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COVARIANCE STRUCTURE OF INTERRUPTED MARKOV MODULATED POISSON PROCESS

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Abstract We consider the covariance structure of an interrupted Markov modulated Poisson process. In this process, periods of on-time and off-time alternate; the on-time interval has a phase-type distribution and the off-time interval has a general one. The off-time period represents the time during which no customers arrive. The on-time period on the other hand, represents the time during which customers do arrive. Here, the arrival rate depends on the phase condition of the on-time interval distribution. We derive the Laplace-Stieltjes transform for the inter-arrival time distribution. Using the results, we study the correlation structures of succeeding inter-arrival times. When the on-time length distribution is hyperexponential, the covariance of succeeding inter-arrival lengths is positive, whereas becomes negative for Erlangian on-time lengths.

1. Introduction

An interrupted Poisson process (IPP) has been proposed as an approximate overflow process from the loss system $M/M/S/S$ by Kuczura [4]. The inter-arrival time distribution of overflows from the $M/M/S/S$ has a hyperexponential form with order $S+1$. As the number of servers becomes larger, the representation becomes more complicated. On the other hand, the inter-arrival time distribution of interrupted Poisson arrivals has a hyperexponential form with order 2 and can easily be dealt with. Because of its simplicity, the interrupted Poisson process has been frequently used to analyze models with overflow inputs. The interrupted Poisson process has sufficient accuracy when its peaked characteristics are not so notable. As the peaked characteristics do become notable, however, the approximation errors become larger. Furthermore, the assumption for the interrupted Poisson process that both on-time and off-time interval distributions are exponential restricts the number of applications. To overcome this restriction, Machihara has proposed a modified interrupted Poisson process in which the off-time period length distribution is hyperexponential [5,6]. Tran-Gia has proposed a more general IPP called the "generalized interrupted Poisson process ($GIPP$)" in which on-time and off-time interval distributions are general and he derived the Laplace-Stieltjes transform (LST) of the forward recurrence time of the inter-arrival time [9]. From this result, he considers the condition under which the $GIPP$ is renewal. When the on-time interval distribution is exponential, the $GIPP$ becomes renewal. Kingman [3] derived the necessary and sufficient condition that a renewal process can be expressed as a doubly stochastic Poisson process. Neuts et al. [8] derived the condition for a Markov modulated Poisson process by a different approach from Kingman's. Tran-Gia's renewality condition can easily be derived by Kingman's result or Neuts' result.

This paper considers an interrupted Markov modulated Poisson process ($IMMPP$) which is a more generalized process of the $GIPP$. In this process, the on-time interval has a phase-type distribution with states $\{1,2,\cdots,m\}$ and Poisson customers arrive at a rate λ_i, when the phase state is i. Note that if the off-time interval also had phase-type characteristics, it would be possible to deal with the arrival process as a Markov modulated Poisson one. We derive the Laplace-Stieltjes transform for the inter-arrival time distribution of this process. Using the results, we study the correlation structures of succeeding inter-arrival times. When the on-time length distribution is hyperexponential, the covariance of succeeding inter-arrival lengths is positive, whereas it becomes negative for Erlangian on-time lengths.

2. Interrupted Markov Modulated Poisson Process

We consider an interrupted Markov modulated Poisson process ($IMMPP$), which is a more generalized process of the interrupted Poisson process [4] or Markov modulated Poisson process [2].

(1) Periods of on-time and off-time alternate; the on-time interval has a phase-type distribution [7] with representation (α, T, T^0) and the off-time interval has a general distribution $G^*(x)$ and its LST is given by $g(s)$. Furthermore, the on-time period length and the off-time period length are independent of each other.

(2) In the off-time period, no customers arrive. In the on-time period, on the other hand Poisson customers do arrive. Here, the arrival rate depends on the phase condition of the on-time period length distribution. We assume that the arrival rate for the phase state i is $\lambda_i(>0)$. When $\lambda_{i_0}=0$ for some phase state i_0, this phase state may be considered to be some sub-period of the off-time period.

