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BASIC THEORY OF SELECTION BY RELATIVE RANK WITH COST

Sigeiti Moriguti Professor Emeritus, University of Tokyo

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Abstract Suppose there are n objects in a row and we want to choose as good an object as possible. We are allowed to observe one by one starting at an end. We may stop at any point and take the object there, but going back is not allowed. The stopping rule is based on the relative rank of the object among those observed so far. The cost of observation, k, is considered to be either zero or positive. This paper presents a basic theory on this problem as well as a set of algorithms which give the optimum stopping rule for given n and k, and the characteristics of the rule.

1. Introduction.

Suppose there are *n* objects and they are ranked from 1(the best) to n(the worst) without any tie. They are arranged in a row. We are allowed to observe them one by one starting at an end. Let the (absolute) rank of the *i*-th object be x_i . Then

 $(1.1) x_1, x_2, \cdots, x_n$

is assumed to be a random permutation of $1, 2, \dots, n$. When we have observed first *i* objects, the relative rank of the *i*-th object is:

(1.2) $y_i = (\text{number of } x_1, \cdots, x_{i-1} \text{ less than } x_i) + 1.$

Now, we consider the stopping rules of the following form:

Rule The stopping criteria s_1, s_2, \dots, s_n are pre-determined. For $i = 1, 2, \dots$, right after observing the first *i* objects, if the relative rank y_i of *i*-th object is less than or equal to s_i , then we stop there and select the *i*-th object. Otherwise we continue the observation.

It is to be noted that the choice of the i-th object is possible only when we have just completed the observation of the first i objects. In other words, if we go on without deciding at that time, it is not allowed to come back later and take the i-th object.

About the stopping criteria, the condition that

$$(1.3) 0 = s_1 \le s_2 \le \dots \le s_n = n$$

would not make any harm. Namely, our criteria are such that the condition gets milder as we go, and if we happen to come to the last object, we are to take the last one in any case.

Possible practical applications would include:

(a) Bathing beauties — There are n beauties in a row at a beach. One walks along, and offers a date to someone he hopes to be of a high rank.

- (b) Marriage There are n candidates for a bride. One interviews one by one, and marries one hopefully with a high rank. Ferguson(1989) describes, in Section 5, an interesting story of Johannes Kepler(1571-1630) in this relation.
- (c) **Development of a new product** There are n proposals of a new product. The company makes the detailed productivity and marketability study one by one, and selects the one hopefully with a high rank.
- (d) Infrastructure projects There are n proposals of infrastructure development projects. The agency makes the feasibility study and cost/benefit analysis one by one, and decides on one hopefully with a high rank.

This type of problems are known also as the "secretary problem". Many papers on this subject in its various modifications are listed in review articles Freeman(1983) and Ferguson(1989). Some papers not mentioned therein are included in the list of references of this paper.

Here, we develop a basic theory in the case of non-negative cost. The probability distribution of the number of observations until stopping, and the probability distribution of the absolute rank of the selected object are among the major topics considered.

2. Probability distribution of the number of observations.

When observation requires no cost, the number of observations may not mean much. But if some cost is required, then the number of observations until stopping becomes of interest. According to our rule, it is a random variable in general. In this Section, let us discuss its probability distribution.

Given the stopping criteria $s_i (i = 1, 2, \dots, n)$, let us consider the probability distribution of the number of observations *i* until stopping.

Let the absolute ranks of the first *i* items be x_1, x_2, \dots, x_i . If we replace them with the relative ranks among them, then we would have any one of the *i*! permutations. Hence, the relative rank y_i of the *i*-th object among them takes the value from 1 to *i* with equal probability 1/i. Namely,

(2.1)
$$P(y_i = j) = 1/i \quad (j = 1, 2, \dots, i).$$

This probability is independent of the values of y_1, \dots, y_{i-1} .

Now, let us define

(2.2)
$$\omega_i$$
 = the event that stopping does not occur at the *i*-th observation or earlier,

and let its probability be denoted by

(2.3)
$$Q(i) = P(\omega_i).$$

Then the probability that stopping occurs right after the *i*-th observation is given by

(2.4)
$$p(i) = P\{\omega_{i-1}, y_i \leq s_i\} = Q(i-1) \cdot s_i/i.$$

And we get a recurrence formula

(2.5)
$$Q(i) = Q(i-1) \cdot (1-s_i/i).$$

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Starting with Q(0) = 1 and using (2.4) and (2.5) successively, we can obtain all the values of p(i) and Q(i) for $i = 1, 2, \dots, n$. In particular, under the assumption $s_n = n$ (cf. (1.3)), we get Q(n) = 0.

