Journal of the Operations Research
Society of Japan
Vol. 36, No. 3, September 1993



# THE PROBLEM WITH APPORTIONMENT

Michel Balinski

*C.N.R.S. and Laboratoire d'Econométrie de l'Ecole Polytechnique, Paris*

(Received July 13, 1992; Revised January 18,1993)

*Abstract* The problem of apportionment is to allocate the seats of a House of Representatives or Diet to states or constituencies proportionally to their respective populations. This paper reviews the problem, its background and pitfalls, and the various approaches that have been applied to its analysis.

## 0. Introduction

The problem of apportionment possesses a deceptively innocent air : to allocate seats in a House of Representatives, Parliament, National Assembly or Diet to states, constituencies, or regions *proportionally* to their respective populations seems so simple a task[1] ! Proportionality is no mystery : the only seemingly minor complication is the need to allocate integer numbers of seats.

And yet, . . ., repeated errors of judgement, implicit and erroneous assumptions, misleading ideas and downright false statements have hounded the problem over the two hundred years of its history. A recent paper [12] that appeared in this journal[2] is no exception. It is subject to pitfalls similar to ones that have trapped distinguished students of the problem in the past, and so motivated the writing of this survey which attempts to set the record straight.

The aim of this paper is to point out that the innocent face of the problem hides a surprising subtlety of behavior, to summarize some of the salient results and to correct certain misunderstandings and errors. Implicit is the plea that at least some of the by now well established concepts be *used* to test any new idea or new method that is formulated *before* it is formally proposed.

In section 1 the problem is defined formally, and in section 2 its background and the best known of the methods used to find solutions are presented. Section 3 describes the "optimization approach" and its difficulties. Section 4 explains the idea of axiomatization as a practical approach to finding equitable methods. An example of the axiomatic approach to arriving at a reasonable concept of "constrained proportionality" in reals - the concept of the "fair shares" of the states - is given in section 5. Section 6 reviews the axiomatic theory of apportionment. In section 7, Huntington's pairwise comparisons analysis is presented in order to compare it with Oyama's [12] "average ratio pairwise transfer" idea whose shortcoming is discussed in Section 8. In section 9 it is shown why a subclass of the divisor methods, the "parametric divisor methods," do not, as has been claimed, adequately "cover" all of the divisor methods. Section 10 takes up the concept of the bias of a method and explains why it is inadequate to analyze one problem and to deduce from it conclusions concerning which method is most equitable. Finally, in section 11, a characterization of Webster's method heretofore never explicitly stated - a "folk theorem" - is proven, and the evidence in support of Webster's method is summarized.

[1] In another guise the problem is to allocate seats to political parties proportionally to their respective vote totals.

[2] *Journal of the Operations Research Society of Japan.*

## 1. The problem

A *problem of apportionment* $(\mathbf{p},\mathbf{f},\mathbf{c},h)$ is given by : an s-vector of *populations* $\mathbf{p} = (p_i) > \mathbf{0}$, an integer *house size* $h \geq 0$, and integer s-vectors $\mathbf{f} = (f_i) \geq \mathbf{0}$ of *floors* and $\mathbf{c} = (c_i) > \mathbf{0}$ of *ceilings*. An *apportionment* (of $h$) is an s-vector of integers $\mathbf{a} = (a_i)$ satisfying

(1) $\Sigma_i a_i = h,\ f_i \leq a_i \leq c_i$ for all $i$.

A problem is *feasible* if there exists an integer vector $\mathbf{a}$ satisfying (1). Necessary and sufficient conditions for feasibility are that $f_i \leq c_i$ and $\Sigma_i f_i \leq h \leq \Sigma_i c_i$. In most practical instances ceilings are not present. However, floors often are : in the United States House of Representatives each state has a floor of 1, in the French Assemblée Nationale each département has a floor of 2, and in the Canadian Parliament the provinces have variable, historically determined floors. Presumably, each constituency of Japan must receive at least one seat.

Letting $p^* = \Sigma_i p_i$ be the total population, define the *quota* of state $i$ to be $q_i = p_i/p^*$. If there were no integer requirement and no floors or ceilings then the desired apportionment is the proportional solution : $\mathbf{a}$=$\mathbf{q}$. But if there are floors and ceilings and no integer requirement, then what? A reasonable idea is to say that the solution should be proportional except to meet a floor or ceiling : the *fair share* of $h$ of state $i$, denoted $r_i$, is defined to be

(2) $r_i = \text{mid}\{f_i, p_i/x, c_i\}$ with $x > 0$ chosen so that $\Sigma_i r_i = h$,

where $\text{mid}\{x, y, z\} = y$ if $x \leq y \leq z$. Sometimes it will be convenient to use the notation $r_i(h)$ to indicate the fair share of house size $h$. The *ideal* solution is the vector of fair shares $\mathbf{r}= (r_i)$ which, in the absence of floors and ceilings, is equal to the vector of the quotas. A more solid axiomatic justification for this definition of the fair shares is given in section 5.

A *method of apportionment* $\Phi$ is a rule or correspondance that assigns at least one apportionment of $h$ to every feasible problem $(\mathbf{p},\mathbf{f},\mathbf{c},h)$. **The central question is** : what method of apportionment $\Phi$ should be chosen?

## 2. Background

The history of the problem is natural enough : necessity breeds invention [8]. In 1792, in response to the first census taken in the United States, Thomas Jefferson and Alexander Hamilton proposed competing but apparently reasonable methods. Jefferson's favored his state of Virginia, then the most populous of the states, the Virginians held the power, so Jefferson's method was chosen (after a first use of the presidential veto) and then used for the ensuing fifty years to apportion the United States House of Representatives.

Both methods are easily explained[3]. *Hamilton's method* is : first, give to each state the integer part of its fair share, $[r_i]^-$ ; then, assign the seats not yet allocated, one per state, to those states having the largest fractions or remainders $r_i - [r_i]^-$. Notice that it is entirely possible for there to be several solutions in cases of "ties" in the remainders.

An entirely different reasoning is found in *Jefferson's method* : find a "divisor" $x$ (a real number) so that the sum of the integers obtained by dividing the populations by $x$ and rounding down gives the required house size $h$, except to meet a floor or ceiling; that is,

choose $x > 0$ so that $\Sigma_i a_i = h$ where $a_i = \text{mid}\{f_i, [p_i/x]^-, c_i\}$.