If the off-time period length has a phase-type distribution, $IMMPP$ is identical to a Markov modulated Poisson process. If we assume that λ_i is constant for all i, we obtain a generalized interrupted Poisson process [9].

Of interest is the inter-arrival time distribution for $IMMPP$. $IMMPP$ is a Markov renewal process, [1] i.e., the inter-arrival time depends on the phase states of the on-time period length distribution. The inter-arrival time distribution $A^*(x)$ has the following $m\times m$ matrix structure in the number of phase states, m. That is,

$$A^*(x)=(A^*_{ij}(x))_{1\le i,j\le m},$$

where $A^*_{ij}(x)=P\{X\le x, Ph(X)=j \mid Ph(0)=i\}$.

The X is the inter-arrival time and $Ph(x)$ is a phase state of the on-time period length distribution. Of course, the following is satisfied.

$$\sum_{j=1}^{m} A^*_{ij}(\infty)=1,$$

for any $i=1,2,\cdots,m$.

Let $\delta_{ij}=1$ if $i=j$ and $\delta_{ij}=0$ if $i\neq j$.

Theorem 2.1 The Laplace-Stieltjes transform (LST) $A(s)$ of $A^*(x)$ is given by

$$\begin{aligned}A(s)&=\{I-(sI+\Lambda-T)^{-1}T^0g(s)\alpha\}^{-1}(sI+\Lambda-T)^{-1}\Lambda\\&=\{sI+\Lambda-T-T^0g(s)\alpha\}^{-1}\Lambda,\end{aligned}\tag{2.1}$$

where

$$\Lambda=(\delta_{ij}\lambda_i), 1\le i,j\le m.$$

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Proof

Assume that a Poisson call arrives at time 0. There are two possibilities for how the next customer arrives after time 0: either there is an arrival during the on-time period which continues from time 0 (Case 1) or there are no arrivals during this period (Case 2).
Case 1 The probability that a customer arrives in $(x, x+dx)$ is given by

$$\exp\{(-\Lambda + T)x\}\Lambda dx$$

Case 2 The probability that the first on-time period and succeeding off-time period end at u and v, respectively, and after that a customer arrives in $(x, x+dx)$ is given by

$$\exp\{(-\Lambda + T)u\}T^0 \frac{dG^*(v-u)}{d(v-u)}\alpha \frac{dA^*(x-v)}{d(x-v)}dx.$$

Now, we obtain

$$\begin{aligned} dA^*(x) &= \exp\{(-\Lambda + T)x\}\Lambda dx \\ &\quad + \{\int_0^x \int_0^v \exp\{(-\Lambda + T)u\}T^0 \frac{dG^*(v-u)}{d(v-u)}\alpha \frac{dA^*(x-v)}{d(x-v)}dudv\}dx. \end{aligned}$$

Taking the Laplace-Stieltjes transform, we obtain (2.1). □

Remark 2.2 When $m = 1$, i.e., the on-time period distribution is exponential, we obtain

$$\begin{aligned} A(s) &= \{1 - (s + \lambda_1 - T)^{-1}T^0 g(s)\alpha\}^{-1}(s + \lambda_1 - T)^{-1}\lambda_1 \\ &= \{s + \lambda_1 - T - T^0 g(s)\alpha\}^{-1}\lambda_1, \end{aligned}$$

and the process becomes a renewal one.

3. Correlation structure of inter-arrival times

Let $\tau_0(=0), \tau_1, \tau_2, \cdots$ denote the arrival epoches of the succeeding interrupted Markov modulated Poisson customers. The covariance of $\tau_i - \tau_{i-1}$ and $\tau_{i+j} - \tau_{i+j-1}$ is defined as

$$Cov(j) = E\{(\tau_i - \tau_{i-1})(\tau_{i+j} - \tau_{i+j-1})\} - E\{\tau_i - \tau_{i-1}\}E\{\tau_{i+j} - \tau_{i+j-1}\}.$$

For sufficiently large i, we obtain

$$\begin{aligned} Cov(j) &= PA'(0)A^{j-1}(0)A'(0)e - (PA'(0)e)(PA^{j-1}(0)A'(0)e) \\ &= PA'(0)A^{j-1}(0)A'(0)e - (PA'(0)e)^2, \end{aligned} \tag{3.1}$$

where $e = {}^t(1, 1, \cdots, 1)$,
$A'(0) = dA(s)/ds\,|_{s=0}$,
and P is an invariant probability vector of $A(0)$.