In many cases, s_i takes on the same value s for a set of i values. Under the condition (1.3), the set is an interval I_s . If we define

then the interval I_s can be expressed as

(2.7)
$$I_s = \{i \mid i_s \le i \le i_{s+1} - 1\}.$$

Here, I_s is understood to be empty when $i_s = i_{s+1}$. Now, we can write (2.5) in the form

(2.8)
$$Q(i) = Q(i-1) \cdot (1-s/i) \quad (i \in I_s)$$

and rewrite it as

(2.9)
$$Q(i) \cdot i(i-1) \cdots (i-s+1) = Q(i-1) \cdot (i-1)(i-2) \cdots (i-s) \quad (i \in I_s).$$

Using the notation

(2.10)
$$i^{(s)} = i(i-1)\cdots(i-s+1),$$

we can get from (2.9)

(2.11)
$$Q(i) \cdot i^{(s)} = \text{const} = Q(i_s - 1) \cdot (i_s - 1)^{(s)} \quad (i \in I_s).$$

In particular, applying (2.11) for $i = s_{i+1} - 1$, we have

(2.12)
$$Q(i_{s+1}-1) \cdot (i_{s+1}-1)^{(s)} = Q(i_s-1) \cdot (i_s-1)^{(s)}.$$

For $i \in I_0 = \{1, 2, \dots, i_1-1\}$, stopping never occurs and therefore Q(i) = 1. Thus $Q(i_1-1) = 1$. Starting with this value, and applying (2.12) successively, we obtain

$$Q(i_2 - 1) = Q(i_1 - 1) \cdot (i_1 - 1)/(i_2 - 1) = (i_1 - 1)/(i_2 - 1),$$

$$Q(i_3 - 1) = Q(i_2 - 1) \cdot (i_2 - 1)^{(2)}/(i_3 - 1)^{(2)} = (i_1 - 1)(i_2 - 2)/(i_3 - 1)^{(2)},$$

.....

(2.13)
$$Q(i_s-1) = (i_1-1)(i_2-2)\cdots(i_{s-1}-s+1)/(i_s-1)^{(s-1)}.$$

Hence the last term of (2.11) can now be written as

$$(2.14) \quad Q(i_s-1) \cdot (i_s-1)^{(s)} = (i_1-1)(i_2-2) \cdots (i_{s-1}-s+1)(i_s-s),$$

so that (2.11) becomes

(2.15)
$$Q(i) = (i_1 - 1)(i_2 - 2) \cdots (i_s - s)/i^{(s)} \quad (i \in I_s).$$

Remark. Formula (2.15) can be derived directly by the following considerations (Watanabe(1965)): — Relative ranks of the first *i* objects can appear as any one of the *i*! permutations with uniform probability 1/i!. Among them, those which correspond to the event ω_i have the characteristics that the object with relative rank

1 is in the first $i_1 - 1$ positions, the object with relative rank 2 is in the first $i_2 - 1$ positions except the position occupied by the object of relative rank $1, \dots$, the object with relative rank s is in the first $i_s - 1$ positions except the positions occupied by the superior s - 1 objects. The remaining i - s objects can be in any positions, in any order, among the unoccupied i - s positions. Hence there are

$$(2.16) (i_1-1)(i_2-2)\cdots(i_s-s)\cdot(i-s)!$$

permutations corresponding to the event ω_i . This leads to the formula (2.15) for the probability Q(i) of ω_i .

Now let us consider the expected number of observations.

Formula (2.4) gives the probability of stopping right after the *i*-th observation. Using it, the expected number of observations until stopping is given by

(2.17)
$$e_0 = \sum_{i=1}^n i \cdot p(i).$$

On the other hand, since Q(n) = 0, we have for any *i*

(2.18)
$$Q(i) = p(i+1) + p(i+2) + \dots + p(n),$$

whence

(2.19)

$$e_{0} = (p(1) + p(2) + p(3) + \dots + p(n)) + (p(2) + p(3) + \dots + p(n)) + (p(3) + \dots + p(n)) + \dots + p(n))$$

$$= 1 + \sum_{i=1}^{n-1} Q(i).$$

It can also be expressed in the form

(2.20)
$$e_0 = (2i_1 - 1) + (i_1 - 1) \left(\frac{1}{i_1} + \frac{1}{i_1 + 1} + \dots + \frac{1}{i_2 - 2}\right) \\ - \sum_{s=3}^{n-1} \frac{Q(i_s - 1) \cdot (i_s - 1)}{(s - 1)(s - 2)}.$$

(See Appendix 1 for the detail.)