---

[3]The methods are defined under the assumption that there are both floors and ceilings. These are the definitions that result from the axiomatic theory described below. In the United States, where only the floor of 1 is operative, the Hamilton method when used was interpreted as follows : base the procedure on the quotas $q_i$ (instead of on the fair shares $r_i$) and first give each state having $q_i \leq 1$ one seat. This procedure might not result in a feasible apportionement! Also, there are cases in United States history where this slight change in procedure yields different solutions (including the United States problem of 1990).

To be entirely correct one must here define $[n]^-$, for $n$ a positive integer, to be either of the values $n$ or $n-1$, for here again there can be ties.

Both methods have been invented and invented again, Jefferson's sometimes in clothes that make it appear to be quite a different method, both have been and are used in one or another country, but both were ultimately rejected in the United States because of behavior that slowly came to light through use. Different problems, different data, yielded differing solutions, and "patterns" and "paradoxes" arose that made reasonable political men look for other methods to apportion seats among regions.

The historical record reveals (at least) six other methods proposed since 1792. In 1822 - in response to the exaggerated advantage that the method of Jefferson was perceived to give to the large states - Representative William Lowndes of South Carolina proposed the same procedure as Hamilton except to assign the seats not yet allocated, one per state, to those states having the largest "adjusted remainders" $(r_i - [r_i]^-)/p_i$. This method, clearly advantageous to the small, has never been used. The remaining five may, with that of Jefferson, be given a unified description [5] [8], although it must be kept in mind that some of them were first proposed in decidedly different terms which make some of them look rather more persuasive.

Let $d$ be a real valued function defined on the nonnegative integers and satisfying

$$(3) \quad \begin{array}{l} a \le d(a) \le a+1, \text{ and} \\ \text{not } \{d(a) = a \text{ and } d(b) = b+1 \text{ for integers } a > 0 \text{ and } b \ge 0\}. \end{array}$$

A *d-rounding* $[z]_d$ of any nonnegative real number $z$ is an integer a defined by

$$(4) \quad [z]_d \begin{array}{ll} = a, & \text{if } d(a-1) < z < d(a), \text{ and} \\ = a \text{ or } a+1, & \text{if } z = d(a). \end{array}$$

The *divisor method based on d* is then

$$(5) \quad \begin{array}{l} \Phi^d(\mathbf{p},\mathbf{f},\mathbf{c},h) = \\ \{a : a_i = \operatorname{mid}(f_i, [p_i/x]_d, c_i) \text{ and } \Sigma_i a_i = h \text{ for some real number } x > 0\}. \end{array}$$

The so-called five traditional methods are all divisor methods defined, respectively, with the following functions $d$ :

**Adams** (1832), or smallest divisors : $d(a) = a$

**Dean** (1832), or harmonic mean : $d(a) = a(a+1)/(a+.5)$

**Hill** (1911), Huntington, equal proportions or geometric mean : $d(a) = \{a(a+1)\}^{1/2}$

**Webster** (1832), major fractions, odd numbers, arithmetic mean or Sainte-Lagüe : $d(a) = a + .5$

**Jefferson** (1792), greatest divisors, d'Hondt, Hagenbach-Bischoff, or highest averages : $d(a) = a + 1$.

The Marquis de **Condorcet** had proposed in his 1792 plan for a French Constitution the divisor method based on $d(a) = a + .4$.

The obvious difficulty is : how to choose among these? There is an infinite choice of divisor methods since any $d$ satisfying (3) will do!

## 3. Optimization

A different, supposedly more modern approach to the problem is via optimisation. Here the idea is to choose a feasible apportionment $\mathbf{a}$ to minimize some expression of the distance between $\mathbf{a}$ and the ideal perfectly proportional solution. But precisely what expression of distance? Hamilton's method gives solutions that minimize $\Sigma_j |a_j - r_j|$, and $\Sigma_j (a_j - r_j)^2$, and indeed any $L_p$ norm $|\mathbf{a}\text{-}\mathbf{r}|$.

Webster's method gives solutions that minimize $\Sigma_j p_j (a_j/p_j - h/p^*)^2$ whereas Hill's method yields solutions that minimize $\Sigma_j a_j (p_j/a_j - p^*/h)^2$. Oyama [12] gives the new result that the divisor method based on $d(a) = a + t$ for $0 \le t \le 1$ minimizes $\Sigma_j p_j \{(a_j + t - .5)/p_j - h/p^*\}^2$, thus generalizing the formula for the Webster method to this particular class of parametric divisor methods. However, he is quite wrong in claiming that very little has been said concerning the optimization approach : **every** one of the 10 functions he discusses, except for the above result, has been studied before (see [8], pp. 102-105).

**Table 1**

| State | Population | Quota | Adams | Dean | Hill | Webst. | Jeff. | Hamilt. |
|---|---|---|---|---|---|---|---|---|
| 1 | 107 658 | 39.401 | 38 | 40 | 40 | 40 | 41 | 40 |
| 2 | 27 744 | 10.154 | 10 | 10 | 10 | 10 | 10 | 10 |
| 3 | 25 17 | 9.215 | 9 | 9 | 9 | 9 | 9 | 9 |
| 4 | 19 951 | 7.302 | 7 | 7 | 7 | 8 | 7 | 7 |
| 5 | 14 610 | 5.347 | 6 | 5 | 6 | 5 | 5 | 5 |
| 6 | 9 225 | 3.376 | 4 | 4 | 3 | 3 | 3 | 4 |
| 7 | 3 292 | 1.205 | 2 | 1 | 1 | 1 | 1 | 1 |
| Total | 207 658 | 76 | 76 | 76 | 76 | 76 | 76 | 76 |

Two of his assertions are false. He states that Webster's solutions minimize $W^* = \Sigma_j|a_j/p_j - h/p^*|$ and that Dean's solutions minimize $D^* = \Sigma_j|p_j/a_j - p^*/h|$ (theorems 3.3 and 3.5). Table 1 gives a counter example. If the unique Dean solution is changed by transferring one seat from state 6 to state 1 then $D^*$ is reduced (from 466.984 to 449.195), wheras if the unique Webster solution is changed by transferring one seat from state 4 to state 1 then $10^5W^*$ is reduced (from 4.056 to 2.998). His "proofs" err in assuming that if $i$ is favored over $j$, that is, if $a_i/p_i > a_j/p_j$ then $a_i/p_i > h/p^* > a_j/p_j$, which is, of course, not necessarily true.

While it is of decided interest to know what measure of distance yields what method, the optimization approach has no more force in resolving the question of what solutions to choose because once again there is an infinity of choice with no criteria by which to judge.

