Journal of the Operations Research Society of Japan
Vol. 36, No. 3, September 1993

A SIMPLE OPTION PRICING MODEL WITH MARKOVIAN VOLATILITIES

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(Received October 8, 1992; Revised April 28, 1993)

Abstract This paper develops a stochastic volatility model, which overcomes the Black-Scholes model tendency to overprice near at-the-money options and underprice deep out or in-the-money options. In contrast to the previous literature that assumes diffusion volatilities, this paper assumes that the volatility follows a Markov chain on a discrete state space. This intuitive approach has easier mathematics and, by taking limit, the diffusion results can be obtained. By generalizing the binomial model to the Markovian volatility model, a recursive pricing scheme is first developed, under a particular assumption on preference to the volatility dynamics, and then a continuous time result by taking limit. The both discrete and continuous models give general conditions under which the call value is increasing in the current volatility. Also, based on the local convexity and concavity of the Black-Scholes equation in volatility, we explain why the deficits of the Black-Scholes equation take place. Some numerical experiments are also given to support our results.

1 Introduction

Black and Scholes (BS) equation [2] has been widely recognized as a useful tool to determine the price of a call option written on a stock whose price fluctuation obeys a geometric Brownian motion process. Despite of the questionable Brownian motion assumption, the very reason that the BS equation is so widely used is its easily computable form. However, it has been pointed out that the BS equation involves problems that are intolerable for practitioners (see, e.g., Rubinstein [26]). For example, the BS equation works poorly (underprices) for deep out or in-the-money options whereas it overprices near at-the-money options (see, e.g., Finnerty [9] and MacBeth and Merville [20]). A natural way to overcome such deficits of the BS equation is to construct a model that allows for the volatility of stock returns to change over time, since the volatility is the only parameter taken from stock prices into the BS equation, and also since there is some empirical evidence that indicates that the volatility does change[1].

Since 1987, many papers concerning option pricing with stochastic volatility have been published, most of which assume that the volatility follows a diffusion process[2]. For example, Hull and White [15] and Wiggins [30] use a geometric Brownian motion as the volatility dynamics, Scott [27] and Stein and Stein [28] an Ornstein-Uhlenbeck process, whereas Johnson and Shanno [17], Bailey and Stulz [1] and Heston [13] assume a square root process (see, e.g., Cox, Ingersoll and Ross [6]) or its slight generalization[3]. Since, in those models, there

[1]Another way to do this may be to use a fat-tailed process as the stock price process. Assuming Naik and Lee's equilibrium pricing [24], Madan and Milne [21] derive an approximate call option price when the variance Gamma (VG) process [22] is used as the uncertainty driving the stock price, and compare it with the BS price. They conclude through numerical experiences that VG values are typically higher than BS values especially for out-of-the-money options.

[2]Föllmer and Schweizer [10] consider a very general model and develop a pricing scheme for contingent claims in an incomplete market by a remaining-risk-minimization strategy.

[3]As Bailey and Stulz [1] point out, the economic relevance of stochastic volatility option models depends

are two assets in the market consisting of one risky stock and one riskless bond and since volatility is not a traded asset, the market is incomplete so that options written on the stock can not be priced by the usual arbitrage arguments (see, e.g., Harrison and Pliska [12]). In order to price such contingent claims, we need to determine the "price of volatility risk". Wiggins [30] uses an intertemporal general equilibrium model of Cox, Ingersoll and Ross [5] and shows that the case of a representative investor with log-utility leads to the price of volatility risk being zero, where the option is assumed to be written on a market portfolio whose price process is uncorrelated to the volatility process (see also Stein and Stein [28]). Heston [13] makes a particular assumption on the price of volatility risk while Bailey and Stulz [1] relate the volatility dynamics to the interest rate dynamics and then make use of a result of Cox, Ingersoll and Ross [6]. Hull and White [15] and others assume that investors' attitude to volatility risk is neutral, i.e., the price of volatility risk is *a priori* assumed to be zero. Each of these assumptions is sufficient to price all contingent claims and they obtain respective partial differential equations which option pricing must satisfy. Furthermore, for the case that the stock price process and the volatility dynamics are independent of each other, Hull and White [15] and Scott [27] show that the solution is given as an integral of the BS equation with respect to the distribution function of the volatility dynamics. Therefore, in principle, the option price can be evaluated numerically. Unfortunately, however, these formulas are very difficult to compute so that they ought to go into either approximations or Monte Carlo simulation. Although they found through extensive numerical works that the stochastic volatility models are better at explaining actual option prices, the computational difficulty often makes the models impractical.

On the other hand, from the points of view of model construction, it is unclear why the volatility dynamics should have a continuous path. For example, many practitioners feel that there are two situations, in one of which a stock price fluctuates up and down quickly and in the other its movement is relatively slow. The two cases, of course, correspond to the situations of high volatility and of low volatility, respectively. In Figure 1, we plotted historical volatilities of daily returns on IBM during 1990, where volatility values are calculated as a moving standard deviation of 20 days. It can be observed that the periods before September and after November have a low level of volatility and the other a high volatility. Although, more generally, we may need to assume multi-levels of volatility[4], a discrete model is attractive because of the following reasons: First, under some regularity conditions, many continuous models can be obtained as limits of the corresponding discrete models. Second, discrete models can describe our intuitions behind more directly, and third, the mathematics used is usually more elementary and transparent. In this paper, assuming that the volatility dynamics follows a Markov chain on a discrete state space, we construct a simple stochastic volatility model and derive an option pricing scheme.

This paper is organized as follows. In the next section, we develop a discrete-time model in which the stochastic law of volatility is described in terms of a discrete-time Markov chain. This model can be regarded as an extension of the binomial model of Cox, Ross and Rubinstein [7]. Under a particular assumption on preference to the volatility dynamics, a recursive pricing scheme is developed for European call options. Based on this formula, we then show that, if the underlying Markov chain is stochastically monotone, the higher current volatility the higher the option price. This conclusion affirms a well recognized phenomenon in actual option prices. In Section 3, we then derive a continuous-time model by shrinking both the size of movement of the stock price and the time scale at the same time, but not the state of volatility. The continuous-time model has a geometric Brownian motion process as the stock price process with a continuous-time Markovian volatility on a discrete state space. Since any diffusion process can be approximated by a birth-death

nontrivially on the assumed volatility dynamics.

[4]For example, in Figure 2, it may be plausible to assume 4 levels of volatility.