Remark. Perhaps this expression is new. In "classical secretary problem" where observation cost is zero, the expected absolute rank of the selected object is less than 4 however large n may be (see Chow et al.(1964)). This remarkable achievement is possible, however, by observing at least about n/4 objects and nearly n/2 objects on the average (i.e. i_1 and e_0 is nearly equal to n/4 and n/2, respectively). It is observed, e.g., in Table 1 below.

Next, under the condition ω_i (cf.(2.2)), the expected number of additional observations until stopping is given by

(2.21)
$$e_i = (1 \cdot p(i+1) + 2 \cdot p(i+2) + \dots + (n-i) \cdot p(n))/Q(i) \\ = (Q(i) + Q(i+1) + \dots + Q(n))/Q(i).$$

From this, using (2.5), we can get a recurrence formula

(2.22)
$$e_{i-1} = e_i \cdot Q(i)/Q(i-1) + 1 = (1 - s_i/i) \cdot e_i + 1.$$

So, we can start with $e_n = 0$ and use (2.22) successively for $i = n, n - 1, \dots$. It will be a better procedure for the numerical computation for finite n. (The last value e_0 thus obtained will obviously be the value of (2.17).)

Remark. Formula (2.22) is in line with (3.24) below which is the recurrence formula derived by Lindley (1961) as the backward induction formula in the Bellman's Dynamic Programming framework.

3. Probability distribution of the absolute rank.

In order to obtain the probability distribution of the absolute rank r of the selected object, let us first consider

(3.1) $f(i,r) = P\{\text{stop right after the } i\text{-th observation, absolute rank } r\}.$

The stopping probability p(i) is given by (2.4). Then the relative rank y_i of the *i*-th object is distributed as (2.1). Stopping there occurs if and only if

$$(3.2) 1 \le y_i \le s_i.$$

The condition that stopping did not occur before is

$$(3.3) y_1 > s_1, y_2 > s_2, \cdots, y_{i-1} > s_{i-1}.$$

But this event is independent of the event that $y_i = j$. Hence we can neglect this condition hereafter.

Now let us consider the probability $P(y_i = j | x_i = r)$, i.e. the probability of $y_i = j$ under the condition $x_i = r$. Suppose that the absolute rank x_i of the *i*-th object is r. Then the set $\{x_1, x_2, \dots, x_{i-1}\}$ of the absolute ranks of the preceding i-1 objects are chosen at random from the remaining n-1 values, so that there are

$$(3.4) \qquad \qquad \begin{pmatrix} n-1\\ i-1 \end{pmatrix}$$

possibilities. The event $y_i = j$ implies that in the set $\{x_1, x_2, \dots, x_{i-1}\}$, there are j-1 values superior to $x_i = r$, and the remaining i - j are inferior to r. Such combination occurs in

(3.5)
$$\binom{r-1}{j-1}\binom{n-r}{i-j}$$

ways. Therefore

(3.6)
$$P(y_i = j \mid x_i = r) = {\binom{r-1}{j-1} {\binom{n-r}{i-j}}} / {\binom{n-1}{i-1}}$$

Since the object with absolute rank r can be at any of the n positions, the probability that it happens to be at the position i is 1/n. Hence the probability that $x_i = r$ and $y_i = j$ occurs together is

(3.7)
$$P(x_i = r, y_i = j) = P(x_i = r) \cdot P(y_i = j \mid x_i = r) \\ = \frac{1}{n} \cdot \binom{r-1}{j-1} \binom{n-r}{i-j} / \binom{n-1}{i-1}.$$

On the other hand, the probability of the event $y_i = j$ is given by (2.1). Therefore the probability that $x_i = r$ under the condition $y_i = j$ is given by

(3.8)
$$P(x_{i} = r \mid y_{i} = j) = P(x_{i} = r, y_{i} = j) / P(y_{i} = j)$$
$$= \left[\frac{1}{n} \cdot \binom{r-1}{j-1} \binom{n-r}{i-j} / \binom{n-1}{i-1}\right] / \frac{1}{i}$$
$$= \binom{r-1}{j-1} \binom{n-r}{i-j} / \binom{n}{i}.$$

Probability (3.1) can now be expressed as

(3.9)
$$f(i,r) = \sum_{j=1}^{s_i} P(\omega_{i-1}, y_i = j) \cdot P(x_i = r \mid y_i = j)$$
$$= Q(i-1) \cdot \sum_{j=1}^{s_i} \frac{1}{i} \binom{r-1}{j-1} \binom{n-r}{i-j} / \binom{n}{i}.$$

It is to be noted here that the binomial coefficients involved are non-zero only when

(3.10)
$$j-1 \le r-1 \text{ and } i-j \le n-r.$$

Hence, for $r < s_i$ the sum in (3.9) is essentially up to j = r. And f(i, r) = 0 for $r > n - i + s_i$.