## 4. Properties

Over the course of many years, however, as practical interested men witnessed the nature of the different solutions that came from changing census data and looked for arguments to sustain their opposition to or support of one or another method, properties concerning the behavior of methods began to be identified. The Jefferson method was abandoned because it was seen to unduly "favor" the largest states : by the United States census populations of 1830, for example, New York with a fair share of 38.593 was accorded 40 seats whereas Vermont with a fair share of 5.646 was accorded 5. In the second half of the 19th century the law stipulated that the "Vinton method of 1850" was to be used[4] - which was nothing other than Hamilton's method. But beginning with the census of 1882 the infamous Alabama paradox exasperated political men's frustrations with mathematics : it was noticed that with that method an **increase** of the size of the house (with no change in populations) could result in a **decrease** in the number of seats allocated to a state - the state of Alabama in that case.

And thus it came to be an accepted "precedent" that the only methods that are admissable for the apportionment of the U.S. House of Representatives are methods that avoid the "Alabama paradox." The notion of having a method that favors neither the

[4] In fact, contrary to the assertion in [12], the details of history reveal that the 1850 law was never strictly followed (see [8], pp. 37-42).

large states nor the small ones but is somehow even-handed between small and large always hovered in the background. This combination of desires led directly to another fundemental error of judgement concerning apportionment.

The debate among mathematicians and others in the 1920s and 1930s focussed on the five traditional methods as being the only ones known that avoided the Alabama paradox. (Note that by its very definition any divisor method must perforce be house monotone.) And these could be lined up from that of Adams to that of Jefferson (as they are given above) in an order that goes from most favorable for the small to most favorable for the large. This has a strict mathematical sense ([8], pp. 118-119), but is easily observed on examples. One of the key arguments for the U.S. Congress's choice of Hill's method was that it is the middle of the five methods : its chief advocate claimed that it "has been mathematically shown to have no bias in favor of either the larger or the smaller States," a 1929 committee of mathematicians of the National Academy of Sciences concluded that "it occupies mathematically a neutral position with respect to emphasis on larger and smaller States," and in 1948 another such committee (including von Neumann) stated its preference for Hill's method because "it stands in a middle position as compared with the other methods" (see [8] pages 54, 56 and 78 for references). Much mirth has been elicited over the difficulty this reasoning would have encountered if the number of methods under consideration had been even. The fact is that no one had even bothered to define what "neutral" or what "bias" means. "Middle" is clear enough but if instead of the five methods the infinite class of divisor methods had been under consideration, the conclusions would have been quite different. Moreover, an applied statistician had conclusively shown in 1927 and 1928 through extensive analysis of the historical census data, and with elaborate tables and diagrams, that "if the main purpose is, as it probably was in the Constitutional Convention of 1787, to hold the balance even between the large and the small States as groups, that end is best secured by the method of [Webster]" (see [8] page 55). This claim of the emminent mathematicians turns out to be a fundemental error in the analysis - and an error which has been perpetuated in practice because Hill's method has been used to apportion the U. S. House of Representatives since 1940, and it is biased in favor of the small by every reasonable definition of the idea of bias that I have seen.

So long as the argument remains without fundemental "criteria" by which to judge the relative merits of different methods the choice of which method to use (in one or another circumstance) will remain *ad hoc* and arbitrary. The primitive idea of a method that has no bias in favor of either the large or the small is perfectly reasonable, but it must be specified. And there are other key ideas concerning the behavior of methods that have emerged over the years that need to be made precise. The role of the mathematics is then to turn the analysis upside down : instead of studying the properties of one or another method, impose the properties and deduce which methods satisfy them. This is the axiomatic method used for practical ends.

## 5. An axiomatisation of fair shares : constrained proportionality in reals

An example of the axiomatic method at work is arriving at a definition of proportionality when constraints are imposed [1],[2]. Given a problem $(\mathbf{p},\mathbf{f},\mathbf{c},h)$ a vector $\mathbf{r} = (r_i)$ is sought that is proportional to $\mathbf{p}$ but that belongs to the set $R = \{\mathbf{r}: \Sigma_i r_i = h \text{ and } \mathbf{f} \leq \mathbf{r} \leq \mathbf{c}\}$.

Let $\Psi$ be a correspondence defined on $(\mathbf{p},\mathbf{f},\mathbf{c},h)$ with values in $R$. The approach consists in postulating qualitative properties that $\Psi$ should enjoy that translate the idea of proportionality and then deduce the analytic definition of $\Psi$.

If the usual definition of proportionality works then it should be the unique solution to the problem. Specifically, $\Psi$ is *exact* : if $(h/\Sigma_i p_i)\mathbf{p} \in R$ then $(h/\Sigma_i p_i)\mathbf{p} = \Psi(\mathbf{p},\mathbf{f},\mathbf{c},h)$.

Any part of a proportional vector should be proportional. If $S \cup T$ is a partition of the indices then $\mathbf{p}^S$ represents that part of the vector $\mathbf{p}$ that corresponds to the indices $S$, and similarly for other vectors and sets of indices. Specifically, $\Psi$ is

*consistent* : if $\mathbf{r}=(\mathbf{r}^S,\mathbf{r}^T) \in \Psi(\mathbf{p},\mathbf{f},\mathbf{c},h)$ then $\mathbf{r}^S \in \Psi(\mathbf{p}^S,\mathbf{f}^S,\mathbf{c}^S,\Sigma_S r_i)$, and if also $\mathbf{t}^S \in \Psi(\mathbf{p}^S,\mathbf{f}^S,\mathbf{c}^S,\Sigma_S r_i)$ then $(\mathbf{t}^S,\mathbf{r}^T) \in \Psi(\mathbf{p},\mathbf{f},\mathbf{c},h)$.

Finally, the values of a vector proportional to $\mathbf{p}$ should change in accordance with the changes in $\mathbf{p}$. Specifically, $\Psi$ is

*monotone* : if for $\mathbf{p}$ and $\mathbf{q}$ satisfying $p_k > q_k$ for some one index $k$ and $p_j = q_j$ otherwise, $\mathbf{r} \in \Psi(\mathbf{p},\mathbf{f},\mathbf{c},h)$ and $\mathbf{t} \in \Psi(\mathbf{q},\mathbf{f},\mathbf{c},h)$ then $r_k \geq t_k$.

These three axioms are sufficient to characterize the fair shares.

*Fair share characterization.* The unique correspondence $\Psi$ satisfying exactness, consistency and monotonicity has as its values the fair shares (2).

The proof is as follows. It is immediate to verify that the fair shares $\mathbf{r}$ as defined in (2) satisfy the three conditions. Since the floors $\mathbf{f}$ and the ceilings $\mathbf{c}$ are fixed throughout mention of them is omitted in what follows to simplify the notation.

