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INVERSE NTH POWER DETECTION LAW FOR WASHBURN'S LATERAL RANGE CURVE

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Abstract A detection law (the conditional detection rate) which corresponds to the model of Washburn's lateral range curve is presented in this paper. The detection law dealt with here is a general one which includes the inverse cube law and the definite range law. The lateral range curve and the sweep width of this law are derived explicitly. Two methods for estimating the parameters of the law are presented and a set of actual data is analyzed. Applying the model, we investigate the forestalling detection problem in a search-and-search situation. Taking account of the forestalling detection by the target, the generalized formulas of the lateral range curve and the sweep width are derived.

1. Introduction

In search theory, the capability of a detection device is usually presented by a conditional detection probability (or rate) given the distance from the searcher to the target. This function of conditional detection probability is called the detection law. If a detection law is given, the sighting potential of a search pattern, the lateral range curve and the sweep width are caluculated from it. Therefore, the detection law is a fundamental and essential quantity to investigate search problems. However, general models of detection laws have not been presented. On the other hand, as for the lateral range curve $PL(x)$, a general model is proposed by Washburn [4] :

$$PL(x) = 1 - \exp(-|\frac{x_0}{x}|^b), \tag{1}$$

where x is the lateral range and x_0 and b are parameters. He also shows its sweep width W as

$$W = 2x_0\Gamma(1-\frac{1}{b}), \tag{2}$$

where $\Gamma(\cdot)$ is the Gamma function defined by $\Gamma(x)=\int_0^\infty t^{x-1}\exp(-t)dt$. The formulas (1) and (2) have wide applicability by selecting parameters x_0 and b, appropriately. However, he did not show any derivation of the formula and any detection law corresponding to Eq.(1). If we want to calculate the sighting potential [3] of searcher's path or to analyze the forestalling detection problem, we must know the detection law instead of the lateral range curve. In this paper, we present a detection law which corresponds to the model of Washburn's lateral range curve (1).

In search theory, the definite range law, the imperfect definite range law and the inverse cube law have been used to investigate search problems. In these detection laws, only the inverse cube law is a decreasing function of distance. This law was given by Koopman [2] in his study on search theory during WW II. The detection rate of the inverse cube law is defined by

$$f(r) = \frac{k}{r^3}. \tag{3}$$

where r is the distance from the searcher to the target and k is a constant specified by the conditions of the circumstance and the target. Usually, k is determined by experiment data. According to Koopman's derivation, this law is a model of visual search and is obtained by assuming that the detection rate is proportional to the solid angle (measured from the searcher) subtended by the target. Because of the nicety of mathematical treatment, the inverse cube law has been used widely for many detection devices to approximate the detection function. However, since Eq.(3) has only one parameter k, it is difficult to approximate the experimental curve sufficiently. This is the reason why we require a more general detection law.

2. The Model of Inverse nth Power Law of Detection

Usually, a detection device has its inherent detection capability such as the effective range r_0 in a circumstance of the search space. For the detection device, the distance in the search space has meaning in the normalized distance r/r_0. Further, usually the certainty of detection which is presented by the detection probability decreases with the distance, but there are many varieties of the dependency on the range at every detection device. Considering these features, we postulate that the detection rate $f(r)$ is decreasing in proportion to the inverse nth power of the normalized range r/r_0: $f(r)=C(r/r_0)^{-n}$. Simplifying the parameters, we define a model of detection rate function by

$$f(r) = \left(\frac{k}{r} \right)^n, \quad k > 0, \quad n > 0. \tag{4}$$

We refer to the detection law given by Eq.(4) as "the inverse nth power law." Comparing Eq.(4) with Eq.(3), we note that this detection law is a natural extension of the inverse cube law. If a searcher searches the target being at a distance of r for time t independently, the detection probability is given by

$$P(r, t) = 1 - \exp(-f(r)\, t). \tag{5}$$

Varying k and n, we show the curve of $P(r,1)$ given by Eqs.(4) and (5) in Fig.1.

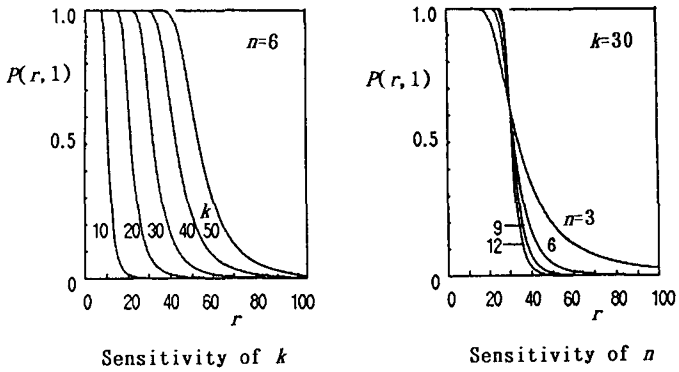


Fig.1 The detection probability $P(r,1)$

From these curves, we can see the role of the parameters k and n in Eq.(4). As seen in Fig.1, the curve of $P(r,1)$ increases in size as k increases (k is the scale parameter of the curve), and the falling gradient of the curve becomes steeper as n increases (n is the shape parameter). Hence, the inverse nth power law can be expected to approximate correctly the actual detection law by selecting these parameters k and n appropriately.

