0}, \ \boldsymbol{y}_3 \geq \mathbf{0}$$

has a solution $(\boldsymbol{y}_1, \boldsymbol{y}_2, \boldsymbol{y}_3, \boldsymbol{y}_4)$.

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Kaoru TONE
Graduate School of Policy Science
Saitama University
Urawa, Saitama 338
Japan