Suppose, then, that $\Psi$ satisfies the properties and let $\mathbf{r}$ be the unique fair share vector (2) for the problem $(\mathbf{p},h)$. Define $\lambda = (\lambda_i) > 0$ by

$$\lambda_i \begin{array}{ll} = f_i/(p_i/x) > 1 & \text{if } f_i > p_i/x, \\ = c_i/(p_i/x) < 1 & \text{if } c_i < p_i/x, \\ = 1 & \text{if } f_i \leq p_i/x \leq c_i. \end{array}$$

By exactness $\mathbf{r} = \Psi((\lambda_i p_i),h)$. If $\lambda_i = 1$ for all $i$ the theorem is proved. Suppose that $\lambda_k \neq 1$ for some $k$, say $\lambda_1 > 1$, and that $\mathbf{r}' \in \Psi((p_1,\lambda_2 p_2,\cdots,\lambda_s p_s),h)$. By monotonicity $r_1' \leq r_1$, but $\lambda_1 > 1$ means $r_1 = f_1$ and therefore $r_1' = r_1 = f_1$. But by consistency $(r_2,\cdots,r_s)$ and $(r_2'\cdots,r_s')$ both belong to $\Psi((\lambda_2 p_2\cdots,\lambda_s p_s),h-f_1)$. Now use exactness to deduce that $r_i' = r_i$ for $i \neq 1$ so $\mathbf{r} = \mathbf{r}'$ and $\mathbf{r} = \Psi((p_1,\lambda_2 p_2,\cdots,\lambda_s p_s),h)$. Now repeat the same argument for every $\lambda_k \neq 1$. This completes the proof.

It is interesting to note that the three properties used in characterizing the fair shares have their exact *qualitative* counterparts in characterizing the most important class of methods of apportionment, but they are very different anatically in that for apportionment the properties concern integer solutions rather than solutions in reals.

## 6. Axiomatics : a review of the theory of apportionment

Certain properties are common to all reasonable methods. They are stated in this paragraph and then always assumed to hold without further mention. If the ideal of proportionality can be met then it should be. A method $\Phi$ is said to be *exact*[5] : if the fair shares $\mathbf{r}$ are integer valued then $\Phi(\mathbf{p},\mathbf{f},\mathbf{c},h) = \mathbf{r}$. The concept of proportionality demands that a method $\Phi$ be *homogeneous* : if $\mathbf{a} \in \Phi(\mathbf{p},\mathbf{f},\mathbf{c},h)$ then $\mathbf{a} \in \Phi(\lambda\mathbf{p},\mathbf{f},\mathbf{c},h)$ for all $\lambda > 0$. By the necessity of fairness - that the names of the states should not effect their apportionments - a method $\Phi$ must be *anonymous* : permuting the populations results in apportionments that are permuted in the same way. Also, as the house size grows apportionments by any method should become "no less proportional" : if $\mathbf{a} \in \Phi(\mathbf{p},\mathbf{f},\mathbf{c},h)$ and $\mathbf{a}^* = \lambda\mathbf{a}$ is integer valued, where $0 < \lambda < 1$ then $\mathbf{a}^* = \Phi(\mathbf{p},\mathbf{f},\mathbf{c},h)$. A method that satisfies these four properties is called *proportional*.

The most telling properties that have been identified across the years are here summarized.

[5]The definition given of a divisor function in [12], page 189 - which ignores the restriction given in (3) - does not guarantee the exactness of divisor methods, and so is deficient in this regard.

A method $\Phi$ is

*house monotone* : if $\mathbf{a} \in \Phi(\mathbf{p},\mathbf{f},\mathbf{c},h)$ implies there exists $\mathbf{a}^* \in \Phi(\mathbf{p},\mathbf{f},\mathbf{c},h+1)$ satisfying $\mathbf{a}^* \geq \mathbf{a}$ for all $h \geq 0$ (for which the problem is feasible[6]).

That is, a method should avoid the Alabama paradox.

A method $\Phi$

*satisfies fair share* : if $\mathbf{a} \in \Phi(\mathbf{p},\mathbf{f},\mathbf{c},h)$ implies $[r_i]^- \leq a_i \leq [r_i]^+$ for all $i$,

where $[x]^-$ and $[x]^+$ are $x$ rounded down and rounded up, respectively. In words, a method should always give to each state either its fair share rounded down or its fair share rounded up. A related but weaker property is for a method $\Phi$ to be

*near fair share* : if for all pairs $i \neq j$, $\mathbf{a} \in \Phi(\mathbf{p},\mathbf{f},\mathbf{c},h)$ implies not $\{|a_i - r_i| > |a_i + 1 - r_i|$ and $|a_j - r_j| > |a_j - 1 - r_j|\}$

This simply says that it should be impossible to transfer one seat from one state to another and thus bring both apportionments closer to their respective fair shares.

Problems may change in ways other than simply a change in the size of the house $h$. But there are several possible alternatives in defining what should happen when the populations of the states change. For example, it is tempting to ask that if the fair share of a state increases from one problem to another then the state's apportionment should not decrease. But it can be shown that no method of apportionment has this property. A method $\Phi$ is

*weakly population monotone* : if $\mathbf{a} \in \Phi(\mathbf{p},\mathbf{f},\mathbf{c},h)$ and $\mathbf{a}^* \in \Phi(\mathbf{p}^*,\mathbf{f},\mathbf{c},h)$ with $p_i^* > p_i$ and $p_j^* = p_j$ for all $j \neq i$ implies $a_i^* \geq a_i$.

The trouble with this definition is that it is not terribly relevant to real problems, for when the data of a problem changes it hardly ever changes in just one piece of data! A more realistic concept is for a method $\Phi$ to be

*population monontone* : if $\mathbf{a} \in \Phi(\mathbf{p},\mathbf{f},\mathbf{c},h)$ and $\mathbf{a}^* \in \Phi(\mathbf{p}^*,\mathbf{f},\mathbf{c},h)$ with $p_i^* > p_i$ and $p_j > p_j^*$ implies not $\{a_i > a_i^*$ and $a_j^* > a_j\}$.

That is, no state that gains in population should give up seats to one that loses population. But population monotonicity implies weak population monotonicity so it is reasonable to only invoke the weaker concept if it suffices.