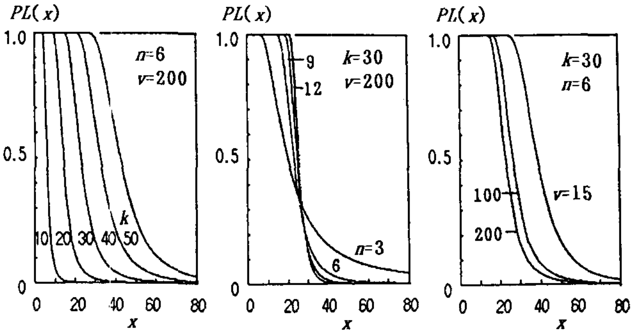
Here, we derive the lateral range curve $PL(x)$ of the inverse nth power law. Considering the searcher traveling with speed v along the infinite straight line with CPA (the closest point of approach) x from the target (x is called the lateral range), we obtain the sighting potential $F(x)$ of the target as

$$F(x) = \int_{-\infty}^{\infty} \frac{k^n}{(x^2+v^2t^2)^{n/2}}\, dt = \frac{k^n\sqrt{\pi}\,\Gamma((n-1)/2)}{v|x|^{n-1}\Gamma(n/2)}. \tag{6}$$

From Eq.(6), the lateral range curve $PL(x)$ is obtained by

$$PL(x) = 1 - \exp(-F(x)) = 1 - \exp\left(-\frac{k^n\sqrt{\pi}\,\Gamma((n-1)/2)}{v|x|^{n-1}\Gamma(n/2)}\right). \tag{7}$$

Varying the parameters k, n and v, we show the curves of $PL(x)$ in Fig.2.



Sensitivity of k Sensitivity of n Sensitivity of v

Fig.2 The lateral range curve $PL(x)$

Integrating the lateral range curve $PL(x)$ with respect to x over $(-\infty,\infty)$, we have the sweep width W of the inverse nth power law.

$$W = 2\Gamma\left(\frac{n-2}{n-1}\right)\left\{\frac{k^n\sqrt{\pi}\,\Gamma((n-1)/2)}{v\Gamma(n/2)}\right\}^{1/(n-1)}. \tag{8}$$

Sensitivity of the parameters k, n and v to the sweep width W are shown in Fig.3.

Here, we define x_0 as

$$x_0 = \left\{\frac{k^n\sqrt{\pi}\,\Gamma((n-1)/2)}{v\Gamma(n/2)}\right\}^{1/(n-1)}. \tag{9}$$

Then, $PL(x)$: Eq.(7) and W: Eq.(8) are rewritten as the following simple expressions.

$$PL(x) = 1 - \exp\left(-\left(\frac{x_0}{|x|}\right)^{n-1}\right), \tag{10}$$

$$W = 2x_0\Gamma\left(\frac{n-2}{n-1}\right). \tag{11}$$

Eqs.(10) and (11) are exactly identical with Washburn's model given by Eqs.(1) and (2). Here if we set $n=3$ in Eqs.(7) and (8), we obtain $PL(x)$ and W of the inverse cube law, and if we consider the limit $n\to\infty$ remaining x_0 constant, we have $PL(x)$ and W of the definite range law. Therefore, the inverse nth power law is a general detection law by which a wide range of detection laws, not only the decreasing type laws but also the cookie-cutter type laws, can be approximated if the parameters k and n are selected appropriately.

3. Estimation of the Parameters k and n from an Experiment

In this section, we discuss two methods for estimating the parameters k and n in Eq.(4) from a set of experiment data. The experiment considered here is as follows.