Figure 1. IBM Historical Volatility

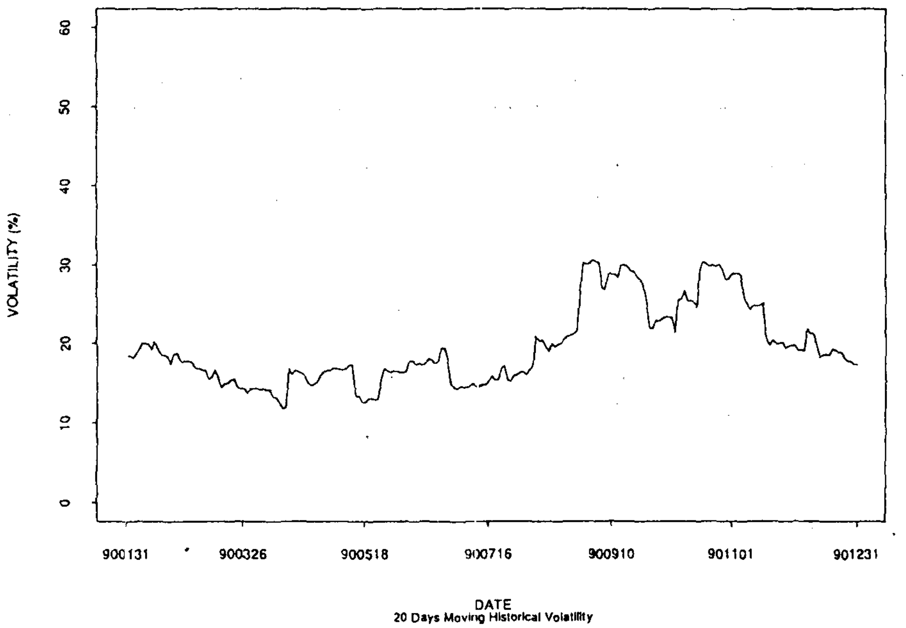
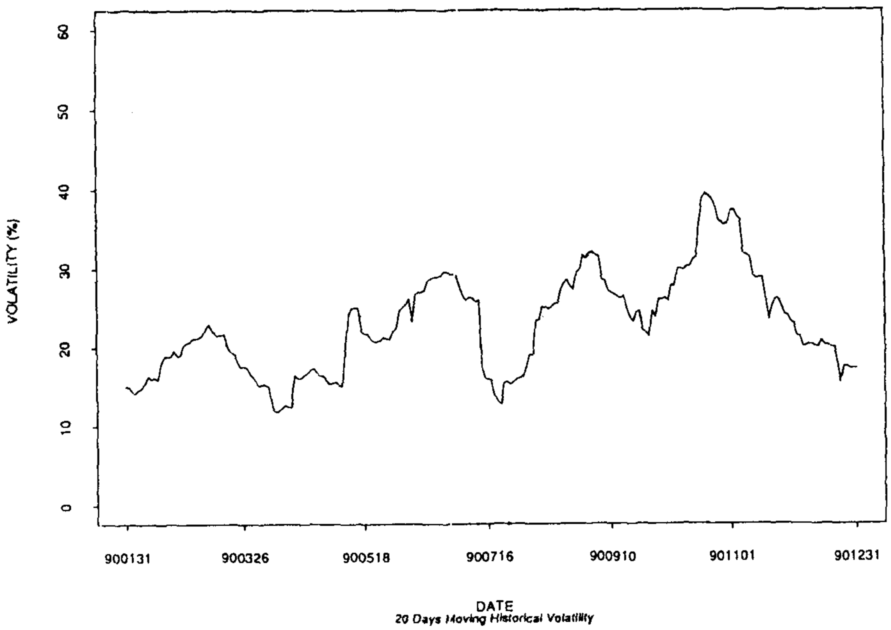


Figure 2. ITT Historical Volatility



process[5], our model includes the existing stochastic volatility models mentioned above as limit. An explicit call valuation formula is given when the volatility can take one of two values. More generally, when possible volatility values are all sufficiently high or sufficiently low so that the BS equation is globally convex or concave in volatility, lower and upper bounds are given for the call value under the Markovian volatility. These results in turn explain explicitly why the BS equation underprices deep out or in-the-money options and overprices near at-the-money options. Some numerical experiments are given to support our results. It is observed that, for realistic parameter values, the BS equation is usually globally convex or concave, and that the bounds on Markovian volatility prices are tight for near at-the-money options maturing within one year. Finally, in Section 4, we conclude this paper.

2 The Discrete-time Model

In this section, we develop the framework of the discrete option pricing model with Markovian volatilities. Throughout this paper, it is assumed that the market is frictionless with no dividends paid, and consists of one risky stock and one riskless bond. The rate of return on the bond over each period is constant, say $\rho - 1$ with $\rho > 1$. To describe the movement of stock price, we introduce a stochastic process that represents the state of some underlying conditions of the market. Let X_n denote the underlying state at time n and suppose that $\{X_n, n = 0, 1, \cdots\}$ follows a discrete-time, irreducible and time-homogeneous Markov chain defined on the state space $\mathcal{S} = \{1, 2, \cdots, m\}$, m possibly infinite, which is governed by the stationary transition probability matrix $\boldsymbol{F} = (\boldsymbol{f}_{ij})$, where

$$f_{ij} = \Pr[X_{n+1} = j | X_n = i], \quad i, j \in \mathcal{S}.$$

The underlying conditions are assumed to include all that affect the change of the stock price. The Markov chain will be called the underlying process of the stock. When $X_n = i$, $i \in \mathcal{S}$, the rate of return on the stock over each period has two possible values: $u_i - 1$ with probability α_i or $d_i - 1$ with probability $1 - \alpha_i$. That is, using the notation in Cox, Ross and Rubinstein [7], the stock price movement is described as

$$S \begin{cases} u_i S, & \text{with probability } \alpha_i, \\ d_i S, & \text{with probability } (1 - \alpha_i), \end{cases} \tag{2.1}$$

where we assume that $u_i > \rho > d_i$ for all $i \in \mathcal{S}$. For a moment, we also assume that the underlying process $\{X_n\}$ is independent of the stock price movement[6].

Let $C_i(n, S, K)$ denote the price of a European call option written on the stock when the current stock price is S, the time to maturity n, the exercise price K and the current underlying state i. It is evident that $C_i(0, S, K) = \max\{S - K, 0\}$ for all $i \in \mathcal{S}$. In order to describe our model explicitly, we begin by the one-period model. In this case, since investors recognize the current underlying state and therefore the stock price movement (2.1), and since the underlying state at the terminal epoch is irrelevant to the option pricing, they can construct the unique hedging portfolio exactly as in Equation (1) of Cox, Ross and Rubinstein [7]. Hence, the price of the call is uniquely determined as

$$C_i(1, S, K) = \rho^{-1}[p_i \max\{u_i S - K, 0\} + (1 - p_i) \max\{d_i S - K, 0\}], \tag{2.2}$$

where the "risk-neutral probabilities" p_i are given by

$$p_i = \frac{\rho - d_i}{u_i - d_i}; \quad 1 - p_i = \frac{u_i - \rho}{u_i - d_i}, \quad i \in \mathcal{S}. \tag{2.3}$$

[5]For example, an Ornstein-Uhlenbeck process is approximated by the Ehrenfest urn model and a Brownian motion by the simple random walk. See, e.g, Karlin and Taylor [18] for details.

[6]More rigorously, we assume that the process $\{X_n\}$ is independent of the number of up jumps in the option's life.

We next consider the two-period model. In this case, since investors do not know the underlying state at the next period, they can not construct a "complete" hedging portfolio. In fact, there are $2m$ possible states at time 1 so that the market is incomplete. Thus, as mentioned in Section 1, we need to give the price of volatility risk by some means. In what follows, we do not take the usual course of equilibrium pricing (see Section 1), but rather make some assumptions on preference to the volatility dynamics. As we shall show, the assumptions make the "martingale probability" unique so as to determine the option price. Not surprisingly, our option pricing turns out to estimate the price of volatility risk to be zero, as in many of former studies[7].

Let $p_j^*(u_i)$ and $p_j^*(d_i)$ satisfy

$$\sum_{j=1}^{m}[u_i p_j^*(u_i) + d_i p_j^*(d_i)] = \rho; \quad \sum_{j=1}^{m}[p_j^*(u_i) + p_j^*(d_i)] = 1. \tag{2.4}$$

Then, $p_j^*(u_i)$ and $p_j^*(d_i)$ are called martingale probabilities corresponding to up and down, respectively, of the stock price at the next period when the underlying state at time 1 is $j \in \mathcal{S}$. According to the martingale theory of Harrison and Kreps [11], if (2.4) holds, the no arbitrage price for the call option is given by

$$C_i(2,S,K) = \rho^{-1}\sum_{j=1}^{m}[p_j^*(u_i)C_j(1,u_iS,K) + p_j^*(d_i)C_j(1,d_iS,K)]. \tag{2.5}$$

Note that there are infinitely many $p_j^*(u_i)$ and $p_j^*(d_i)$ that satisfy (2.4).