A method of apportionment is a rule of fair division. Any part of a "fair" division should be "fair." If $S \cup T$ is a partition of the set of states, then $\mathbf{a}^S$ represents that part of the vector $\mathbf{a}$ that corresponds to the states of $S$, and similarly for other vectors and sets of states. A method $\Phi$ is

*consistent* : if $\mathbf{a} = (\mathbf{a}^S,\mathbf{a}^T) \in \Phi(\mathbf{p},\mathbf{f},\mathbf{c},h)$ implies $\mathbf{a}^S \in \Phi(\mathbf{p}^S,\mathbf{f}^S,\mathbf{c}^S,\Sigma_S a_i)$, and if also $\mathbf{b}^S \in \Phi(\mathbf{p}^S,\mathbf{f}^S,\mathbf{c}^S,\Sigma_S a_i)$ then $(\mathbf{b}^S,\mathbf{a}^T) \in \Phi(\mathbf{p},\mathbf{f},\mathbf{c},h)$.

In words : if a method apportions $\mathbf{a}^S$ to the states in $S$ then the same method applied to apportioning $h_S = \Sigma_S a_i$ seats among the states in $S$ with the same data will admit the same result ; and moreover, if the method applied to this subproblem admits another solution then the method applied to the entire problem also admits the corresponding alternate solution. Proportionality is a consistent idea. But consistency is very much more fundemental. In terms of the the "fair divison" of an inheritance it says that if the distribution of houses, paintings, automobiles, furniture, etc., among the inheritors is "fair" then if any subgroup of them were to pool what they jointly receive and if the same rule of allocation were applied to the pooled goods to distribute them among the subgroup then the same distribution would result - and if another also did then that would yield an alternative solution to the entire problem.

---

[6]This caveat will not be repeated subsequently but must be understood as being present.

In the context of the history of the problem in the United States it is clear that a method must be house monotone - the hoopla in Congress over the Alabama paradox has set this precedent. However, examples show that not one of the five traditional divisor methods satisfy fair share; and, indeed, there is no divisor method that satisfies fair share. This fact made my co-author H. P. Young and me believe that the major challenge was to find a method that is house monotone and satisifies fair share. The result was the ***quota method*** [4] :

(i) $\mathbf{a} = \mathbf{f}$ for house size $h = \Sigma_i f_i$

(ii) if $\mathbf{a}$ is an apportionment for $h$, then $\mathbf{a}^*$ is an apportionment for $h+1$ with $a_k^* = a_k + 1$ for some one $k$ that maximizes $p_j/(a_j+1)$ over all $j$ satisfying $r_j(h+1) > a_j$, and $a_j^* = a_j$ for $j \neq k$.

Here was the ideal rule, the one that achieved both properties that seemed the most important! Dismissed in [12] as "a new complicated scheme," it is easy enough to describe : it assigns an extra seat to that state that deserves it most - measured by the value of $p_j/(a_j+1)$ - but among those who will not thereby violate the upper fair share $[r_j(h+1)]^+$. Regretably we did not immediately investigate its other properties, and so added another error of judgement to the history of the problem. For after a time - after having spent some effort at looking at solutions and seeing something seemed odd about them - we found that the entire class of methods that are house monotone and satisfy fair share ([6],[13]) can be easily described and that not one of these methods is population monotone! (Admittedly finding an example exhibiting the lack of population monotonicity was not a trivial task.) As important, not one of them is even consistent. This was a significant lesson : the axiomatic approach is fine but one needs to compute and to *look* at the results for many problems and so give oneself the chance to see that perhaps other unstated axioms are not satisfied. This realization led us to investigate more deeply both the historical record to learn what properties might be important to the users and the mathematics . . . and to reject the quota method as a method for apportioning seats.

There is a host of results describing what methods are realized by what combinations of axioms (see [8]). I will only cite several of the most important.

*Impossibility theorem* [5]. There is no method of apportionment that is consistent and satisfies fair share.

The situation is not that "as yet, no method has been found to satisfy [the several 'natural' requirements for an acceptable apportionment method] simulaneously in the general case" ([12], p. 188, underlining added) : it is that there is no such method, a very different statement.

*Divisor method characterization* [5]. The only methods of apportionment that are consistent and weakly population monotone are the divisor methods.

This gives powerful reasons for considering divisor methods as the only ones that are reasonable candidates for choice.

Consistency implies of a method that if one knows how any pair of states share any number of seats then the method is completely specified. Thus the problem of how to fairly apportion seats among 130 provinces is reduced to how to fairly share any number of seats between any pair of states, a seemingly much easier problem.

## 7. Pairwise comparisons

E.V. Huntington ([10],[11]) carried out the first serious study of the problem of apportionment. He concentrated on the aspect of consistency just mentioned : the equity in representation between each *pair* of states. Given a "trial" apportionment $\mathbf{a}$, he reasoned, one seat should be transferred from the relatively over-represented state $i$ to the relatively under-represented one $j$ - thus from $i$ to $j$ if $a_i/p_i > a_j/p_j$ - whenever this decreases the "discrepancy" in representation between the two. If no transfer between any pair of states can reduce the discrepancy then this "stable" apportionment is the solution. The question then becomes : *what* measure of discrepancy? Huntington considered and analyzed every possible "linear" measure of the difference between $a_i/p_j$ and $a_j/p_j$ such as $|a_i/p_i - a_j/p_j|$, $|p_i/a_i - p_j/a_j|$, their relative difference $|a_i/p_i - a_j/p_j|/\min\{a_i/p_i, a_j/p_j\}$, $|p_i/p_j - a_i/a_j|$, etc. He found that either the measure does not guarantee a stable solution (as is the case for the last of the measures given in the previous sentence) or the measure gives solutions identical to one of the five traditional divisor methods. He fixed on the relative difference measure because "it is clearly the relative or percentage difference, rather than the mere absolute difference, which is significant," and, as was described above, he claimed the method that resulted was "unbiased". By concentrating on pairs as he did Huntington restricted his study to divisor methods; but his choice of measures of difference between pairs of states unfortunately limited him to consideration of only the five traditional methods and this seems ultimately to have led him (and others) astray.

## 8. Average ratio pairwise transfer

Oyama [12] takes the same tack as Huntington but restricts its application to the case when one state is over-represented absolutely and the other under-represented absolutely, that is, when $a_i/p_j \geq h/\Sigma_i p_i \geq a_j/p_j$. He remarks that it "possibly implies that our stable region is 'larger than' Huntington's one" and that "our new rule provides deep insights to traditional apportionment methods" (see [12], p. 194). Consider the example given in table 2. Let $t_i = (a_i/p_i) \times 10^3$. In this example the most over-represented down to the most under-represented state by the common solution of the methods of Adams, Dean, Hill and Webster are

$$t_5 = 2.232 > t_4 = 1.998 > t_3 = 1.996 > (h/\Sigma_i p_i) \times 10^3 = 1.701 > t_1 = 1.700 > t_2 = 1.667.$$

It is impossible to make a transfer of one seat from any one state to any other and thereby reduce $|a_i/p_i - a_j/p_j|$ since this is a Webster solution. Consider, however, the apportionment $\mathbf{a}^* = (50, 51, 2, 2, 2)$, for which $t_1 = 1.666$ and $t_2 = 1.700$ : it also is "stable" by Oyama's "average ratio pairwise transfer" criterion - so indeed his stable region is not ***possibly*** larger, it ***is*** larger - but does one wish to admit as "stable" an apportionment which gives more seats to one state than to another one with larger population? The only effect of restricting the possibility of transfers to pairs of states one of which is under-represented and the other over-represented is to allow additional apportionments as in this example : it admits alternative solutions that ignore inequities among the over-represented states or among the under-represented ones. How this new rule provides insights, let alone deep ones, remains an unexplained mystery.