Once the values of (3.9) are determined, the probability of r is determined as the marginal distribution. Thus, the absolute rank of the selected object is r with probability

(3.11)
$$f(r) = \sum_{i=i_1}^n f(i,r) \quad (r = 1, 2, \cdots, n).$$

In particular, for r = 1, the sum in (3.9) reduces to a single term and we get

(3.12)
$$f(1) = \sum_{i=i_1}^{n} Q(i-1)\frac{1}{n}.$$

Noting that $Q(i_1 - 1) = 1$ and Q(n) = 0, and considering (3.3), it becomes

(3.13)
$$f(1) = (e_0 - i_1 + 1)/n.$$

It can be shown (see Appendix 2 for details) that

(3.14)
$$f(i,r) = Q(i-1)/n \quad (1 \le r \le s_i),$$

and

(3.15)
$$g(i,r) \equiv f(i,r) - f(i,r+1) \\ = \frac{Q(i-1)}{i} {r-1 \choose s_i - 1} {n-r-1 \choose i - s_i - 1} / {n \choose i} \quad (s_i \le r \le n - i + s_i).$$

Therefore, for a fixed value of i,

(3.16)
$$\frac{g(i,r+1)}{g(i,r)} = \frac{r}{r-s_i+1} \cdot \frac{n-r-i+s_i}{n-r-1} \quad (s_i \le r \le n-i+s_i-1)$$

and

(3.17)
$$g(i,s_i) = \frac{Q(i-1)}{i} \binom{n-s_i-1}{i-s_i-1} / \binom{n}{i}$$

Starting with (3.17), successive application of (3.16) will give us all g(i,r) $(r = s_i + 1, s_i + 2, \cdots)$. Then, starting with f(i,r) = 0 $(r = n - i + s_i + 1)$, successive application of

(3.18)
$$f(i,r) = f(i,r+1) + g(i,r) \quad (r = n - i + s_i, \cdots, s_i)$$

will give us all the f(i, r)'s efficiently, and the coincidence of the value of $f(i, s_i)$ with (3.14) will work as a check.

After getting all the values f(i,r), we can easily compute (3.11). On the other hand, there are another set of formulae based on the partial sum for $i \in I_s(s = 1, 2, \cdots)$. (See Appendix 3 for the details.)

Remark. Formulae (3.14), (3.15), etc. in this Section were first derived in Watanabe (1965). They provide an efficient algorithm to compute the p.d.f. and c.d.f. of r, the absolute rank of the selected object.

Now let us derive the expected rank of the selected object.

Under the condition that, after the *i*-th observation, the relative rank y_i of the *i*-th object is j, the expected absolute rank is given, from (3.8), by

(3.19)
$$E(x_{i} | y_{i} = j) = \sum_{r=1}^{n} r \cdot P(x_{i} = r | y_{i} = j)$$
$$= \sum_{r=j}^{n-i+j} r\binom{r-1}{j-1}\binom{n-r}{i-j} / \binom{n}{i}$$
$$= \left[j / \binom{n}{i} \right] \sum_{r=j}^{n-i+j} \binom{r}{j} \binom{n-r}{i-j}.$$

But, since

(3.20)
$$\sum_{r=j}^{n-i+j} {r \choose j} {n-r \choose i-j} = {n+1 \choose i+1},$$

we get

(3.21)
$$E(x_i \mid y_i = j) = \frac{n+1}{i+1}j.$$

The expected rank of the object selected with the stopping criteria $s_i (i = 1, 2, \dots)$ is obtained, from (3.21), (2.1), and (2.2), as

(3.22)
$$E(r) = \sum_{i=i_1}^{n} Q(i-1) \cdot \frac{1}{i} \cdot \frac{n+1}{i+1} \sum_{j=1}^{s_i} j$$
$$= \sum_{i=i_1}^{n} Q(i-1) \cdot \frac{n+1}{i(i+1)} \cdot \frac{s_i(s_i+1)}{2}.$$

Let c_i denote the expected rank under the condition ω_i defined in (2.2), i.e.

$$(3.23) c_i = E(r \mid \omega_i).$$

Then we get, using (2.5), the recurrence formula

(3.24)
$$c_{i-1} = \frac{1}{Q(i-1)} \sum_{k=i}^{n} Q(k-1) \frac{n+1}{k(k+1)} \cdot \frac{s_k(s_k+1)}{2}$$
$$= \frac{n+1}{i(i+1)} \frac{s_i(s_i+1)}{2} + \frac{Q(i)}{Q(i-1)} c_i$$
$$= \frac{(n+1)s_i(s_i+1)}{2i(i+1)} + \left(1 - \frac{s_i}{i}\right) c_i.$$

For i=n, (3.24) gives

$$(3.25) c_{n-1} = (n+1)/2.$$

Starting with this, successive application of (3.24) will give the values of c_{n-2}, \dots, c_0 . It can easily be seen that $c_i = c_0 (i \in I_0)$ and the value of c_0 is equal to (3.22).