Consider a representative investor who has a utility function $U(i,j)$ to the movement from state i to state j of the underlying process $\{X_n\}$. This utility function is cardinal (see, e.g., Ingersoll [16]) but, other than this, no restriction is made for a moment[8] (see Lemma 2.1 below). Since the investor knows the stochastic law $\boldsymbol{F} = (\boldsymbol{f}_{ij})$ of the Markov chain $\{X_n\}$ and the risk-neutral probabilities p_i in (2.3) for the one-period case, it may be plausible to assume that

$$p_j^*(u_i) = p_i f_{ij} U(i,j); \quad p_j^*(d_i) = (1-p_i) f_{ij} U(i,j). \tag{2.6}$$

Then, the two-period option price (2.5) is given by the discounted expectation of the utility times the one-period option price. To be more specific, let R be a random variable that, given $X_0 = i$, equals u_i with probability p_i or d_i with probability $1-p_i$. Then, the two-period price is given by

$$C_i(2,S,K) = \rho^{-1}E[U(i,X_1)C_{X_1}(1,RS,K)|X_0 = i]. \tag{2.7}$$

Intuition behind (2.7) is the following. If the investor gets high (low, respectively) utility by the transition from state i to state X_1 then the one-period price $C_{X_1}(1,RS,K)$ is evaluated with a higher (lower) weight $U(i,X_1)$. On the other hand, if the investor does not care the transition, then he will value the call only based on the expectation. As a consequence, this case leads to

$$C_i(2,S,K) = \rho^{-1}\left[p_i\sum_{j=1}^{m} f_{ij}C_j(1,u_iS,K) + (1-p_i)\sum_{j=1}^{m} f_{ij}C_j(1,d_iS,K)\right]. \tag{2.8}$$

This, of course, corresponds to the case that the investor's attitude to volatility risk is neutral, i.e. the price of volatility risk is zero.

[7] If $m < \infty$, the remaining-risk-minimization strategy of Föllmer and Schweizer [10] and Hofmann, Platen and Schweizer [14] also give the same price.

[8] Since the analysis that follows is irrelavent to the utility of wealth, no restriction on the von Neumann and Morgenstern utility function is made either.

The next lemma shows that if $U(i,j)$ is of a particular form then the martingale probabilities bear risk-neutrality.

Lemma 2.1. Suppose that the martingale probabilities are given as in (2.6). If $U(i,j) = U(j)/U(i)$, where $U(i)$ is bounded, and if the Markov chain $\{X_n\}$ is recurrent, then $U(i,j) = 1$ for all $i,j \in \mathcal{S}$, i.e., the investor's preference to the movement of the underlying process is indifference.

Proof. From (2.4), (2.6) and the assumptions, one has

$$\sum_{j=1}^{m} f_{ij} U(j) = U(i), \quad i \in \mathcal{S}. \tag{2.9}$$

But, since the Markov chain is recurrent, the only bounded solution to (2.9) is constant (see Blackwell [3]). Hence, $U(i,j) = 1$. □

The assumption $U(i,j) = U(j)/U(i)$ in Lemma 2.1 may be justified as follows. For each state $i \in \mathcal{S}$, some "potential utility" $U(i)$ is assigned. If the current state is preferable to the investor, the transition to state j may be relatively less preferable. If, on the other hand, the current state is not preferable, he may be relatively more satisfied by any transition. The assumption $U(i,j) = U(j)/U(i)$ will describe this sort of situation. It is noted that, under the condition of Lemma 2.1, the investor's attitude to volatility risk is neutral and the option price is given by (2.8).

In order to develop the multi-period model, we assume that the investor's utility to the movement of the underlying process $\{X_n\}$ is Markovian and stationary in time. That is, if the current state of $\{X_n\}$ is i with history $\{i_0, \cdots, i_{n-1}\}$, then the utility by the transition to state j is merely given by $U(i,j)$ [9]. Then, under the conditions of Lemma 2.1, we obtain the "risk-neutral world," and hence the call price with Markovian volatilities is recursively given as in (2.8) for general n. The next theorem summarizes the above discussion.

Theorem 2.1. Suppose that the investor's utility to the movement of the underlying process is Markovian and stationary in time, and suppose that the martingale probabilities are given as in (2.6). If $U(i,j) = U(j)/U(i)$, where $U(i)$ is bounded, and if the Markov chain $\{X_n\}$ is recurrent, then the call option price is determined by the recursion formula

$$C_i(n,S,K) = \rho^{-1} \sum_{j=1}^{m} f_{ij}[p_i C_j(n-1, u_i S, K) + (1-p_i) C_j(n-1, d_i S, K)], \tag{2.10}$$

starting with $C_i(0,S,K) = \max\{S-K, 0\}$. □

Remark 2.1. The recursive pricing scheme developed in Theorem 2.1 has a numerical intractability, because the computational complexity grows according to $(2m)^n$. However, it is useful as a milestone in the development of our Markovian volatility option pricing model, as we shall see below. □

In what follows, we investigate some properties of the option pricing with Markovian volatilities given by (2.10). The Markov chain $\{X_n\}$ is said to be stochastically monotone if its transition probabilities f_{ij} satisfy

$$\sum_{j=1}^{\ell} f_{ij} \ge \sum_{j=1}^{\ell} f_{kj} \quad \text{for all } i \le k \text{ and for all } \ell \in \mathcal{S} \tag{2.11}$$

(see, e.g., Keilson [19]). The next result supports a well recognized phenomenon in actual option prices.

[9] In general, the utility may depend on the path of $\{X_n\}$ and the time. For such a general case, it will be given by $U_n(i_0, \cdots, i_{n-1}, i, j)$.

Theorem 2.2 Suppose that $u_1 \geq \cdots \geq u_m > \rho > d_m \geq \cdots \geq d_1$. If the underlying Markov chain is stochastically monotone, then $C_i(n, S, K)$ is non-increasing in $i \in \mathcal{S}$.

Proof. The proof proceeds by induction. For $n = 1$, note from (2.2) and (2.3) that $\rho C_i(1, S, K)$ is evaluated at $x = \rho S$ of the straight line

$$y_i = \frac{g(u_i S) - g(d_i S)}{u_i S - d_i S}(x - d_i S) + g(d_i S), \quad d_i S \leq x \leq u_i S,$$

where $g(x) = \max\{x - K, 0\}$. Since $g(x)$ is convex, the straight lines are ordered as $y_1 \geq \cdots \geq y_m$ for $d_m S \leq x \leq u_m S$, if u_i and d_i are given as in the theorem. Hence, the theorem follows for $n = 1$. Now, suppose that $C_i(n-1, S, K) \geq C_j(n-1, S, K)$ for $i < j$. Recalling Merton's result [23] that $C_k(n-1, S, K)$ are convex in S, the same argument as for $n = 1$ goes through to prove that

$$\begin{aligned} p_i C_k(n-1, u_i S, K) &+ (1 - p_i) C_k(n-1, d_i S, K) \\ &\geq p_j C_k(n-1, u_j S, K) + (1 - p_j) C_k(n-1, d_j S, K), \quad i < j, \end{aligned}$$

from which

$$\rho C_j(n, S, K) \leq \sum_{k=1}^{m} f_{jk}[p_i C_k(n-1, u_i S, K) + (1 - p_i) C_k(n-1, d_i S, K)]. \tag{2.12}$$

Also, from (2.10) and since $f_{ik} = \sum_{\ell=1}^{k} f_{i\ell} - \sum_{\ell=1}^{k-1} f_{i\ell}$ with understanding that the empty sum equals 0, one has

$$\begin{aligned} \rho C_i(n, S, K) = \sum_{k=1}^{m} \sum_{\ell=1}^{k} f_{i\ell}[p_i\{C_k(n-1, u_i S, K) - C_{k+1}(n-1, u_i S, K)\} \\ +(1 - p_i)\{C_k(n-1, d_i S, K) - C_{k+1}(n-1, d_i S, K)\}]. \end{aligned} \tag{2.13}$$

Since $C_k(n-1, S, K)$ is non-increasing in k by the induction hypothesis, the curly bracket terms in the right hand side of (2.13) are all non-negative. The desired monotonicity of $C_i(n, S, K)$ then follows from (2.11) through (2.13), proving the theorem. □

A couple of remarks are suitable at this point.