**Table 2**

| State | Population | Quota | Adams=Dean=Hill=Webst | Jefferson |
|---|---|---|---|---|
| 1 | 30 007 | 51.045 | 51 | 52 |
| 2 | 29 994 | 51.023 | 50 | 52 |
| 3 | 1 002 | 1.705 | 2 | 1 |
| 4 | 1 001 | 1.703 | 2 | 1 |
| 5 | 896 | 1.524 | 2 | 1 |
| Total | 62 900 | 107 | 107 | 107 |

## 9. "Parametric divisor methods"

**Table 3**

| State | Population | Quota | Hill | $t \in$ $[0, .388]$ | $t \in$ $[.389, .425]$ | $t \in$ $[.426, 1]$ |
|---|---|---|---|---|---|---|
| 1 | 42 444 | 1.434 | 2 | 2 | 1 | 1 |
| 2 | 73 470 | 2.482 | 2 | 3 | 3 | 2 |
| 3 | 134 148 | 4.532 | 4 | 4 | 5 | 5 |
| 4 | 164 343 | 5.552 | 6 | 5 | 5 | 6 |
| Total | 414 405 | 14 | 14 | 14 | 14 | 14 |

A *parametric divisor method* [12] is a divisor method based on a function $d$ of form $d(a) = a+t$, where $0 \leq t \leq 1$, which is a special case of (3). Thus of the "traditional five" the methods of Adams ($t = 0$), Webster ($t = 0.5$) and Jefferson ($t = 1$) are parametric methods; and Condorcet's ($t = 0.4$) is as well. The simplicity of the mathematical definition of such a method seems to be its chief feature for there is no qualitative characterization that distinguishes a parametric divisor method from the general class of divisor methods. However, we are told that "the parametric method covers all the six traditional methods . . . by changing the parameter t from 0.0 to 1.0" ([12], p. 214, underlining added). What "covers" means is left undefined. It seems that in some sense the reader is to feel that it "suffices" to consider these methods : presumably they produce all reasonably conceivable solutions. And yet, there are problems for which there is a divisor method apportionment that is unattainable by ***any*** parametric divisor method. Table 3 gives an example of a problem where the unique Hill apportionment can be obtained by ***no*** parametric divisor method. It is a simple exercise to prove that if **a** is an apportionment of $h$ for two parametric divisor methods, one based on $t_1$ and the other on $t_2$ ($\geq t_1$), then it is also an apportionment of $h$ for all parametric divisor methods based on $t$, where $t_1 \leq t \leq t_2$. Thus, the table contains *all* parametric divisor method solutions.

## 10. Bias

Oyama [12] carries out extensive analyses of the various different apportionments of the 512 seats of the House of Representatives among Japan's 130 political constituencies according to the 1985 populations. On the basis of his analysis he concludes :

"We believe that the [method of Hamilton] is the most unbiased method." (p. 206)

A parametric method "should be taken into account for the parameter" $0.3 \leq t \leq 0.5$ since[7] $t$ larger than 0.5 makes solutions "too favorable to larger constituencies" and $t$ smaller than 0.3 "too favorable to smaller constituencies" (p. 207).

"We would like to strongly recommend [the parametric divisor method] with parameter value" $0.46 \leq t \leq 0.48$ (p. 214).

In making these assertions he falls into the same traps as others have before him. There is no definition of what is meant by an "unbiased" method, no specific criterion is given by which to judge whether a method is too favorable for one or another class of constituencies, the entire analysis concerns the *one* specific problem in question and no others. Therefore, given another problem it is impossible to know what conclusions might be drawn - perhaps Webster's method would seem the "most unbiased," perhaps a parameter value below 0.4 and above 0.55 would make solutions too favorable for the smaller or the larger

[7] Oyama [12] defines parametric divisor methods as here but then switches terms by talking about $1-t$ rather than $t$. This explains the abbreviated citations.

constituencies respectively, perhaps it would be better to chose a parametric method with parameter value $0.51 \leq t \leq 0.53$. Nor could one know what judgements would result if another specific problem were in hand for no criteria are defined.

**Table 4**

| Constituency | Population | Fair share (of 512) | Fair share (of 16) | $t \in$ [.46, .48] | $t = .5$ |
|---|---|---|---|---|---|
| SITM-1 | 1 526 507 | 6.452 | 6.502 | 6 | 7 |
| AITI-1 | 1 057 501 | 4.470 | 4.504 | 4 | 5 |
| IWTE-2 | 586 719 | 2.480 | 2.499 | 3 | 2 |
| NGNO-1 | 585 569 | 2.475 | 2.494 | 3 | 2 |
| Total | 3 756 296 | | 16 | 16 | 16 |

It turns out that all parametric divisor method apportionments for the 1985 Japan Diet problem for $t$ in the range $0.46 \leq t \leq 0.48$ are one and the same, and it differs from the Webster apportionment ($t = 0.5$) in exactly 4 constituencies as shown in table 4. It seems that the range $0.46 \leq t \leq 0.48$ is preferred because "it gives almost the same assignment as [Hamilton's method]" (p. 214) - and it agrees with Hamilton's in the four constituencies of table 4. But this range of parameters will not necessarily coincide in the same manner for another problem. Looking at the fair shares of the 512 seats it does indeed seem as though the apportionment (6,4,3,3) is less "biased" than the apportionment (7,5,2,2); but a closer look at what the fair shares of the 16 seats that are shared among the four gives a different picture. Since the methods are consistent the solutions to the subproblem of table 4 must be the same. Still more to the point is the way each pair of constituencies share the number of seats they get together. The corresponding fair shares are given in table 5.