The invariant vector P satisfies the equations

$$P\Lambda^{-1}(T + T^0\alpha) = (0, 0, \cdots, 0) \tag{3.2}$$

and

$$P\Lambda^{-1}T^0\alpha(-T)^{-1} = P\Lambda^{-1}. \tag{3.3}$$

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Proof From the definition of P, $P(\Lambda - T - T^0\alpha)^{-1}\Lambda = P$. Post-multiplying by $\Lambda^{-1}(\Lambda - T - T^0\alpha)$, we get (3.2).

Post-multiplying by $(-T)^{-1}$ in (3.2), we get (3.3). □

Now, we prepare the following equations in order to derive Theorem 3.1.

$$\frac{\alpha}{\alpha(-T)^{-1}e} = \frac{P\Lambda^{-1}(-T)}{P\Lambda^{-1}e} \tag{3.4}$$

and

$$\frac{P(\Lambda - T)^{-1}(-T)}{P\Lambda^{-1}e} = \frac{\alpha(\Lambda - T)^{-1}\Lambda}{\alpha(-T)^{-1}e}. \tag{3.5}$$

Proof From (3.3), we obtain

$$\begin{aligned} P\Lambda^{-1}T + P\Lambda^{-1}T^0\alpha &= P\Lambda^{-1}T + \frac{P\Lambda^{-1}e}{\alpha(-T)^{-1}e}\alpha \\ &= (0, 0, \cdots, 0). \end{aligned}$$

From (3.4), we obtain

$$\begin{aligned} \frac{\alpha(\Lambda - T)^{-1}\Lambda}{\alpha(-T)^{-1}e} &= \frac{P\Lambda^{-1}(-T)(\Lambda - T)^{-1}\Lambda}{P\Lambda^{-1}e} \\ &= \frac{P\Lambda^{-1}(-T)(I - \Lambda^{-1}T)^{-1}}{P\Lambda^{-1}e} \\ &= \frac{P(I - \Lambda^{-1}T)^{-1}\Lambda^{-1}(-T)}{P\Lambda^{-1}e} \\ &\text{(from the commutativity of } \Lambda^{-1}T \text{ and } (I - \Lambda^{-1}T)^{-1}) \\ &= \frac{P(\Lambda - T)^{-1}(-T)}{P\Lambda^{-1}e}. \quad \square \end{aligned}$$

Theorem 3.1 Covariances of inter-arrival times are given by

$$\begin{aligned} Cov(j) =& (P\Lambda^{-1} - x'(0)\alpha)(\Lambda - T)^{-1}\Lambda A^{j-1}(0)(\Lambda - T)^{-1}(I + y'(0)T)e \\ & - \{P(\Lambda - T)^{-1}(I + y'(0)T)e\}^2, \end{aligned} \tag{3.6}$$

where

$$x(s) = (1 - g(s)\alpha(sI + \Lambda - T)^{-1}T^0)^{-1}g(s)P(sI + \Lambda - T)^{-1}T^0$$

and

$$y(s) = (1 - g(s)\alpha(sI + \Lambda - T)^{-1}T^0)^{-1}g(s)\alpha(sI + \Lambda - T)^{-1}\Lambda e.$$

Here, $y(s)$ is the LST of the time length distribution from the beginning epoch of the off-time period to the arrival epoch of the first interrupted modulated Poisson customer.