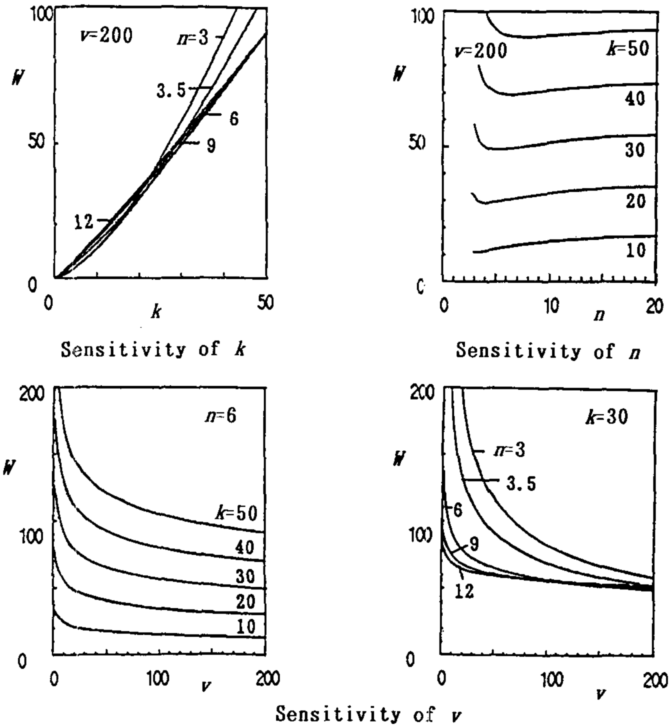


Fig. 3 The sweep width W

The searcher approaches the target from far away along a straight line with lateral range $x=0$. When the searcher detects the target, he records the detection range R, then turns back to the starting point and repeats the same trial again and again. Let r_i be the detection range on the ith trial, $i=1,2,\cdots,N$, and without any loss of generality, we assume $r_i \geq r_{i+1}$, $r_{N+1}=0$.

The expected values of $\{r_i\}$ and $\{r_i^2\}$ are calculated by

$$E_0(R) = \frac{\sum_i r_i}{N}, \tag{12}$$

$$E_0(R^2) = \frac{\sum_i r_i^2}{N}. \tag{13}$$

The curve of cumulative fraction is defined as a piecewise linear curve constructed by connecting points $\bar{G}_0(r_i)$, $i=1,2,\cdots,N$, sequentially given by

$$\bar{G}_0(r_i) = \frac{i}{N}. \tag{14}$$

$\bar{G}_0(r_i)$ is the experimental c.d.f. (tail) of detection range: $\Pr(R \geq r_i) = \bar{G}_0(r_i)$.

Assuming Eq.(4) and calculating the sighting potential of the course (y,∞) with $x=0$, we obtain the theoretical c.d.f. of the detection range corresponding to $\bar{G}_0(r_i)$ given by Eq.(14) as follows.

$$\bar{G}(y) = 1 - \exp\left(-\int_y^\infty \frac{f(r)}{v}\,dr\right) = 1 - \exp\left(-\frac{k^n}{v(n-1)|y|^{n-1}}\right). \tag{15}$$

From Eq.(15), p.d.f. $g(y)$ is obtained by

$$g(y) = -\frac{d\bar{G}(y)}{dy} = \frac{k^n}{v|y|^n}\exp\left(-\frac{k^n}{v(n-1)|y|^{n-1}}\right). \tag{16}$$

Using Eq.(16), we can calculate the first and the second order moment of the detection range corresponding to $E_0(R)$; Eq.(12) and $E_0(R^2)$; Eq.(13), respectively.

$$E(R) = \int_0^\infty y g(y)\,dy = \{ \frac{k^n}{v(n-1)} \}^{1/(n-1)} \Gamma(\frac{n-2}{n-1}), \quad n > 2, \tag{17}$$

$$E(R^2) = \int_0^\infty y^2 g(y)\,dy = \{ \frac{k^n}{v(n-1)} \}^{2/(n-1)} \Gamma(\frac{n-3}{n-1}), \quad n > 3. \tag{18}$$

Here, we consider methods for estimating (k, n). Although we examine the least squares method to Eq.(15), it seems to be impractical since equations to the least square estimators (k, n) are very complicated. We propose here two methods : the maximum likelihood method and the moment matching method. Hereafter we call them simply the ML and the MM method, respectively.

A. The ML method.

Since p.d.f. of the detection range is given by Eq.(16), the joint density function of $\{r_i\}$ (Likelihood) is derived as

$$L(\{r_i\}) = \Pi_i g(r_i) = \frac{k^{nN}}{v^N} (\Pi_i r_i)^{-n} \exp(- \frac{k^n}{v(n-1)} \Sigma_i r_i^{1-n}).$$

To obtain the estimators k and n maximizing $L(\{r_i\})$, $\log L(\{r_i\})$ is partially differentiated with respect to k and n, and then those are set to zero. We have

$$\frac{\partial \log L(\{r_i\})}{\partial k} = 0: \ \frac{Nn}{k} - nk^{n-1} \frac{\Sigma_i r_i^{1-n}}{v(n-1)} = 0, \tag{19}$$

$$\frac{\partial \log L(\{r_i\})}{\partial n} = 0: \ N\log(k) - \Sigma_i \log(r_i) - \frac{k^n}{v(n-1)^2}[(n-1)\Sigma_i\{r_i^{1-n}\log(k/r_i)\} - \Sigma_i r_i^{1-n}] = 0. \tag{20}$$