Remark 2.2. It is well known that the BS equation is non-decreasing with respect to volatility (see, e.g., Cox and Rubinstein [8]). However, this comparability result holds only in the sense of statics, since the volatility there is held fixed over time. In practice where the volatility is changing over time, it is observed that the higher current volatility the higher the option price. Theorem 2.2 affirms it if the underlying Markov chain is stochastically monotone. Since any strong Markov process whose sample path is continuous can be approximated by a stochastically monotone Markov chain (see, e.g., Keilson [19]), the result in Theorem 2.2 seems to explain, to some extent, the above well recognized phenomenon in practice. □

Remark 2.3. Suppose that the conditions in Theorem 2.2 hold and the current underlying state is 1. Then, as the time goes by, the underlying process is moving to a higher state with high probability (see, e.g., Keilson [19]). Since the volatility in a higher state is smaller, the value of the call option might be gradually degraded. Therefore, an investor who holds an American call option written on the stock may have an incentive to exercise his right before the maturity. However, as Merton [23] has proved in great generality, rational investors never exercise before the maturity. Indeed, it can be easily verified that $C_i(n, S, K) \geq \max\{S - K, 0\}$ for any $i \in \mathcal{S}$ and $n = 1, 2, \cdots$. □

Remark 2.4. So far, we have assumed that the rate of return ρ on the bond as well as u_i and d_i, $i \in \mathcal{S}$, does not change in time. It is possible, as far as concerned with the pricing formula (2.10), that these parameters are allowed to change over time, depending on the whole history. Furthermore, the transition probabilities f_{ij} of the underlying Markov chain can also be dependent on the history. However, as noted in Remark 2.1, the computational complexity becomes much harder, making these generalizations impractical. Finally, we note that the price of a European put option can be obtained by a straightforward extension of the so-called "put-call parity". □

Suppose that the Markov chain $\{X_n\}$ is ergodic and let $\boldsymbol{\pi} = (\boldsymbol{\pi}_i)$ be the stationary distribution of $\{X_n\}$. $\boldsymbol{\pi}$ is given as the unique row vector satisfying the equation $\boldsymbol{\pi} = \boldsymbol{\pi F}$ with $\sum_{i=1}^{m} \pi_i = 1$. Let

$$\bar{u} = \sum_{i=1}^{m} \pi_i u_i; \quad \bar{d} = \sum_{i=1}^{m} \pi_i d_i. \tag{2.14}$$

Now, consider the following situation. Suppose that the underlying process is *not* recognized and the rate of return on the stock is estimated by a historical data that consists of a sufficiently long period. Then, the estimated rates of return will be given as in (2.14). Suppose that the binomial model of Cox, Ross and Rubinstein [7] is used to price the call. We call this price the CRR value. It is then of great interest to know the relation between CRR and the average value AVG$=\sum_{i=1}^{m} \pi_i C_i(n, S, K)$. Note that the latter represents the average price of the calls for a sufficiently long period, if the underlying process *is* recognized. If the former is greater than the latter then the CRR value overprices, while it underprices otherwise. Unfortunately, however, there seems no mathematical relation between the two prices in the discrete-time model in contrast to the continuous-time counterpart (see Theorem 3.3 for the continuous-time model).

In Example 2.1 below, we perform some numerical experiment for the discrete-time model. However, as indicated in Remark 2.1, we do not intend to show the practical usefulness of this model, but rather infer some intuitions behind.

Example 2.1. In this example, we assume that the underlying conditions have only two states, say good for state 1 and bad for state 2 (cf. Figure 1). The current stock price S is fixed to be 50. We consider two cases where the transition probabilities of the underlying Markov chain $\{X_n\}$ are given by $f_{11} = 0.7$ and $f_{22} = 0.5$ as Case 1, and $f_{11} = f_{22} = 0.4$ as Case 2. Note that $\{X_n\}$ is stochastically monotone if and only if $f_{11} + f_{22} \geq 1$. Hence, Case 1 assumes a stochastically monotone Markov chain while Case 2 does not. In Table 1, various values of $C_i(n, S, K)$, $i = 1, 2$, AVG$=\sum_{i=1}^{2} \pi_i C_i(n, S, K)$ and CRR are listed[10]. It is explicitly observed that $C_1(n, S, K) > C_2(n, S, K)$ in Case 1, as it should be, and even in Case 2, although this case needs not be. Also, it is interesting to note that the CRR underprices all the cases with even the situation $C_2(n, S, K) >$CRR sometimes. □

3 The Continuous-time Model

In this section, we derive an option pricing for the continuous-time model as a limit of (2.10). For this purpose, we let h be the length of each period and assume it sufficiently small. For the underlying process $\{X_n\}$, we set $f_{ij} = q_{ij} h$ for $i \neq j$. The discrete-time Markov chain $\{X_n\}$ then converges in distribution to a continuous-time Markov chain $\{X(t), t \geq 0\}$, on the same state space $\mathcal{S}$, governed by the infinitesimal generator $\boldsymbol{Q} = (\boldsymbol{q}_{ij})$ with $q_{ii} = -q_i$ and $q_i = \sum_{j \neq i} q_{ij}$. Throughout this section, we assume that q_i is bounded to eliminate inessential technical difficulties. Now, let μ_i and σ_i denote the instantaneous mean rate of return and the volatility of the stock price of the continuous-time model, respectively, when the underlying process is in state i. Then, assuming that the underlying process is independent of the stock price process (see footnote 6 for details), the exactly same

[10] All the calculations in this paper are carried out by Splus software package on Sun workstation.

Parameters	$r = 1.05, u_1 = 1.10, u_2 = 1.07, d_1 = 0.80, d_2 = 0.90$					
	K	n	$C_1(n,S,K)$	$C_2(n,S,K)$	AVG	CRR
	40	2	13.8813	13.7544	13.8337	13.8252
		3	15.6841	15.5292	15.6260	15.5749
		4	17.2989	17.1986	17.2613	17.1810
		5	18.8317	18.7586	18.8043	18.7916
		6	20.3482	20.2679	20.3181	20.2759
Case 1	45	2	9.7254	9.4387	9.6179	9.3979
		3	11.6402	11.3819	11.5434	11.5323
$\pi_1 = \frac{5}{8}$		4	13.5680	13.3131	13.4724	13.4088
$\pi_2 = \frac{3}{8}$		5	15.2525	15.0615	15.1809	14.9865
		6	16.8181	16.7064	16.7762	16.7348
	50	2	6.4003	5.5093	6.0662	6.0138
		3	8.1316	7.6840	7.9638	7.5932
		4	9.9218	9.5783	9.7930	9.7686
		5	11.9117	11.5752	11.7855	11.7158
		6	13.6206	13.3653	13.5249	13.2921
	55	2	3.1954	2.0765	2.7758	2.7697
		3	5.3542	4.5176	5.0405	4.9801
		4	6.9818	6.4764	6.7923	6.4223
		5	8.6759	8.2909	8.5316	8.4745
		6	10.6554	10.2658	10.5093	10.4322
	60	2	0.2205	0.0000	0.1378	0.0000
		3	2.7564	1.7287	2.3710	2.3669
		4	4.6659	3.9176	4.3853	4.3174
		5	6.1675	5.6873	5.9874	5.5920
		6	7.8113	7.3985	7.6565	7.5722
	K	n	$C_1(n,S,K)$	$C_2(n,S,K)$	AVG	CRR
	40	2	13.8421	13.7615	13.8119	13.7968
		3	15.6150	15.5225	15.5803	15.5124
		4	17.2543	17.1804	17.2266	17.1594
		5	18.7861	18.7437	18.7702	18.7514
		6	20.2902	20.2421	20.2722	20.2167
Case 2	45	2	9.6876	9.4784	9.6091	9.3622
		3	11.5430	11.3739	11.4796	11.4521
$\pi_1 = \frac{1}{2}$		4	13.4440	13.2855	13.3845	13.2903
$\pi_2 = \frac{1}{2}$		5	15.1441	15.0106	15.0940	14.9360
		6	16.7512	16.6751	16.7226	16.6590
	50	2	6.1869	5.5879	5.9623	5.8216
		3	8.0427	7.7004	7.9143	7.3918
		4	9.7536	9.5455	9.6756	9.6216
		5	11.7264	11.5085	11.6447	11.5339
		6	13.4477	13.2787	13.3843	13.1012
	55	2	2.9264	2.1748	2.6445	2.5367
		3	5.0875	4.5102	4.8710	4.7203
		4	6.8421	6.4480	6.6943	6.1690
		5	8.4528	8.2314	8.3698	8.2542
		6	10.4084	10.1481	10.3108	10.1864
	60	2	0.1206	0.0000	0.0787	0.0000
		3	2.4271	1.7189	2.1615	2.0578
		4	4.3564	3.8425	4.1637	4.0109
		5	5.9947	5.5981	5.8460	5.3115
		6	7.5368	7.2975	7.4471	7.2800