4. Optimal stopping rule.

So far, we derived various formulae for an arbitrary set of the stopping criteria s_i (i = 1, 2, ..., n). Now let us consider optimizing the criteria. The framework we adopt here is the backward induction of Bellman's Dynamic Programming, i.e. determining s_i at each step so that c_{i-1} will be minimized for given c_i .

If the cost of observation is negligible, then our objective will be to minimize c_0 , the expected absolute rank of the selected object. In order to achieve this, we ought to choose such value s_i at each step *i* that will minimize (3.24). Rewriting (3.24) in the form

(4.1)
$$c_{i-1} = c_i + \frac{1}{i} \sum_{j=1}^{s_i} \left(\frac{n+1}{i+1} j - c_i \right),$$

it becomes obvious that s_i is to be chosen as the largest integer j which satisfies

(4.2)
$$\frac{n+1}{i+1}j \le c_i,$$

namely

(4.3)
$$s_i = \operatorname{trunc}\left[\frac{i+1}{n+1} \cdot c_i\right],$$

where trunc[x] means the truncated value, that is the greatest integer not exceeding x. (Note that, for given c_i , (4.1) will be minimized when all the positive terms are excluded from the summation.)

Starting with $s_n = n, c_{n-1} = (n+1)/2$, and using (4.3) and (3.24) alternately, we can get $s_i, c_{i-1} (i = n - 1, n - 2, \dots, 1)$ which give us the optimal stopping rule and the expected rank c_0 attained by that rule. This result coincides with what has long been known (see Chow et al.(1964), for instance).

Next, let us consider the case where the cost of observation is not negligible. Put

(4.4)
$$k = \frac{\text{cost of observing one object}}{\text{loss of getting one lower expected rank}},$$

and let our objective be to minimize

(4.5)
$$t_0 = c_0 + k \cdot e_0.$$

(Strictly speaking, $c_0 - 1$ would be the real loss. But there will be no harm in considering only c_0 , and call t_0 the total "loss" including the expected cost of observations.) In order to achieve this, we combine c_i in (3.23) and e_i in (2.21) and put

$$(4.6) t_i = c_i + k \cdot e_i.$$

Then the recurrence formula for t_i is obtained from (3.24) and (2.22) as

(4.7)
$$t_{i-1} = \frac{(n+1)s_i(s_i+1)}{2i(i+1)} + k + \left(1 - \frac{s_i}{i}\right)t_i.$$

Then, by similar arguments as (4.1) through (4.3), the value of s_i which minimizes t_{i-1} is given by

(4.8)
$$s_i = \operatorname{trunc}\left[\frac{i+1}{n+1}t_i\right].$$

Starting with $t_{n-1} = (n+1)/2+k$, and using (4.8) and (4.7) alternately, we can determine $s_i, t_{i-1} (i = n-1, n-2, \dots, 1)$. The result will be the optimal stopping rule and the optimized expected "total loss".

This treatment was done by Watanabe(1965) probably for the first time.

5. Numerical examples.

For values of n which are not extremely large, the computation discussed above can be done easily with computer. Here we present the results obtained for n = 25 with k = 0 and 0.5, together with some graphs.

Please note that the cumulative probabilities

(5.1)
$$F(r) = f(1) + f(2) + \dots + f(r) \quad (r = 1, 2, \dots, n)$$

are also shown in the tables and the graphs.

Note. For some reasons, s(i), c(i), e(i) and t(i) are used in the Tables and Figures for s_i, c_i, e_i and t_i .

In Table 1 and Fig. 1, it is interesting to note that $s_i = 0$ for $i = 1, 2, \dots, 7$, so that one should keep observing about 1/4 of the population without taking any action (just getting information). Then, $s_i = 1$ up to i = 13, meaning that one takes an object only if it turns out to be the best among all those observed so far. At the end of this phase, one has already observed about half the population. After that the criterion gets milder and milder.

The behaviour of p_i and Q(i) may appear to be a little irregular at first glance. But if one notices the change of s_i at certain points, that feeling will probably disappear.

Fig. 2 differs from Fig. 1 in several features, due to the cost of observation. First, "information only" period reduces considerably, and the criteria get milder much sooner. Second, both c_i and t_i decrease for small i and turn to increasing at a certain value of i. About the seeming irregularity of p_i and Q(i), the same comments apply here as in the case of Fig. 1.