Solving k from Eq.(19), we have

$$k = (\frac{Nv(n-1)}{\Sigma_i r_i^{1-n}})^{1/n}. \tag{21}$$

Substituting Eq.(21) into Eq.(20), we have

$$N(n-1)\Sigma_i\{r_i^{1-n}\log(r_i)\} + N\Sigma_i r_i^{1-n} - (n-1)\Sigma_i r_i^{1-n} \Sigma_i \log(r_i) = 0. \tag{22}$$

Unfortunately, Eq.(22) of n cannot be solved analytically, therefore it must be solved numerically, e.g., by Newton method, and then k is obtained from Eq.(21).

B. The MM method.

In the MM method, we use conditions: $E_0(R) = E(R)$ given by Eqs.(12) and (17) and $E_0(R^2) = E(R^2)$ given by Eqs.(13) and (18). These conditions are written as

$$\{ \frac{k^n}{v(n-1)} \}^{1/(n-1)} \Gamma(\frac{n-2}{n-1}) = E_0(R), \tag{23}$$

$$\{ \frac{k^n}{v(n-1)} \}^{2/(n-1)} \Gamma(\frac{n-3}{n-1}) = E_0(R^2), \quad n > 3. \tag{24}$$

k is solved from Eq.(23) as

$$k = [v(n-1)\{ \frac{E_0(R)}{\Gamma((n-2)/(n-1))} \}^{n-1}]^{1/n}. \tag{25}$$

Substituting k into Eq.(24), we have the next equation.

$$\Gamma(\frac{n-2}{n-1})^2 - a\Gamma(\frac{n-3}{n-1}) = 0, \quad a = \frac{E_0(R)^2}{E_0(R^2)}. \tag{26}$$

To solve Eq.(26) numerically, $\Gamma(\cdot)$ is approximated by a polynomial expression and Eq.(26) is solved for n by applying Newton method. Then, k is obtained by substituting n into Eq.(25).

4. An Example

In this section, the ML and the MM methods are applied to a set of actual data which is obtained from an experiment of an airplane searching for a small target on the sea by a radar. The data are shown in Table 1.

Applying the ML and the MM methods to the data given by Table 1, we obtain the estimators k and n shown in Table 2. The accuracy of the Newton method for n is 10^{-4}.

Table 1. Data of detection range

Conditions	Target: a small buoy with the corner reflector. Detection device: a radar of aircraft. Height of the searcher: 1500 ft. Search speed: 200 kt. Sea state: 3.
Detection range	37,37,28,34,30,30,37,37,25,30,37,37,37,34,42,35,37,33, 37,36,32,31,42,35,31,37,37,31,35,32,39,41,36,40,42,43, 42,45,38,42,30,31,39,40 nm (nautical mile).
Summary	N=44, max.range=45 nm, min.range=25 nm, $E(R)$=35.9 nm, $E(R^2)$=1311 nm^2, a=0.985.

Table 2. Estimated k and n

	k	n	MSD
The ML method	52.7	8.3	7.234×10^{-3}
The MM method	47.9	12.1	3.676×10^{-3}

In order to comfirm the fitness of $\bar{G}_0$ and $\bar{G}$, we examine the Kolmogrov-Smirnov test at the 0.05 level. Consequently, they are not rejected by the test. In Table 2, MSD (the mean square deviation) of $\bar{G}_0$ from $\bar{G}$ is also shown.

$$\mathrm{MSD} = \frac{\Sigma_i\{\bar{G}_0(r_i)-\bar{G}(r_i)\}^2}{N} .$$

MSD is a measure of fitness of the estimators to the experimental data $\bar{G}_0$. Hence, the MM method is better than the ML method in this case.

Fig.4 shows c.d.f. of detection range $\bar{G}(r)$ given by Eq.(15) using the estimated parameters by the MM method. In this figure, the points plotted by · are the points