Table 1: Option Prices Calculated by Discrete Model

argument as in Cox, Ross and Rubinstein [7] shows that the stock price process converges in distribution to a stochastic process $\{S(t), t \geq 0\}$ satisfying the stochastic differential equation

$$\frac{dS(t)}{S(t)} = \mu_{X(t)}dt + \sigma_{X(t)}dW(t), \tag{3.1}$$

where $\{W(t), t \geq 0\}$ denotes the standard Brownian motion process independent of $\{X(t)\}$. For the bond price, we set $\rho = e^{rh}$ in the discrete-time model. The bond price then converges to $B(t)/B(0) = e^{rt}$.

Let $C_i(T, S, K)$ be the price of a European call of the continuous-time model, where S is the current stock price, T the time to maturity, K the exercise price and i the current state of the underlying process. $C_i(T, S, K)$ can be obtained as a limit of (2.10). The proof of the next theorem is standard and is omitted. Note that the instantaneous mean rates μ_i are irrelevant to option pricing.

Theorem 3.1. In the above continuous-time model, the option price satisfies the following set of partial differential equations:

$$\sum_{j=1}^{m} q_{ij}C_j - \frac{\partial C_i}{\partial T} + \frac{1}{2}\sigma_i^2 S^2 \frac{\partial^2 C_i}{\partial S^2} + rS\frac{\partial C_i}{\partial S} - rC_i = 0, \quad i \in \mathcal{S}. \tag{3.2}$$

Remark 3.1. If state i is absorbing so that $q_{ij} = 0$ for all $j \in \mathcal{S}$, then (3.2) for $i \in \mathcal{S}$ coincides with the partial differential equation obtained by Black and Scholes [2]. Also, since $C_i(T, S, K)$ is derived from (2.10) as a limit, the properties obtained in Section 2 hold under appropriate assumptions. For example, if the continuous-time Markov chain $\{X(t)\}$ is stochastically monotone (see, e.g., Keilson [19]) and if $\sigma_1 > \cdots > \sigma_m$, then we have $C_1(T, S, K) \geq \cdots \geq C_m(T, S, K)$. Since any diffusion process can be approximated by a birth-death process (see footnote 5), which is necessarily stochastically monotone, the above result holds for any diffusion volatility model, provided that the state of volatility is appropriately ordered. □

Another look at the solution of (3.2) is as follows. Let t_i, $i \in \mathcal{S}$, denote the random total time that $\{X(t)\}$ is in state i during the time interval $[0, T]$. Of course, $\sum_{i=1}^{m} t_i = T$. Given a realization of $\{X(t), 0 \leq t \leq T\}$, it is well known that the logarithm of the solution $S(T)$ of (3.1) is distributed by the normal distribution having the variance $\sigma^2(T)T$, where

$$\sigma^2(T) = \frac{1}{T}\sum_{j=1}^{m} t_j\sigma_j^2, \tag{3.3}$$

and the option price is given by $\mathrm{BS}(\sigma^2(T))$. Here, $\mathrm{BS}(x)$ denotes the option price obtained from the BS equation having volatility $\sqrt{x}$, i.e.

$$\mathrm{BS}(x) = S\Phi(\xi) - Ke^{-rT}\Phi(\xi - \sqrt{xT}); \quad \xi = \frac{\log(S/K) + rT}{\sqrt{xT}} + \frac{\sqrt{xT}}{2}, \tag{3.4}$$

where $\Phi(\xi) = \int_{-\infty}^{\xi} \phi(u)du$ with $\phi(u) = \frac{1}{\sqrt{2\pi}}e^{-u^2/2}$. Since $\{W(t)\}$ and $\{X(t)\}$ are independent of each other, it follows that

$$C_i(T, S, K) = E[\mathrm{BS}(\sigma^2(T))|X(0) = i], \quad i \in \mathcal{S}, \tag{3.5}$$

which agrees with the expression obtained in Hull and White [15] and Scott [27]. Unfortunately, however, the probability distribution of $\sigma^2(T)$ given $X(0) = i$ is hard to obtain except for $m = 2$, i.e., the underlying process has 2 states. For $m = 2$, it is known (see page 178 in Ross [25]) that,

$$\Pr[t_1 = T | X(0) = 1] = e^{-q_1 T} \tag{3.6}$$

and, for $s < T$,

$$\begin{aligned} &\Pr[t_1 \le s | X(0) = 1] \\ = &\sum_{n=1}^{\infty} e^{-(q_1+q_2)T} \frac{[(q_1+q_2)T]^n}{n!} \\ &\times \sum_{k=1}^{n} {}_nC_{k-1} \left(\frac{q_2}{q_1+q_2}\right)^{k-1} \left(\frac{q_1}{q_1+q_2}\right)^{n-k+1} \tilde{\beta}(s/T; k, n-k+1), \end{aligned} \tag{3.7}$$

where $\tilde{\beta}$ denotes the beta distribution

$$\tilde{\beta}(s/T; k, n-k+1) = \sum_{i=k}^{n} {}_nC_i \left(\frac{s}{T}\right)^i \left(1 - \frac{s}{T}\right)^{n-i}. \tag{3.8}$$

Hence, $C_i(T, S, K)$ in (3.5) can be obtained from (3.6) through (3.8) as

$$\begin{aligned} &C_1(T, S, K) \\ = &\int_0^{T+} \mathrm{BS}\left(\sigma_2^2 + \frac{\sigma_1^2 - \sigma_2^2}{T} s\right) d\Pr[t_1 \le s | X(0) = 1] \\ = &\mathrm{BS}(\sigma_1^2) - \frac{S(\sigma_1^2 - \sigma_2^2)}{2\sqrt{2\pi}} \int_0^{T-} \frac{\Pr[t_1 \le s | X(0) = 1]}{\sqrt{\sigma_2^2 T + (\sigma_1^2 - \sigma_2^2)s}} \\ &\times \exp\left\{ -\frac{[\log(S/K) + rT + (\sigma_2^2 T + (\sigma_1^2 - \sigma_2^2)s)/2]^2}{2(\sigma_2^2 T + (\sigma_1^2 - \sigma_2^2)s)} \right\} ds. \end{aligned} \tag{3.9}$$

The last equality in (3.9) follows from integration by parts and since, by an algebra using (3.4), one has

$$\mathrm{BS}'(x) = \frac{S\sqrt{T}}{2\sqrt{x}} \phi(\xi) \tag{3.10}$$

(cf. page 221 of Cox and Rubinstein [8]). $C_0(T, S, K)$ can be obtained similarly.