**Table 5**

| Constituency | Population | Fair share (of 9) |
|---|---|---|
| SITM-1 | 1 526 507 | 6.501 |
| IWTE-2 | 586 719 | 2.499 |

| Constituency | Population | Fair share (of 9) |
|---|---|---|
| SITM-1 | 1 526 507 | 6.505 |
| NGNO-1 | 585 569 | 2.495 |

| Constituency | Population | Fair share (of 7) |
|---|---|---|
| AITI-1 | 1 057 501 | 4.502 |
| IWTE-2 | 586 719 | 2.498 |

| Constituency | Population | Fair share (of 7) |
|---|---|---|
| AITI-1 | 1 057 501 | 4.505 |
| NGNO-1 | 585 569 | 2.495 |

Viewed in this light the parametric methods with range $0.46 \leq t \leq 0.48$ look less attractive. In every instance the Webster apportionment rounds the fair shares of the total each pair of constituencies receive together by the usual rule to determine apportionments; namely, round them to the closest integer. In this sense Webster's is the "most similar" to

Hamilton's. Moreover, it is the only divisor method identical to it over all 2 and 3 state problems.

The essence of the idea of "bias" concerns the behavior of methods over <u>many</u> problems. Every *one* problem is full of inequities. For example, the fair share (under the assumption that each constituency must have at least one seat) of the Japanese constituency labelled TKYO-6 is 3.419 and yet every apportionment calculated in Oyama's paper gives 3 seats to this constituency : so are all those methods biased against TKYO-6? Clearly the methods have nothing against TKYO-6! The data of the problem in question is such that in *this* case this constituency will receive fewer seats than its fair share. But one can hope that in another case the same constituency will receive more than its fair share, so that over the "long run" it will receive on average the average of its fair shares. A method $\Phi$ is

*unbiased* : if the average of the apportionments of each constituency is equal to the average of its fair shares.

This leaves the question : average over what set of problems? It is a thorny question because there is no large pool of problems to analyze and there is no natural probability distribution from which to draw random problems. In the face of this difficulty various "models" have been studied.

It is trivial to propose an unbiased method : assign the $h$ seats at random with probabilities proportional to the fair shares. In this case none of the other desirable properties is guaranteed. It is easy to argue that the method of Hamilton is unbiased : the magnitudes of the fractional remainders of the fair shares are independent of the magnitudes of their integer parts so there is no tendency for a larger or a smaller state to be advantaged. But Hamilton's method fails to satisfy basic properties, most notably consistency and several monotonicity concepts.

If consistency is accepted as a rock bottom requirement then knowing how any two states share any number of seats is enough to specify a method. One approach to "bias" is to fix a pair of populations $p_1 > p_2 > 0$ and to assume that each number of seats between 0 and $p_1 + p_2$ are equally likely to be shared between them. Taking averages over this set of two state problems one has the result that the only divisor method that is unbiased for all choices of $(p_1, p_2)$ is Webster's. The advantage of this model is that no assumption need be made regarding the distribution of the populations. Its disadvantage is that the average is taken over sizes of house that are not realistic.

A more reasonable approach is to fix an apportionment $\mathbf{a}$ and to consider the set of all problems $R(\mathbf{a})$ whose fair shares give the apportionment $\mathbf{a}$ for a particular divisor method $\Phi^d$ :

$R(\mathbf{a}) = \{\mathbf{r} > 0 : d(a_i) \geq r_i \geq d(a_i - 1)\}$

Assume that every choice of $\mathbf{r}$ within the "box" $R(\mathbf{a})$ is equally likely. This is equivalent to assuming that every normalized population $\mathbf{p}$ for which $\mathbf{a} \in \Phi^d(\mathbf{p}, \mathbf{f}, \mathbf{c}, h)$ is equally likely. It means one is assuming a uniform distribution *locally*, so again no assumption is made regarding the overall distribution of apportionment problems.

*Characterization of unbiased method* [7],[8]. Webster's is the unique unbiased divisor method.

No one model or definition can establish beyond all doubt that one method or another is "unbiased" : the conclusion depends upon the model and the definition. Nevertheless, the evidence that Webster's method is indeed unbiased is overwhelming both on theoretical and experimental grounds. An experimental estimate of the "bias" of an apportionment solution $\mathbf{a}$ to a problem with fair shares $\mathbf{r}$ can be made as follows (see [8], p. 126-128). If $\mathbf{x}$ is a vector and $T$ a subset of the indices, define $x_T = \Sigma_T x_i$. Let $S$ be the set of the $[s/3]^-$ smallest states and $L$ the set of the $[s/3]^-$ largest states. The set of states $S$ is collectively better represented than the set of states $L$ if $a_S/r_S > a_L/r_L$ and less well represented if the inequality is reversed. The *percentage bias in favor of the small* of any apportionment is defined to be the relative difference between those terms, that is, $100(a_S/r_S - a_L/r_L)/(a_L/r_L)$. Applying this to the 21 House of Representatives apportionment problems of United States history one can count the

**Table 6. Number of times the small favored over 21 problems of U.S. history.**

| Adams ($t = 0$) | Dean | Condorcet ($t = 0.4$) | Hill | $t = 0.46$ |
|---|---|---|---|---|
| 21 | 16 | 16 | 14 | 11 |

| $t = 0.48$ | Webster ($t = 0.5$) | $t = 0.52$ | $t = 0.54$ | $t = 0.6$ | Jefferson ($t = 1$) |
|---|---|---|---|---|---|
| 10 | 8 | 7 | 6 | 4 | 0 |

number of times the small states are favored (table 6) and compute the average bias over the 21 problems (table 7). The 0.31% bias of the Webster method in favor of the large states is surprisingly close to the predicted value of 0.0%.

**Table 7.**
**Average of the percentage biases in favor of the small over 21 problems of U.S. history.**

| Adams (t=0) | Dean | Condorcet (t=0.4) | Hill | t=0.46 |
|---|---|---|---|---|
| 17.50 | 4.36 | 3.96 | 2.55 | 1.61 |

| $t = 0.48$ | Webster ($t = 0.5$) | $t = 0.52$ | $t = 0.54$ | $t = 0.6$ | Jefferson ($t = 1$) |
|---|---|---|---|---|---|
| 0.58 | -0.31 | -0.91 | -1.16 | -3.02 | -16.55 |

Of course, the choice of the sets of small states and large states is arbitrary : different choices give different numerical results, but by and large one obtains the same qualitative results. Also, one should beware of the statistical significance of these numbers because the choice of house size changed over history and it was undoubtedly correlated with data concerning the populations of the states.