Proof

Equation (2.1) gives

$$\begin{aligned} A(s) =& (sI + \Lambda - T)^{-1}\Lambda \\ &+ (1 - g(s)\alpha(sI + \Lambda - T)^{-1}T^0)^{-1}(sI + \Lambda - T)^{-1}T^0 g(s)\alpha(sI + \Lambda - T)^{-1}\Lambda. \end{aligned} \tag{3.7}$$

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By first multiplying (3.7) by P, which is an invariant probability vector of $A(0)$ we obtain

$$PA(s) = P(sI + \Lambda - T)^{-1}\Lambda + x(s)\alpha(sI + \Lambda - T)^{-1}\Lambda. \quad (3.8)$$

Then, by multiplying (3.7) by $e = {}^t(1, 1, \cdots, 1)$, we get

$$A(s)e = (sI + \Lambda - T)^{-1}\Lambda e + y(s)(sI + \Lambda - T)^{-1}T^0. \quad (3.9)$$

Now, we have

$$\begin{aligned} x(0) &= (1 - \alpha(\Lambda - T)^{-1}T^0)^{-1}P(\Lambda - T)^{-1}T^0 \\ &= \frac{P(\Lambda - T)^{-1}T^0}{\alpha(\Lambda - T)^{-1}\Lambda e} \qquad (\text{From } \alpha(\Lambda - T)^{-1}T^0 + \alpha(\Lambda - T)^{-1}\Lambda e = 1) \\ &= \frac{P\Lambda^{-1}e}{\alpha(-T)^{-1}e} \qquad (\text{From } (3.5)) \\ &= P\Lambda^{-1}T^0. \qquad (\text{From } (3.3)) \end{aligned}$$

This gives

$$\begin{aligned} -PA'(0) &= (P + x(0)\alpha)(\Lambda - T)^{-2}\Lambda - x'(0)\alpha(\Lambda - T)^{-1}\Lambda \\ &= (P + P\Lambda^{-1}T^0\alpha)(\Lambda - T)^{-2}\Lambda - x'(0)\alpha(\Lambda - T)^{-1}\Lambda \\ &= (P + P\Lambda^{-1}(-T))(\Lambda - T)^{-2}\Lambda - x'(0)\alpha(\Lambda - T)^{-1}\Lambda \ (\text{From } (3.3)) \\ &= (P\Lambda^{-1} - x'(0)\alpha)(\Lambda - T)^{-1}\Lambda. \end{aligned}$$

Since $y(s)$ is the LST of the distribution, $y(0) = 1$.
Therefore, (3.9) gives

$$\begin{aligned} -A'(0)e &= (\Lambda - T)^{-2}(\Lambda e + T^0) - y'(0)(\Lambda - T)^{-1}T^0 \\ &= (\Lambda - T)^{-1}(I + y'(0)T)e. \end{aligned}$$

Thus, we obtain Theorem 3.1 from (3.1). □

When T and Λ are commutative, (3.6) can be transformed into a simpler form. We now prepare the following equations to obtain the simpler form in Theorem 3.2. If $T\Lambda = \Lambda T$, we obtain

$$x(s) = \frac{P\Lambda^{-1}e}{\alpha(-T)^{-1}e}y(s) \quad (3.10)$$

and

$$-PA'(0) = P(I + y'(0)T)(\Lambda - T)^{-1}. \quad (3.11)$$

Proof Since

$$\begin{aligned} &\alpha(sI + \Lambda - T)^{-1}\Lambda \\ &= \frac{\alpha(-T)^{-1}e}{P\Lambda^{-1}e}P\Lambda^{-1}(-T)(sI + \Lambda - T)^{-1}\Lambda \qquad (\text{From } (3.4)) \\ &= \frac{\alpha(-T)^{-1}e}{P\Lambda^{-1}e}P(sI + \Lambda - T)^{-1}(-T), \qquad (\text{From } T\Lambda = \Lambda T) \end{aligned}$$

we get (3.10) from the definitions of $x(s)$ and $y(s)$.