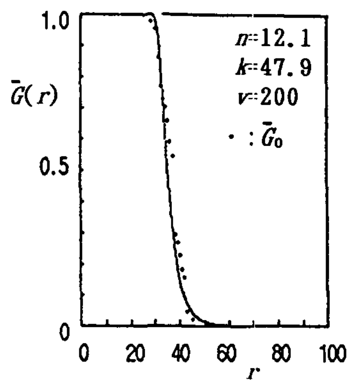
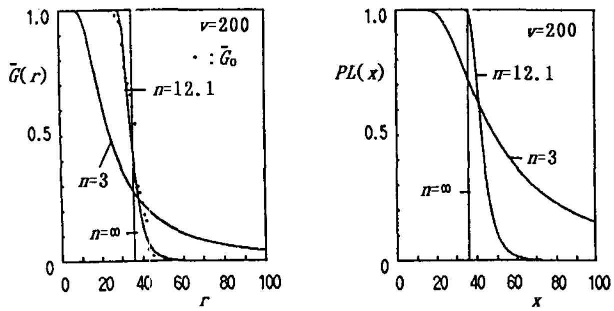


Fig.4 $\bar{G}(r)$ estimated by the MM method of experiment data $\{\bar{G}_0(r_i)\}$ given by Eq.(14). It should be noted that the theoretical c.d.f. $\bar{G}(r)$ of the inverse nth power law agrees well with the experiment values.

If the parameters of the inverse cube law and the definite range law are estimated from the experiment data so as to equate the theoretical mean detection range with the mean of the experiment data, we obtain the estimators (k, n) as shown in Table 3. In Table 3, we also show the sweep width W calculated by Eq.(8). The curves of $\bar{G}(r)$ with the parameters given by Table 3 for each detection law are visualized in Fig.5. As seen in Fig.5 and Table 3, the inverse nth power law agrees very well with the experiment data beyond comparison with the other detection laws. Furthermore, it should be noted that an inappropriate detection law leads to considerable error in the sweep width, even if the parameter is estimated by actual data of detection range. Therefore, it is very important to determine the detection law correctly when we analyze a search problem.

Table 3. Comparison of detection laws

Detection law	Parameters	MSD of $\bar{G}_0$	W
Inverse nth power law	k=47.9, n=12.1	3.676×10^{-3}	87 nm
Inverse cube law	k=54.8, n=3	4.807×10^{-2}	144
Definite range law	x_0=35.9, n=∞	2.890×10^{-2}	72



$\bar{G}(r)$ $PL(x)$

Fig.5 Comparison of detection laws

5. The Lateral Range Curve and the Sweep Width in Two-Sided Search

Analysis of the forestalling detection problem in a two-sided search situation is an important application of the detection law. Koopman [2,3] derived the forestalling lateral range curve and the forestalling sweep width in an encounter of the searcher with the target using detection devices of the inverse cube law. In this section, we generalize the model to the inverse nth power law. We consider a two-sided search situation in which the searcher and the target are searching for each other using detection devices of the inverse nth power law with parameters (k_S, n) and (k_T, m), respectively. The searcher is assumed to approach to the target along a straight line with the lateral range x. The probability of forestalling detection in

$[t, t+\Delta t]$ by the searcher is given by

$$p_S(x, t)\Delta t = \frac{k_S^n}{(x^2+v^2t^2)^{n/2}} \exp\{-\int_{-\infty}^{t}\left(\frac{k_S^n}{(x^2+v^2z^2)^{n/2}} + \frac{k_T^m}{(x^2+v^2z^2)^{m/2}}\right)dz\}\Delta t, \quad (27)$$

where the origin of time is taken at the CPA time and time before CPA is defined as negative. Integrating $p_S(x, t)$ with respect to t over $(-\infty, \infty)$, the forestalling lateral range curve is obtained.

$$PL_S(x) = \int_{-\infty}^{\infty} p_S(x, t)\,dt. \quad (28)$$

The forestalling sweep width is derived by integrating $PL_S(x)$ with respect to x over $(-\infty, \infty)$.

$$W_S(\alpha) = \int_{-\infty}^{\infty} PL_S(x)\,dx. \quad (29)$$

Here, we consider a search-and-search situation in which when the target forestalls the searcher in detection, the target immediately reacts in either of two ways, as a friend or a foe. Then, the parameter k_S of the searcher's detection device is assumed to change to βk_S during the rest of the encounter. If the target is a friend, such as a victim in the ocean waiting for rescue, the detectability of the searcher's detection device is improved by the target's reaction, and then $\beta > 1$. On the other hand, if it is a hostile target, the detectability of the searcher is decreased by the target's alertness, then $0 \leq \beta \leq 1$. In this situation, the searcher can detect the target before the forestalling by the target, but another way of detection is also possible, in which the target forestalls the searcher in detection and get into alert status and then the searcher detects the target during the rest of the encounter. The detection probability $PL_S(x)$ of the former is given by Eq.(28) and the detection probability $PL_{TS}(x)$ of the latter is obtained by

$$PL_{TS}(x) = \int_{-\infty}^{\infty} p_T(x, t)\{1-\exp(-\int_{t}^{\infty}\frac{(\beta k_S)^n}{(x^2+v^2z^2)^{n/2}}\,dz)\}dt, \quad (30)$$

where $p_T(x, t)$ is given by interchanging (k_S, n) and (k_T, m) in Eq.(27). (The more general expressions of $PL_S(x)$ and $PL_{TS}(x)$ are discussed by Iida [1].) Therefore, the generalized lateral range curve $PL(x)$ of the searcher is derived by

$$PL(x) = PL_S(x) + PL_{TS}(x). \quad (31)$$

The generalized sweep width taking account of the forestalling detection by the target is obtained by integrating $PL(x)$ with respect to x over $(-\infty, \infty)$.