We next obtain some bounds for (3.5) that are useful in estimating the true value of the option price. For this purpose, the next lemma is the key. In what follows, we write

$$v(r, T, S, K) = \frac{2}{T}(\sqrt{1 + (\log(S/K) + rT)^2} - 1), \tag{3.11}$$

and assume $\sigma_1 > \cdots > \sigma_m$.

Lemma 3.1. The BS equation (3.4) is convex in x for $x \le v(r, T, S, K)$ and is concave otherwise.

Proof. From (3.10), one has

$$\mathrm{BS}''(x) = S\sqrt{T}\left[\frac{-1}{4x\sqrt{x}}\phi(\xi) + \frac{1}{2\sqrt{x}} \frac{d\xi}{dx} \phi'(\xi)\right].$$

Since $\phi'(u) = -u\phi(u)$ and since

$$\frac{d\xi}{dx} = \frac{1}{2x}\left[-\frac{\log(S/K) + rT}{\sqrt{xT}} + \frac{\sqrt{xT}}{2}\right],$$

it follows that

$$\mathrm{BS}''(x) = \frac{S\sqrt{T}}{4x\sqrt{x}}\left(-1 + \frac{(\log(S/K)+rT)^2}{xT} - \frac{xT}{4}\right)\phi(\xi).$$

The lemma now follows at once. □

Let $\tau_{ij}(T) = E[t_j|X(0)=i]$, i.e., $\tau_{ij}(T)$ denotes the mean time that $X(t)$ is in state j during the time interval $[0,T]$ starting from $X(0)=i$, and let $\sigma_i^2(T) = E[\sigma^2(T)|X(0)=i]$. Then, from (3.3), one has

$$\sigma_i^2(T) = \frac{1}{T}\sum_{j=1}^{m}\tau_{ij}(T)\sigma_j^2. \tag{3.12}$$

Also, let

$$\eta_i = \frac{\sigma_i^2(T)-\sigma_m^2}{\sigma_1^2-\sigma_m^2}, \quad i\in\mathcal{S}. \tag{3.13}$$

Note that $\eta_i\sigma_1^2 + (1-\eta_i)\sigma_m^2 = \sigma_i^2(T)$. The next theorem provides an upper and lower bounds for (3.5).

Theorem 3.2. For any $i\in\mathcal{S}$, if $\sigma_1^2 \le v = v(r,T,S,K)$ then

$$\mathrm{BS}(\sigma_i^2(T)) \le C_i(T,S,K) \le \eta_i\mathrm{BS}(\sigma_1^2) + (1-\eta_i)\mathrm{BS}(\sigma_m^2), \tag{3.14}$$

whereas if $\sigma_m^2 \ge v$ then

$$\mathrm{BS}(\sigma_i^2(T)) \ge C_i(T,S,K) \ge \eta_i\mathrm{BS}(\sigma_1^2) + (1-\eta_i)\mathrm{BS}(\sigma_m^2), \tag{3.15}$$

where $\sigma_i^2(T)$ and η_i are given in (3.12) and (3.13), respectively.

Proof. If $\sigma_1^2 \le v$, since the event $\sigma^2(T)\le\sigma_1^2$ is certain, $\mathrm{BS}(x)$ for this case is convex in x from Lemma 3.1. Hence, the first inequality in (3.14) holds by applying Jensen's inequality to (3.5). To prove the second inequality, let τ be the random variable distributed as $\Pr[\tau=\sigma_1^2]=\eta_i$ and $\Pr[\tau=\sigma_m^2]=1-\eta_i$. Then, since $\sigma_m^2\le\sigma^2(T)\le\sigma_1^2$ with probability one, we have $\sigma^2(T)\prec_c\tau$, where $\prec_c$ denotes the stochastic convex ordering (see, e.g., Stoyan [29]). The desired inequality follows, since $\mathrm{BS}(x)$ is convex in x for this case so that

$$E[\mathrm{BS}(\sigma^2(T))|X(0)=i] \le E[\mathrm{BS}(\tau)] = \eta_i\mathrm{BS}(\sigma_1^2)+(1-\eta_i)\mathrm{BS}(\sigma_m^2).$$

(3.15) follows similarly. □

Remark 3.2. It is easy to see that

$$\tau_{ij}(T) = \sum_{k\ne i}\int_0^T q_{ik}e^{-q_it}(\delta_{ij}t+\tau_{kj}(T-t))dt + e^{-q_iT}\delta_{ij}T, \tag{3.16}$$

where $\delta_{ij}=1$ for $i=j$ and $\delta_{ij}=0$ otherwise. Taking the Laplace transform in (3.16) with respect to T, one has

$$\tilde{\tau}_{ij}(s) \stackrel{\text{def}}{=} \int_0^\infty e^{-sT}\tau_{ij}(T)dT = \sum_{k\ne i}\frac{q_{ik}}{s+q_i}\tilde{\tau}_{kj}(s) + \frac{\delta_{ij}}{s(s+q_i)}.$$

It follows that, in the matrix form,

$$\tilde{\boldsymbol{\tau}}_j(\boldsymbol{s}) = \frac{1}{\boldsymbol{s}}(\boldsymbol{sI}-\boldsymbol{Q})^{-1}\boldsymbol{e}_j, \tag{3.17}$$

where, $\tilde{\tau}_j(s)$ denotes the column vector whose components are $\tilde{\tau}_{ij}(s)$, $\boldsymbol{e}_j$ the jth unit vector, $\boldsymbol{I}$ the identity matrix, and $\boldsymbol{Q} = (\boldsymbol{q}_{ij})$ the infinitesimal generator governing the Markov chain $\{X(t)\}$. $\tau_{ij}(T)$ are then obtained by inverting (3.17). For example, when $m = 2$,

$$\tau_{11}(T) = \frac{q_2}{q_1+q_2}T + \frac{q_1}{(q_1+q_2)^2}[1 - e^{-(q_1+q_2)T}], \quad \tau_{12}(T) = T - \tau_{11}(T),$$

$\tau_{22}(T)$ and $\tau_{21}(T)$ are obtained by q_1 and q_2 being replaced with q_2 and q_1, respectively, in the above equation. □

Suppose that the continuous-time Markov chain $\{X(t)\}$ is ergodic and let $\boldsymbol{\pi} = (\boldsymbol{\pi}_j)$ be the stationary distribution of $\{X(t)\}$. Define $\bar{\sigma}^2 = \sum_{j=1}^m \pi_j \sigma_j^2$. In the same spirit as for the discrete-time model, we want to compare $\sum_{j=1}^m \pi_j C_j(T,S,K)$ with $\mathrm{BS}(\bar{\sigma}^2)$. The next result follows from Jensen's inequality.