Table 1. Computed results for $n = 25$ and $k = 1$	Table 1.	Computed	results	for n	= 25	and	k =	0
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s(i <u>)</u>	c(1-1)) e(i-1)	t(1-1)	Q(1)	p(i)	r	f(r)	F(r)
0	3,1167	14,4586	3.1167	1.0000	.0000	1	. 29834	. 29834
õ	3,1167	13.4586	3,1167	1.0000	.0000	$\overline{2}$.22834	.52669
õ	3,1167	12.4586	3.1167	1.0000	.0000	3	.16595	.69264
ō	3.1167	11.4586	3.1167	1.0000	.0000	4	.11412	.80676
õ	3.1167	10.4586	3.1167	1.0000	.0000	5	.07264	.87940
Ō	3.1167	9.4586	3.1167	1.0000	.0000	6	.04284	.92223
õ	3.1167	8.4586	3.1167	1.0000	.0000	7	.02533	.94756
1	3.1167	7.4586	3.1167	.8750	.1250	8	.01538	.96294
1	3.1492	7.3812	3.1492	.7778	.0972	9	.00995	.97289
1	3.2179	7.1789	3.2179	.7000	.0778	10	.00632	.97921
1	3.3128	6.8654	3.3128	.6364	.0636	11	.00401	.98322
1	3.4274	6.4520	3.4274	.5833	.0530	12	.00316	.98638
1	3.5572	5.9476	3.5572	.5385	.0449	13	.00204	.98842
2	3.6988	5.3599	3.6988	.4615	.0769	14	.00101	.98942
2	3.8820	5.0866	3.8820	.4000	.0615	15	.00098	.99040
2	4.1042	4.7153	4.1042	.3500	.0500	16	.00096	.99136
3	4.3628	4.2460	4.3628	.2882	.0618	17	.00096	. 99232
3	4.6786	3.9416	4,6786	.2402	.0480	18	.00096	.99328
4	5.0669.	3.5299	5.0669	.1896	.0506	19	.00096	.99424
4	5.5515	3.2046	5.5515	.1517	.0379	20	.00096	.99520
5	6.1655	2.7557	6.1655	.1156	.0361	21	.00096	.99616
7	6.9843	2.3043	6.9843	.0788	.0368	22	.00096	.99712
9	8.1335	1.9130	8.1335	.0480	.0308	23	.00096	.99808
12	9.8800	1.5000	9.8800	.0240	.0240	24	.00096	.99904
25	13.0000	1.0000	13.0000	.0000	.0240	25	.00096	1.00000
	s(1) 0 0 0 0 0 0 0 0 0 0 0 0 0	$\begin{array}{c} {\rm s(i)} {\rm c(i-1)} \\ 0 3.1167 \\ 0 3.1167 \\ 0 3.1167 \\ 0 3.1167 \\ 0 3.1167 \\ 0 3.1167 \\ 0 3.1167 \\ 1 3.1167 \\ 1 3.1167 \\ 1 3.128 \\ 1 3.2179 \\ 1 3.2179 \\ 1 3.3128 \\ 1 3.4274 \\ 1 3.5572 \\ 2 3.6988 \\ 2 3.8820 \\ 2 4.1042 \\ 3 4.3628 \\ 3 4.6786 \\ 4 5.0669 \\ 4 5.5515 \\ 5 6.1655 \\ 7 6.9843 \\ 9 8.1335 \\ 12 9.8800 \\ 25 13.0000 \end{array}$	$\begin{array}{cccccccccccccccccccccccccccccccccccc$	$ \begin{array}{cccccccccccccccccccccccccccccccccccc$	$ \begin{array}{cccccccccccccccccccccccccccccccccccc$	$ \begin{array}{cccccccccccccccccccccccccccccccccccc$	$ s(i) \ \ c(i-1) \ \ e(i-1) \ \ t(i-1) \ \ Q(i) \ \ p(i) \ \ r \\ 0 \ \ 3.1167 \ 14.4586 \ \ 3.1167 \ \ 1.0000 \ \ .0000 \ \ 2 \\ 0 \ \ 3.1167 \ 13.4586 \ \ 3.1167 \ 1.0000 \ \ .0000 \ \ 2 \\ 0 \ \ 3.1167 \ 12.4586 \ \ 3.1167 \ 1.0000 \ \ .0000 \ \ 3 \\ 0 \ \ 3.1167 \ 12.4586 \ \ 3.1167 \ 1.0000 \ \ .0000 \ \ 4 \\ 0 \ \ 3.1167 \ 11.4586 \ \ 3.1167 \ 1.0000 \ \ .0000 \ \ 4 \\ 0 \ \ 3.1167 \ 10.4586 \ \ 3.1167 \ 1.0000 \ \ .0000 \ \ 5 \\ 0 \ \ 3.1167 \ 10.4586 \ \ 3.1167 \ 1.0000 \ \ .0000 \ \ .0000 \ \ 5 \\ 0 \ \ 3.1167 \ 10.4586 \ \ 3.1167 \ 1.0000 \ \ .0000 \ \ .0000 \ \ 5 \\ 0 \ \ 3.1167 \ 1.4586 \ \ 3.1167 \ 1.0000 \ \ .0000 \ \ .0000 \ \ 5 \\ 0 \ \ 3.1167 \ 1.4586 \ \ 3.1167 \ 1.0000 \ \ .0000 \ \ .0000 \ \ 5 \\ 0 \ \ 3.1167 \ 1.4586 \ \ 3.1167 \ 1.0000 \ \ .0000 \ \ .0000 \ \ .0000 \ \ 7 \\ 1 \ \ 3.1167 \ 7.4586 \ 3.1167 \ 1.0000 \ .0000 \ \ .0000 \ \ .0000 \ \ .0000 \ \ .0000 \ \ .0000 \ .0000 \ .00000 \ .00000 \ .0000 \ .00000 \ .00000 \ .00000$	$ \begin{array}{c ccccccccccccccccccccccccccccccccccc$