$$W(\alpha, \beta) = \int_{-\infty}^{\infty} PL(x)\,dx. \quad (32)$$

Unfortunately, we cannot obtain the integrations of Eqs.(28),(30) and (32) in closed form. Hence, we must culculate them numerically. Here if the detection devices of the searcher and the target are same kind sensors such as visual, usually the shape parameters n and m of the detection law have almost same value. If we assume $n = m$, then we can derive the explicit formulas. Specifically, setting $n = m$ in Eqs.(27) and (28) and calculating the integral, we obtain

$$PL_S(x) = \int_{-\infty}^{\infty} p_S(x, t)\,dt = \frac{k_S^n}{k_S^n + k_T^n}[1-\exp(-\frac{(k_S^n+k_T^n)\sqrt{\pi}\,\Gamma((n-1)/2)}{v|x|^{n-1}\Gamma(n/2)})].$$

Rewriting k_i by W_i given by Eq.(8) (the sweep width in one-sided search) respectively, and setting $\alpha = W_T/W_S$, we obtain the simplified expression of the above.

$$PL_S(x) = \frac{1}{\alpha^{n-1}+1}\,[1-\exp\{-(\alpha^{n-1}+1)(\frac{W_S}{2\Gamma((n-2)/(n-1))|x|})^{n-1}\}]. \quad (33)$$

Substituting $PL_S(x)$ into Eq.(29), we have

$$W_S(\alpha) = W_S(\frac{k_S^n}{k_S^n+k_T^n})^{(n-2)/(n-1)} = W_S(\frac{1}{\alpha^{n-1}+1})^{(n-2)/(n-1)}. \quad (34)$$

If we substitute $n=3$ in Eqs.(33) and (34), we obtain the formulas of $PL_S(x)$ and $W_S(\alpha)$ of the inverse cube law given by Koopman [2].

Calculating Eq.(30) by setting $n=m$ and rewriting k_i by W_i and $\alpha=W_T/W_S$, we obtain

$$PL_{TS}(x)=\frac{\alpha^{n-1}}{\alpha^{n-1}+1}\left[1+\frac{\beta^{n}}{\alpha^{n-1}-\beta^{n}+1}\exp\left(-(\alpha^{n-1}+1)\left(\frac{W_S}{2\Gamma((n-2)/(n-1))|x|}\right)^{n-1}\right)-\frac{\alpha^{n-1}+1}{\alpha^{n-1}-\beta^{n}+1}\exp\left(-\beta^{n}\left(\frac{W_S}{2\Gamma((n-2)/(n-1))|x|}\right)^{n-1}\right)\right] \quad \text{if } \alpha^{n-1}-\beta^{n}+1\neq 0, \tag{35}$$

$$PL_{TS}(x)=\frac{\alpha^{n-1}}{\alpha^{n-1}+1}\left[1-\exp\left(-(\alpha^{n-1}+1)\left(\frac{W_S}{2\Gamma((n-2)/(n-1))|x|}\right)^{n-1}\right)\right]-\left(\frac{\alpha W_S}{2\Gamma((n-2)/(n-1))|x|}\right)^{n-1}\exp\left(-(\alpha^{n-1}+1)\left(\frac{W_S}{2\Gamma((n-2)/(n-1))|x|}\right)^{n-1}\right) \quad \text{if } \alpha^{n-1}-\beta^{n}+1=0.$$

Substituting Eqs.(33) and (35) into Eq.(31), we obtain

$$PL(x)=1-\frac{1-\beta^{n}}{\alpha^{n-1}-\beta^{n}+1}\exp\left(-(\alpha^{n-1}+1)\left(\frac{W_S}{2\Gamma((n-2)/(n-1))|x|}\right)^{n-1}\right)-\frac{\alpha^{n-1}}{\alpha^{n-1}-\beta^{n}+1}\exp\left(-\beta^{n}\left(\frac{W_S}{2\Gamma((n-2)/(n-1))|x|}\right)^{n-1}\right) \quad \text{if } \alpha^{n-1}-\beta^{n}+1\neq 0, \tag{36}$$

$$PL(x)=1-\left(1+\left(\frac{\alpha W_S}{2\Gamma((n-2)/(n-1))|x|}\right)^{n-1}\right)\exp\left(-(\alpha^{n-1}+1)\left(\frac{W_S}{2\Gamma((n-2)/(n-1))|x|}\right)^{n-1}\right) \quad \text{if } \alpha^{n-1}-\beta^{n}+1=0.$$

Fig.6 shows the sensitivity analysis of the parameters n, α and β for $PL(x)$ given by Eq.(36). As seen in this figure, the curves of $PL(x)$ have a shape such as a two-step function for $n=8$ and 12. We shall discuss this distinctive feature of $PL(x)$ curves later in Section 6.