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Equation (3.11) can be obtained from

$$\begin{aligned}-PA'(0) &= (P\Lambda^{-1} - x'(0)\alpha)(\Lambda - T)^{-1}\Lambda \\ &= \{P\Lambda^{-1} - y'(0)\frac{P\Lambda^{-1}e}{\alpha(-T)^{-1}e}\alpha\}(\Lambda - T)^{-1}\Lambda \qquad \text{(From (3.10))} \\ &= \{P\Lambda^{-1} - y'(0)P\Lambda^{-1}(-T)\}(\Lambda - T)^{-1}\Lambda \qquad \text{(From (3.4))} \\ &= P\Lambda^{-1}(I + y'(0)T)(\Lambda - T)^{-1}\Lambda.\end{aligned}$$

Substituting (3.10) and (3.11) into (3.6) derives following theorem.

Theorem 3.2 If $T\Lambda = \Lambda T$, we obtain

$$\begin{aligned}Cov(j) =& P(I + y'(0)T)(\Lambda - T)^{-1}A^{j-1}(0)(I + y'(0)T)(\Lambda - T)^{-1}e \\ & - \{P(I + y'(0)T)(\Lambda - T)^{-1}e\}^2.\end{aligned} \tag{3.12}$$

In particular,

$$Cov(1) = P\{(I + y'(0)T)(\Lambda - T)^{-1}\}^2 e - \{P(I + y'(0)T)(\Lambda - T)^{-1}e\}^2.$$

Now, we consider two typical examples in which the on-period length distribution is either hyperexponential or Erlangian.

Corollary 3.3 When the on-period length distribution is hyperexponential, the following is satisfied.

$$Cov(1) \geq 0 \text{ and } Cov(1) \geq Cov(2).$$

$Cov(1) = 0$ is satisfied only if $\lambda_i = \lambda$ for any i, and r_i are constant for any i; that is, the on-time period length distribution is exponential or $y'(0) = -\lambda^{-1}$; that is, the off-time period length is zero.

Proof

In this case, we can write

$$T = (-\delta_{ij}r_j) \qquad 1 \leq i,j \leq m, \qquad T^0 = {}^t(r_1, r_2, \cdots, r_m),$$

and

$$\alpha = (k_1, k_2, \cdots, k_m).$$

Since $T\Lambda = \Lambda T$, from Theorem 3.2 we obtain

$$\begin{aligned}Cov(1) &= \sum_{i=1}^{m} p_i\Big(\frac{1 - y'(0)r_i}{\lambda_i + r_i}\Big)^2 - \Big(\sum_{i=1}^{m} p_i\frac{1 - y'(0)r_i}{\lambda_i + r_i}\Big)^2 \\ &= \sum_{j=i+1}^{m}\sum_{i=1}^{m-1} p_i p_j\Big(\frac{1 - y'(0)r_i}{\lambda_i + r_i} - \frac{1 - y'(0)r_j}{\lambda_j + r_j}\Big)^2 \\ &= \sum_{j=i+1}^{m}\sum_{i=1}^{m-1} p_i p_j\frac{\{\lambda_i + r_i - \lambda_j - r_j + y'(0)(r_i\lambda_j - r_j\lambda_i)\}^2}{(\lambda_i + r_i)^2(\lambda_j + r_j)^2} \geq 0.\end{aligned} \tag{3.13}$$

Now, let us consider $Cov(1) - Cov(2)$.

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Since

$$A(0) = (\Lambda - T)^{-1}\Lambda + (1 - \alpha(\Lambda - T)^{-1}T^0)^{-1}(\Lambda - T)^{-1}T^0\alpha(\Lambda - T)^{-1}\Lambda$$
$$= (\Lambda - T)^{-1}\Lambda + \frac{(\Lambda - T)^{-1}T^0\alpha(\Lambda - T)^{-1}\Lambda}{\alpha(\Lambda - T)^{-1}\Lambda e},$$

we obtain

$$Cov(1) - Cov(2) = P(I + y'(0)T)(\Lambda - T)^{-1}(I - A(0))(I + y'(0)T)(\Lambda - T)^{-1}e$$
$$= P(I + y'(0)T)(\Lambda - T)^{-2}\left(- T - \frac{T^0\alpha(\Lambda - T)^{-1}\Lambda}{\alpha(\Lambda - T)^{-1}\Lambda e}\right)(I + y'(0)T)(\Lambda - T)^{-1}e$$
$$= P(I + y'(0)T)(\Lambda - T)^{-2}\left(- T - \frac{T^0P(\Lambda - T)^{-1}(-T)}{P(\Lambda - T)^{-1}T^0}\right)(I + y'(0)T)(\Lambda - T)^{-1}e.$$