Applying the same analysis to the one Japanese problem studied in [12] one obtains the results given in table 8. This one problem appears indeed to "force" solutions that are biased in favor of the large instead of the small, so the parametric method with t = 0.46 or 0.48 seems indeed "better" than t = 0.5. But taking the parametric method with t = 0.4 is even "better" than that, so why not choose it instead of Oyama's recommendation of t = 0.46 or 0.48! The fact is that every single problem is biased : the analysis and the choice of a method must depend on its behavior over many problems. Oyama concludes : "Although Balinski and Young say that [Webster's method] is the only unbiased divisor method, we believe that generally [Webster's method] is still more favorable to larger constituencies *since most numerical examples violate the [fair share] property*" ([12], p. 214, italics added). But to analyze either a single problem or many problems or "all" problems via some model that incorporates the sense of "all" one needs a precise concept of what one means by "bias" and no such concept is defined in [12]! On what does "we believe" depend?

**Table 8.**
**Percentage biases in favor of the small in Japans House, current populations, h=512.**

| Adams (t=0) | Dean | Condorcet (t=0.4) | Hill | t=0.46 |
|---|---|---|---|---|
| 15.68 | 3.78 | -0.48 | -0.48 | -1.88 |

| t=0.48 | Webster | t=0.52 (t=0.5) | t=0.54 | t=0.6 | Jefferson (t=1) |
|---|---|---|---|---|---|
| -1.88 | -4.65 | -4.65 | -4.65 | -6.02 | -14.01 |

Where Oyama has observed that "most numerical examples violate the quota property" is another mystery : I am not aware of a single problem defined on the basis of actual population data where a Webster apportionment violates fair share. Indeed, the "box model" (based on the distribution of seats in the United States House by the census data of 1970) was used to estimate the probability of each of the five traditional divisor methods to violate fair share by Monte Carlo simulation ([8], p. 81 and pp. 131-132) : whereas the Adams and Jefferson methods violate fair share practically every time, Webster's does with (estimated) probability 0.00061. No divisor method simultaneously violates upper fair share in one state and lower fair share in another, so a most unusual distribution of populations would be required for Webster's method to violate fair share (in contrast to Adams's and Jefferson's highly biased methods).

## 11. Webster's method

Conceptually it seems a much easier problem to decide how two states or constituencies should share a fixed number of seats between them than to decide how three, or fifty or a hundred and thirty states should share a fixed number of seats among them. And I believe that most people would agree that the best way, the fairest way, and the most intuitive way to share a fixed number of seats between *two* states is *simple rounding*, i.e., to compute their fair shares and round to the closest integer. Let us say that a method is consistent with simple rounding if in each of its apportionments *every* two states share the number of seats they receive together by simple rounding. A very powerful argument in favor of Webster's method is that it is the only method consistent with simple rounding.

Formally, let $[[x]]$ be the closest integer to $x$, and $[[n+1/2]]$ be either $n$ or $n+1$ when $n$ is integer. Say that

$(a_i, a_j)$ is a *simple rounding* of the problem $(\mathbf{p},\mathbf{f},\mathbf{c},h)$ if

$a_k = \mathrm{mid}\{f_k, [[r_k]], c_k\}$ where $r_k = p_k\{(a_i+a_j)/(p_i+p_j)\}$ for $k = i, j$.

A method $\Phi$ is

*consistent with simple rounding* : if $\mathbf{a} \in \Phi(\mathbf{p},\mathbf{f},\mathbf{c},h)$ implies $(a_i, a_j)$ is a simple rounding of the problem for every pair $i, j$.

*Characterization theorem.* Webster's is the unique method of apportionment that is consistent with simple rounding.

The proof is immediate : Webster's method is consistent, and Webster's method applied to two states is simple rounding.

The weight of the evidence thus suggests that *the method of Webster is most appropriate for the apportionment of representation to constituencies, states or districts* :

• It is the unique method that apportions seats so that each one state receives its fair share of the total shared with every other state rounded in the usual way (that is, rounded at 0.5).

- It is the unique method that is consistent, proportional and unbiased.
- It is the unique method that is consistent and is near fair share.
- It is house monotone and population monotone.
- It fails to satisfy fair share only with a negligibly small probability.

Therefore, for all practical purposes, it does indeed satisfy *all* of the desirable properties.

**Acknowledgement.**

I am indebted to two anonymous referees for their corrections and helpful suggestions.

## References

[1] Y. Aït-Sahalia, Michel Balinski and Gabrielle Demange (1990), "The adjustment of contigency tables : two new approaches," in Paul Champsaur et al, editors, *Essays in Honor of Edmond Malinvaud* (vol. 3), The MIT Press, Cambridge, Mass. and London, 99-114.

[2] M. L. Balinski and Gabrielle Demange (1989), "Algorithms for proportional matrices in reals and integers," *Mathematical Programming* 45, 193-210.

[3] M. L. Balinski and Gabrielle Demange (1989a), "An axiomatic approach to proportionality between matrices," *Mathematics of Operations Research* 14, 1989, 700-719.

[4] M. L. Balinski and H. P. Young (1975), "The quota method of apportionment," *American Mathematical Monthly* 82, 701-730.

[5] M. L. Balinski and H. P. Young (1977), "On Huntington methods of apportionment," *SIAM Journal on Applied Mathematics* 33, 607-618.

[6] M. L. Balinski and H. P. Young (1979), "Quotatone apportionment methods," *Mathematics of Operations Research* 4, 31-38.

[7] M. L. Balinski and H. P. Young (1980), "The Webster method of apportionment," *Proceedings of the National Academy of Sciences*, USA 77, no. 1, 1-4.

[8] Michel L. Balinski and H. Peyton Young (1982), *Fair Representation : Meeting the Ideal of One Man, One Vote*, Yale University Press, New Haven, Ct. and London. Japanese translation, Chikura-Shobo Publishing Company, Tokyo, 1987.

[9] M. L. Balinski and H. P. Young (1983), "Apportioning the United States House of Representatives," *Interfaces* 13.4, 35-43.

[10] E. V. Huntington (1921), "The mathematical theory of the apportionment of Representatives," *Proceedings of the National Academy of Sciences*, U.S.A., 7, 123-127.

[11] E. V. Huntington (1928), "The apportionment of Representatives in Congress," *Transactions of the American Mathematical Society* 30, 85-110.

[12] Tatsuo Oyama, (1991), "On a parametric divisor method for the apportionment problem," *Journal of the Operations Research Society of Japan* 34, 187-221.

[13] J. W. Still (1979), "A class of new methods for Congressional apportionment," *SIAM Journal on Applied Mathematics* 37, 401-418.

Michel BALINSKI : Laboratoire d'Econométrie
de l'Ecole Polytechnique
1 rue Descartes, 75005 Paris, France