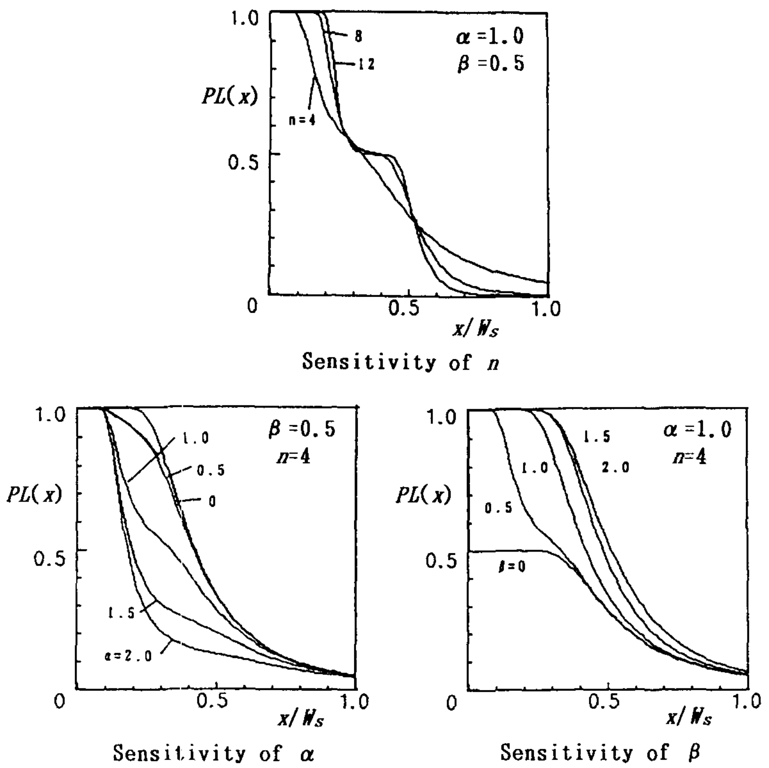


Fig.6 Sensitivity analysis of $PL(x)$

Substituting Eq.(36) into Eq.(32), we have

$$W(\alpha,\beta)=\frac{(1-\beta)(\alpha^{n-1}+1)^{1/(n-1)}+\alpha^{n-1}\beta^{n/(n-1)}}{\alpha^{n-1}-\beta^{n}+1}\,W_S \quad \text{if } \alpha^{n-1}-\beta^{n}+1\neq 0, \tag{37}$$
$$=\frac{(n-2)\alpha^{n-1}+(n-1)}{(n-1)(\alpha^{n-1}+1)^{(n-2)/(n-1)}}\,W_S \quad \text{if } \alpha^{n-1}-\beta^{n}+1=0.$$

To see the sensitivity of α and β in $W(\alpha,\beta)$, curves of $W(\alpha,\beta)$ normalized by W_S are shown in Fig.7 for n =4 and 12.