Fig. 1. Graphical presentation of Table 1 (n = 25, k = 0).

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i ,	s(i)	c(i-1)	e(1-1)	t(1-1)	Q(1)	p(i)	r	f(r)	F(r)
1	0	4.9722	6.1872	8.0658	1.0000	.0000	1	.16749	.16749
2	0	4.9722	5.1872	7.5658	1.0000	.0000	2	.15082	. 31831
3	1	4.9722	4.1872	7.0658	.6667	.3333	3	.13198	.45029
4	1	4.2083	4.7808	6.5987	.5000	.1667	4	.11228	.56257
5	1	3.8777	5.0411	6.3982	.4000	.1000	5	.09285	.65542
6	1	3.7638	5.0513	6.2895	. 3333	.0667	6	.07500	.73042
7	1	3.7737	4.8616	6.2045	.2857	.0476	7	.05951	.78993
8	2	3.8610	4.5052	6.1136	.2143	.0714	8	.04685	.83679
9	2	3.7036	4.6736	6.0403	.1667	.0476	9	.03689	.87367
10	2	3.6474	4.7232	6.0090	.1333	.0333	10	.02909	.90276
11	2	3.6729	4.6540	5.9999	.1091	.0242	11	.02303	.92579
12	2	3.7669	4.4660	5.9999	.0909	.0182	12	.01829	.94408
13	3	3.9203	4.1592	5.9999	.0699	.0210	13	.01445	.95853
14	3	3.9821	4.1069	6.0355	.0549	.0150	14	.01132	.96985
15	3	4.1226	3.9542	6.0998	.0440	.0110	15	.00879	.97865
16	4	4.3408	3.6928	6.1872	.0330	.0110	16	.00672	.98537
17	4	4.5132	3.5904	6.3084	.0252	.0078	17	.00502	.99039
18	4	4.7908	3.3874	6.4845	.0196	.0056	18	.00364	.99403
19	5 '	5.1822	3.0696	6.7170	.0144	.0052	19	.00252	.99656
20	6	5.6401	2.8087	7.0444	.0101	.0043	20	.00165	.99820
21	6	6.2001	2.5839	7.4921	.0072	.0029	21	.00098	.99918
22	8	7.0257	2.2174	8.1343	.0046	.0026	22	.00050	.99969
23	9	8.1335	1.9130	9.0900	.0028	.0018	23	.00020	.99989
24	12	9.8800	1.5000	10.6300	.0014	.0014	24	.00006	.99994
25	25	13.0000	1.0000	13,5000	.0000	.0014	25	.00006	1.00000

Table 2. Computed results for n = 25 and k = 0.5.



Fig. 2. Graphical presentation of Table 2 (n = 25, k = 0.5).

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As for the distribution of r, both Fig. 1 and Fig. 2 show the same feature of skewness like geometric distribution. But the achievement is poorer in Fig. 2 than in Fig. 1, reflecting $c_0 = 5$ for k = 0.5 versus $c_0 = 3$ for k = 0.

6. Final remarks.

The "basic" theory of selection by relative rank with cost is now complete. Generalization in various directions are conceivable and has already been done rather extensively. (See the list of references, especially review articles Freeman (1983) and Ferguson (1989)).

Although the results developed here are based on a rather simple se-up, they will hopefully be useful in more general cases also, in setting the directions to go when observation cost is not negligible.

The asymptotic theory when $n \to \infty$ has only partly been done (e.g. Chow et al.(1964)). The author believes that the cases for low, medium, and high cost must be handled separately. (It is suggested by Moriguti (1992).) Those results will be published successively in the near future.