(from (3.5))

For any non-negative p_i and q_i which satisfy

$$\sum_{i=1}^{m} p_i = 1 \ and \ \sum_{i=1}^{m} q_i = 1$$

and any non-negative x_i and $y_i (i = 1, 2, \cdots, m)$, we have

$$\sum_{i=1}^{m} p_i y_i x_i - \sum_{i=1}^{m} p_i y_i \sum_{i=1}^{m} q_i x_i = \sum_{j=i+1}^{m} \sum_{i=1}^{m-1} (x_i - x_j)(p_i q_j y_i - p_j q_i y_j). \tag{3.14}$$

Substituting

$$(p_1, p_2, \cdots, p_m) = P, \ x_i = \frac{1 - y'(0) r_i}{\lambda_i + r_i}, \ y_i = \frac{r_i x_i}{\lambda_i + r_i}$$

and

$$q_i = \frac{p_i r_i}{\lambda_i + r_i} \Big/ \sum_{j=1}^{m} \frac{p_j r_j}{\lambda_j + r_j}$$

into (3.14), we obtain

$$Cov(1) - Cov(2) = \sum_{j=i+1}^{m} \sum_{i=1}^{m-1} \frac{p_i r_i}{\lambda_i + r_i} \frac{p_j r_j}{\lambda_j + r_j} (x_i - x_j)^2 \Big/ \sum_{l=1}^{m} \frac{p_l r_l}{\lambda_l + r_l} \geq 0. \quad \square$$

Corollary 3.4 When the on-period length distribution is m-Erlangian and $\Lambda = \lambda I$, we obtain $Cov(1) \leq 0$. $Cov(1) = 0$ only if $m = 1$; that is, the on-time period length distribution is exponential or $y'(0) = -\lambda^{-1}$; that is, the off-time period length is zero.

Proof

In this case, we can write

$$T = \begin{pmatrix} -m\mu & m\mu & \cdot & \cdot & \cdot \\ \cdot & -m\mu & m\mu & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -m\mu & m\mu \\ \cdot & \cdot & \cdot & \cdot & -m\mu \end{pmatrix},$$

$$T^0 = {}^t(0, 0, \cdots\cdots, m\mu), \qquad \alpha = (1, 0, \cdots\cdots, 0).$$

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Since

$$P = m^{-1}(1, 1, \cdots, 1),$$

it follows from Theorem 3.2 and

$$(I + y'(0)T)(\Lambda - T)^{-1} = \Lambda^{-1}\{I + (I + y'(0)\Lambda)T(\Lambda - T)^{-1}\}$$

that

$$Cov(1) = \lambda^{-2}\{1 + \lambda y'(0)\}^2\left\{P(\lambda I - T)^{-1}(-T)(\lambda I - T)^{-1}T^0 - (P(\lambda I - T)^{-1}T^0)^2\right\}$$

$$= \frac{\mu^2(1 + \lambda y'(0))^2\left\{m^2 a^{m-1} - \left(\frac{1 - a^m}{1 - a}\right)^2\right\}}{\lambda^2(\lambda + m\mu)^2} \leq 0, \qquad (3.15)$$

where

$$a = \frac{m\mu}{\lambda + m\mu}.$$

In particular, when $m = \infty$, that is, the on-time period is constant,

$$Cov(1) = -\frac{\mu^2(1 + \lambda y'(0))^2}{\lambda^4} \quad \square$$

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