Theorem 3.3. If $\sigma_1^2 \le v = v(r,T,S,K)$ then

$$\mathrm{BS}(\bar{\sigma}^2) \le \sum_{j=1}^m \pi_j C_j(T,S,K), \tag{3.18}$$

whereas if $\sigma_m^2 \ge v$ then

$$\mathrm{BS}(\bar{\sigma}^2) \ge \sum_{j=1}^m \pi_j C_j(T,S,K). \tag{3.19}$$

Proof. From the first inequality in (3.14), one has

$$\sum_{j=1}^m \pi_j \mathrm{BS}(\sigma_j^2(T)) \le \sum_{j=1}^m \pi_j C_j(T,S,K). \tag{3.20}$$

But, since $\mathrm{BS}(x)$ is convex in x if $\sigma_1^2 \le v$, the left hand side in (3.20) is greater than or equal to $\mathrm{BS}(\sum_{j=1}^m \pi_j \sigma_j^2(T))$. On the other hand, since

$$\boldsymbol{s}^2 \tilde{\boldsymbol{\tau}}_j(\boldsymbol{s}) - \boldsymbol{s}\boldsymbol{Q}\tilde{\boldsymbol{\tau}}_j(\boldsymbol{s}) = \boldsymbol{e}_j \tag{3.21}$$

from (3.17) and since $\boldsymbol{\pi}'\boldsymbol{Q} = 0$, premultiplying $\boldsymbol{\pi}'$ to (3.21) yields $\boldsymbol{s}^2\boldsymbol{\pi}'\tilde{\boldsymbol{\tau}}_j(\boldsymbol{s}) = \boldsymbol{\pi}_j$, where $'$ denotes the transpose. In the real domain, this means that $\sum_{i=1}^m \pi_i \tau_{ij}(T) = \pi_j T$. It follows from (3.12) that

$$\bar{\sigma}^2 = \sum_{j=1}^m \pi_j \sigma_j^2(T). \tag{3.22}$$

Therefore, (3.18) follows. The proof of (3.19) is similar. □

Remark 3.3. It is well known (see, e.g., page 269 of Çinlar [4]) that $\pi_j = \lim_{T\to\infty} \tau_{ij}(T)/T$ for any $i \in \mathcal{S}$ so that, from (3.12),

$$\bar{\sigma}^2 = \lim_{T\to\infty} \frac{1}{T}\sum_{j=1}^m \tau_{ij}(T)\sigma_j^2 = \lim_{T\to\infty} \sigma_i^2(T).$$

Hence, $\bar{\sigma}^2$ can be interpreted as the mean of the volatility during one unit of time in the long run. Also, it is known that

$$\bar{\sigma}^2 = \lim_{T\to\infty} \sigma^2(T) = \lim_{T\to\infty} \frac{1}{T}\sum_{j=1}^m t_j \sigma_j^2 \tag{3.23}$$

K	T	Case 1			Case 2		
q_1		0.3			0.4		
q_2		0.4			0.3		
π_1		4/7			3/7		
π_2		3/7			4/7		
K	T	v	$\sigma_1^2(T)$	$\sigma_2^2(T)$	v	$\sigma_1^2(T)$	$\sigma_2^2(T)$
40	0.1	0.5134	0.0396	0.0106	0.5134	0.0394	0.0104
	0.3	0.1859	0.0387	0.0117	0.1859	0.0383	0.0113
	0.5	0.1207	0.0380	0.0127	0.1207	0.0373	0.0120
	1.0	0.0726	0.0364	0.0148	0.0726	0.0352	0.0136
	3.0	0.0441	0.0325	0.0200	0.0441	0.0300	0.0175
49	0.1	0.0063	0.0396	0.0106	0.0063	0.0394	0.0104
	0.3	0.0040	0.0387	0.0117	0.0040	0.0383	0.0113
	0.5	0.0040	0.0380	0.0127	0.0040	0.0373	0.0120
	1.0	0.0048	0.0364	0.0148	0.0048	0.0352	0.0136
	3.0	0.0092	0.0325	0.0200	0.0092	0.0300	0.0175
50	0.1	0.0002	0.0396	0.0106	0.0002	0.0394	0.0104
	0.3	0.0007	0.0387	0.0117	0.0007	0.0383	0.0113
	0.5	0.0012	0.0380	0.0127	0.0012	0.0373	0.0120
	1.0	0.0024	0.0364	0.0148	0.0024	0.0352	0.0136
	3.0	0.0071	0.0325	0.0200	0.0047	0.0321	0.0175
51	0.1	0.0022	0.0396	0.0106	0.0022	0.0394	0.0104
	0.3	0.0001	0.0387	0.0117	0.0001	0.0383	0.0113
	0.5	0.0000	0.0380	0.0127	0.0000	0.0373	0.0120
	1.0	0.0008	0.0364	0.0148	0.0008	0.0352	0.0136
	3.0	0.0053	0.0325	0.0200	0.0053	0.0300	0.0175
60	0.1	0.3124	0.0396	0.0106	0.3124	0.0394	0.0104
	0.3	0.0931	0.0387	0.0117	0.0931	0.0383	0.0113
	0.5	0.0496	0.0380	0.0127	0.0496	0.0373	0.0120
	1.0	0.0178	0.0364	0.0148	0.0178	0.0352	0.0136
	3.0	0.0004	0.0325	0.0200	0.0004	0.0300	0.0175
$\bar{\sigma}^2$		0.0271			0.0229		
Other Parameters		$S = 50, r = 0.05, \sigma_1^2 = 0.04, \sigma_2^2 = 0.01$					

Table 2: Parameter Values

with probability one, independent of the initial state. □

The content of Theorem 3.3 has an important implication. As explained in Section 2, (3.18) is a situation in which the BS equation underprices while it overprices if (3.19) holds. For deep out or in-the-money options with relatively short maturities, $\log(S/K)$ is big in the magnitude so that $v = v(r, T, S, K)$ in (3.11) is relatively large. Hence, it is likely that the BS equation underprices for such options. On the other hand, for near at-the-money options, $\log(S/K)$ is negligible so that the v is, by Taylor's expansion, about r^2T. This value is very small for options with reasonable length of maturities and the BS equation is likely to overprice. Finally, we note that if $\{X(t)\}$ is ergodic then, since (3.23) holds, $C_j(T, S, K)$ converges to $\mathrm{BS}(\bar{\sigma}^2)$ with probability one as $T \to \infty$.

Example 3.1. In this example, we consider the continuous-time model with 2 underlying states as in Example 2.1. The current stock price S is fixed to be 50 and we consider two cases where the transition intensities of the Markov chain $\{X(t)\}$ are given by $q_{12} = 0.3$ and $q_{21} = 0.4$ as Case 1, and $q_{12} = 0.4$ and $q_{21} = 0.3$ as Case 2. Table 2 lists the parameter values and $\boldsymbol{\pi}$, v in (3.11), $\sigma_i^2(T)$, $i = 1, 2$, and $\bar{\sigma}^2$. Note that, for near at-the-money options ($K = 49, 50, 51$ in this example), v is smaller than $\sigma_2^2 = 0.01$ so that the condition for both (3.15) and (3.19) holds, and for the other values of K the opposite condition holds except for $T \geq 1$ of $K = 60$. Also, it can be observed that $\sigma_i^2(T)$ converge to $\bar{\sigma}^2$ as T becomes large, but the speed of convergence seems slow. Hence, the effect from the volatility's change over time can not be neglected for options with short length of maturities. This means that we should expect the current underlying state to affect the price of such options considerably. In Table 3, various values of $\mathrm{BS}(\sigma_i^2(T))$ and

$$\mathrm{EX}(\eta_i) = \eta_i \mathrm{BS}(\sigma_1^2) - (1-\eta_i)\mathrm{BS}(\sigma_2^2),$$