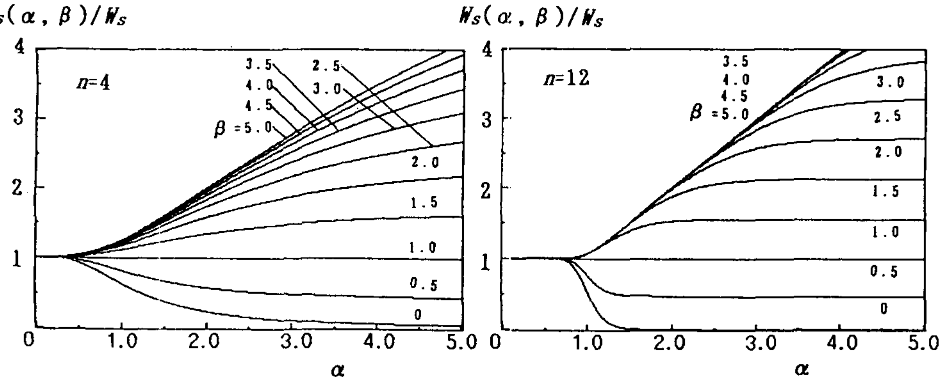


Fig.7 Sensitivity analysis of $W(\alpha,\beta)$

Here, we consider special cases of α and β. If we set α=0 in Eqs.(36) and (37), we obtain Eqs.(7) and (8), respectively. Since α=0 means the blind target, the search situation becomes the one-sided search. Hence this result is natural. Eqs.(7) and (8) are also obtained by setting β=1 in Eqs.(36) and (37). β=1 implies that the target's forestalling detection is ineffective to change the searcher's detection capability. Hence, this case also becomes the one-sided search. Next, we examine the case β=0. Setting β=0 in Eqs.(36) and (37), we obtain Eqs.(33) and (34), respectively. This results are not surprising since β=0 implies the searcher's detection device becomes ineffective by the target's forestalling detection and the searcher's detection is restricted to his detection before the forestalling by the target.

From the above, we can say that Eqs.(36) and (37) are the generalized formulas including Eqs.(7),(33) and Eqs.(8),(34), respectively, as the special cases of the parameters.

6. Discussions

In this section, several discussions are presented concerning the results obtained in the previous sections.

(1) As seen in Fig.6, the curves of $PL(x)$ have a shape as a two-step function for n =8 and 12. To explain this figure, we examine the components of $PL(x)$; $PL_S(x)$ and $PL_{TS}(x)$ for n =12, α=1.0, β=0.5. These are shown in Fig.8. We can elucidate the distinctive feature of $PL(x)$ by this figure. As seen in Fig.8, the curves of $PL_S(x)$ and $PL_{TS}(x)$ decrease from $1/(\alpha^{n-1}+1)$ and $\alpha^{n-1}/(\alpha^{n-1}+1)$, respectively, to zero with a steep slope if n is large, and the range x/W_S of $PL_{TS}(x)$ is reduced considerably compare with $PL_S(x)$. As shown in Eq.(31), the curve of $PL(x)$ is the summation of these two curves: $PL_S(x)$ and $PL_{TS}(x)$, and therefore, the curve of $PL(x)$ exhibits a shape as a step function.

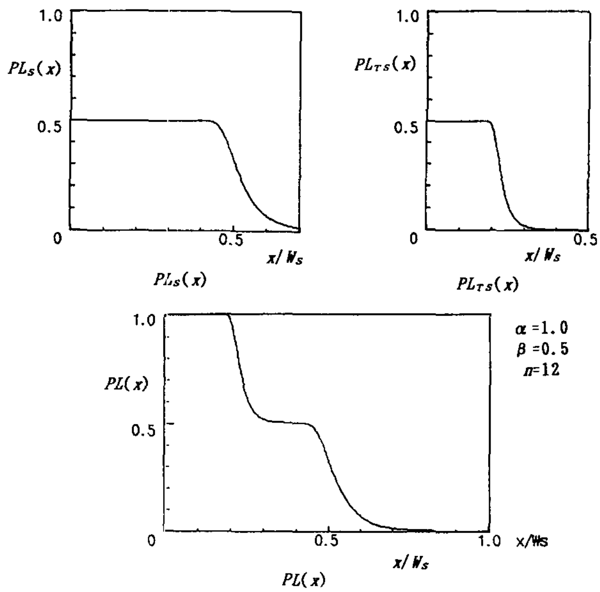


Fig. 8. $PL_S(x)$, $PL_{TS}(x)$ and $PL(x)$

(2) Washburn discussed in Chapter 4 of his book: *Search and Detection* [4] that the generalized lateral range curve given by Eq. (1) could be measured experimentally by generating a sequence of straight line encounters with the target for each of which "detection" or "non-detection" are recorded. Then, the ML method will be used to estimate the parameters x_0 and b in Eq. (1). However, this method may be impractical in real world applications since it needs too many data to obtain $PL(x)$ curve experimentally at many x's. The remarkable thing of the method described in Section 3 of this report is that we can estimate the parameters k and n of the detection law without considering any lateral ranges other than x =0 and can derive the lateral range curve by caluculation from the estimated (k, n).

(3) In Section 4, we analyze the set of actual data applying the inverse nth power law. As shown in Table 2, the MM method gives a smaller MSD than the ML method in this case. However, this conclusion is not always valid. In other cases, the ML method is better than the MM method. Therefore, the method of the best estimator must be investigated in future.

(4) In some actual detection device, sometimes the detection at range zero is not certain. In this case, the inverse nth power law discussed here cannot be applied to formulate the detection capability. Furthermore, the detection capability of some detection device may depend on the aspect of the target in addition to the distance. In these cases, we must investigate another model of detection law and it is an important problem to be studied in future.

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