$i = 1, 2$, and $\mathrm{BS}(\bar{\sigma}^2)$ are listed. If $v \geq \sigma_1^2 = 0.04$, $\mathrm{BS}(\sigma_i^2(T))$, $i = 1, 2$, provide lower bounds for $C_i = C_i(T, S, K)$ while $\mathrm{EX}(\eta_i)$ upper bounds. If $v \leq \sigma_2^2 = 0.01$, on the other hand, the $\mathrm{BS}(\sigma_i^2(T))$ provide upper bounds while $\mathrm{EX}(\eta_i)$ lower bounds. The bounds are surprisingly tight for near at-the-money options with relatively short maturities, say within a year. Also, we can see that the price C_1 is considerably higher than the price C_2 for such options. For long maturity options, we may need to calculate (3.9) directly[11]. The last column termed AVG in Table 3 describes either upper or lower bounds of $\pi_1 C_1 + \pi_2 C_2$ according to whether $v \leq \sigma_2^2$ or $v \geq \sigma_1^2$, respectively. For example, when $K = 50$, AVG is calculated as

$$\mathrm{AVG} = \sum_{i=1}^{2} \pi_i \mathrm{BS}(\sigma_i^2(T)),$$

which, from (3.15), is greater than or equal to $\pi_1 C_1 + \pi_2 C_2$. Therefore, from Table 3, one has $\pi_1 C_1 + \pi_2 C_2 \leq \mathrm{BS}(\bar{\sigma}^2)$, which in turn implies that the BS equation overprices for this case. It can be explicitly observed that the BS equation overprices near at-the-money options whereas underprices deep out or in-the-money options with short maturities. Also, we can observe that $C_2 \leq \mathrm{BS}(\bar{\sigma}^2) \leq C_1$ in this example. Note that, in this example, only the case $T = 1.0$ for $K = 60$ satisfies neither $v \geq \sigma_1^2$ nor $v \leq \sigma_2^2$. □

4 Concluding Remarks

In this paper, we develop a Markovian volatility model, where the stochastic law of volatility is described in terms of a discrete state Markov chain. A discrete-time model is first constructed and a recursion formula for option pricing is derived. Based on the formula, we show that if the Markov chain is stochastically monotone then the higher current volatility the higher the option price. This result can be extended for any strong Markov process with continuous sample path (this includes diffusion processes), so that the result explains, to some extent, a well recognized phenomenon in actual option prices. This result is well known in practice but, to the authors' best knowledges, no theoretical explanation has been made in the literature. We then derive a continuous-time model as a limit of the discrete-time model where the stock price process follows a geometric Brownian motion process and the volatility dynamics follows a Markov chain on a discrete state space. If the stock price process is independent of the volatility (see footnote 6 for details), an option pricing formula is given as an integral of the BS equation. An explicit call valuation formula is given when the volatility takes one of two values, and some easily computable upper and lower bounds are obtained to estimate option prices based on the local convexity and concavity of the BS equation. The bounds are shown to be very tight especially for near at-the-money options with relatively short maturities, say within a year. Also, using the bounds, we succeed in explaining explicitly why the BS equation overprices near at-the-money options whereas it underprices deep out or in-the-money options.

Acknowledgment. The authors wish to thank the referees for their constructive comments which improve the original manuscript considerably. They also thank Hideki Iwaki for valuable discussions. One of the author (M.K.) is supported in part by the fund endowed to the Research Association for Financial Engineering by the Toyo Trust and Banking Co. Ltd., Mito Securities Co. Ltd., and the Dai-ichi Mutual Life Insurance Co.

[11]Calculation of (3.9) takes much time if $(q_1 + q_2)T$ is large. For, we need to take truncation point in (3.7) large so that the triple summation there becomes to matter.

	K	T	BS($\sigma_1^2(T)$)	EX(η_1)	BS($\sigma_2^2(T)$)	EX(η_2)	BS($\overline{\sigma}^2$)	AVG
Case 1	40	0.1	10.1948	10.1948	10.1947	10.1947	10.1947	10.1947
		0.3	10.6042	10.6059	10.5812	10.5827	10.5865	10.5944
		0.5	11.0516	11.0580	10.9648	10.9732	10.9950	11.0144
		1.0	12.1923	12.2117	11.9250	11.9643	12.0526	12.0778
		3.0	16.1943	16.2411	15.7369	15.8222	15.9885	15.9982
	49	0.1	1.9553	1.9538	1.4415	1.4389	1.7621	1.7351
		0.3	3.0778	3.0694	2.2087	2.1942	2.7547	2.7053
		0.5	3.9168	3.8997	2.8715	2.8468	3.5265	3.4688
		1.0	5.5762	5.5371	4.3393	4.3089	5.1133	5.0461
		3.0	10.3520	10.2684	9.3284	9.2192	9.9331	9.9133
	50	0.1	1.3768	1.3745	0.7761	0.7706	1.1628	1.1193
		0.3	2.5166	2.5060	1.5675	1.5458	2.1730	2.1099
		0.5	3.3588	3.3379	2.2335	2.1983	2.9470	2.8765
		1.0	5.0196	4.9730	3.7035	3.6574	4.5341	4.4556
		3.0	9.8036	9.7023	8.7261	8.5870	9.3651	9.3418
	51	0.1	0.9233	0.9213	0.3453	0.3408	0.7137	0.6756
		0.3	2.0279	2.0167	1.0551	1.0318	1.6775	1.6110
		0.5	2.8568	2.8342	1.6901	1.6498	2.4335	2.3568
		1.0	4.5024	4.4502	3.1276	3.0697	3.9998	3.9132
		3.0	9.2762	9.1582	8.1489	7.9811	8.8198	8.7931
	60	0.1	0.0024	0.0025	0.0000	0.0001	0.0002	0.0014
		0.3	0.1510	0.1557	0.0021	0.0100	0.0604	0.0872
		0.5	0.4663	0.4743	0.0365	0.0608	0.2544	0.2821
		1.0	1.4531	1.4419	0.4208	0.4474	1.0370	1.0043
		3.0	5.4446	5.2629	4.0985	3.8214	4.9070	4.6451

	K	T	BS(σ_1^2)	EX(η_1)	BS(σ_2^2)	EX(η_2)	BS($\overline{\sigma}^2$)	AVG
Case 2	40	0.1	10.1948	10.1948	10.1947	10.1947	10.1947	10.1947
		0.3	10.6033	10.6056	10.5812	10.5823	10.5834	10.5913
		0.5	11.0474	11.0558	10.9646	10.9709	10.9805	11.0037
		1.0	12.1727	12.1979	11.9190	11.9505	11.9991	12.0457
		3.0	16.0976	16.1577	15.6598	15.7389	15.8336	15.8926
	49	0.1	1.9532	1.9512	1.4382	1.4363	1.6877	1.6589
		0.3	3.0669	3.0558	2.1911	2.1806	2.6205	2.5664
		0.5	3.8943	3.8719	2.8359	2.8189	3.3540	3.2895
		1.0	5.5191	5.4686	4.2560	4.2404	4.8765	4.7973
		3.0	10.1608	10.0596	9.1036	9.0104	9.5782	9.5567
	50	0.1	1.3745	1.3715	0.7716	0.7675	1.0779	1.0300
		0.3	2.5051	2.4911	1.5468	1.5309	2.0281	1.9575
		0.5	3.3351	3.3078	2.1933	2.1682	2.7628	2.6826
		1.0	4.9600	4.8996	3.6117	3.5840	4.2831	4.1896
		3.0	7.4793	7.3731	6.0624	6.0956	6.7967	6.6696
	51	0.1	0.9211	0.9184	0.3412	0.3379	0.6313	0.5897
		0.3	2.0162	2.0015	1.0337	1.0165	1.5293	1.4548
		0.5	2.8326	2.8028	1.6475	1.6185	2.2430	2.1554
		1.0	4.4408	4.3731	3.0295	2.9926	3.7384	3.6343
		3.0	8.9239	8.0175	7.8943	7.7469	8.4282	8.3356
	60	0.1	0.0024	0.0025	0.0000	0.0000	0.0001	0.0010
		0.3	0.1473	0.1535	0.0017	0.0077	0.0357	0.0641
		0.5	0.4527	0.4634	0.0306	0.0498	0.1783	0.2115
		1.0	1.4005	1.3865	0.3637	0.3919	0.8354	0.8242
		3.0	5.2010	4.9760	3.7836	3.5345	4.4383	4.3911

Table 3: Upper and Lower Bounds

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