Appendix 1. Derivation of formula (2.20)

Formula (2.19) can be rewritten as

(A1.1)
$$e_0 = 1 + \sum_{s=0}^n \sum_{i \in I_s} Q(i),$$

by dividing the set of all values of i into sets $I_s(s = 0, \dots, n)$ as defined in (2.7).

Since Q(i) = 1 for all $i \in I_0$, we have

(A1.2)
$$\sum_{i \in I_0} Q(i) = (i_1 - 1) \cdot 1 = i_1 - 1.$$

For s = 1, (2.15) becomes $Q(i) = (i_1 - 1)/i$ $(i \in I_1)$, so that

(A1.3)
$$\sum_{i \in I_1} Q(i) = (i_1 - 1) \sum_{i \in I_1} \frac{1}{i} = (i_1 - 1) \left(\frac{1}{i_1} + \frac{1}{i_1 + 1} + \dots + \frac{1}{i_2 - 1} \right).$$

For $s \ge 2$, using (2.15), we get

(A1.4)
$$\sum_{i \in I_s} Q(i) = (i_1 - 1)(i_2 - 2) \cdots (i_s - s) \sum_{i \in I_s} \frac{1}{i(i - 1) \cdots (i - s + 1)}.$$

But

(A1.5)
$$\sum_{i \in I_s} \frac{1}{i(i-1)\cdots(i-s+1)} = \frac{1}{s-1} \sum_{i \in I_s} \left[\frac{1}{(i-1)\cdots(i-s+1)} - \frac{1}{i(i-1)\cdots(i-s+2)} \right] = \frac{1}{s-1} \left[\frac{1}{(i_s-1)\cdots(i_s-s+1)} - \frac{1}{(i_{s+1}-1)\cdots(i_{s+1}-s+1)} \right]$$

Substituting (A1.2), (A1.3) and (A1.4) into (A1.1), and using (A1.5), we get

(A1.6)

$$\begin{aligned} e_{0} &= 1 + (i_{1} - 1) + (i_{1} - 1) \left(\frac{1}{i_{1}} + \frac{1}{i_{1} + 1} + \dots + \frac{1}{i_{2} - 1} \right) \\ &+ \sum_{s=2}^{n} \frac{1}{s - 1} \left[\frac{(i_{1} - 1)(i_{2} - 2) \cdots (i_{s} - s)}{(i_{s} - 1)(i_{s} - 2) \cdots (i_{s} - s + 1)} - \frac{(i_{1} - 1)(i_{2} - 2) \cdots (i_{s} - s)}{(i_{s+1} - 1)(i_{s+1} - 2) \cdots (i_{s+1} - s + 1)} \right] \\ &= i_{1} + (i_{1} - 1) \left(\frac{1}{i_{1}} + \frac{1}{i_{1} + 1} + \dots + \frac{1}{i_{2} - 1} \right) + \frac{(i_{1} - 1)(i_{2} - 2)}{i_{2} - 1} \\ &+ \sum_{s=3}^{n-1} \left[-\frac{1}{s - 2} \cdot \frac{(i_{1} - 1)(i_{2} - 2) \cdots (i_{s-1} - s + 1)}{(i_{s} - 1)(i_{s} - 2) \cdots (i_{s} - s + 2)} + \frac{1}{s - 1} \cdot \frac{(i_{1} - 1)(i_{2} - 2) \cdots (i_{s} - s + 1)}{(i_{s} - 1)(i_{s} - 2) \cdots (i_{s} - s + 1)} \right]. \end{aligned}$$

Now, the quantity between the last brackets can be transformed into

$$\frac{(i_1-1)(i_2-2)\cdots(i_{s-1}-s+1)}{(s-1)(s-2)(i_s-1)(i_s-2)\cdots(i_s-s+1)} \{-(s-1)(i_s-s+1)+(s-2)(i_s-s)\}$$
$$= -\frac{Q(i_s-1)\cdot(i_s-1)}{(s-1)(s-2)},$$

(cf. (2.13)). Also,

(A1.8)
$$\frac{(i_1-1)(i_2-2)}{i_2-1} = (i_1-1) - \frac{i_1-1}{i_2-1}.$$

Hence (A1.6) becomes

(A1.9)
$$e_0 = (2i_1 - 1) + (i_1 - 1)\left(\frac{1}{i_1} + \frac{1}{i_1 + 1} + \dots + \frac{1}{i_2 - 2}\right) - \sum_{s=3}^{n-1} \frac{Q(i_s - 1) \cdot (i_s - 1)}{(s - 1)(s - 2)},$$

which is (2.20).

Appendix 2. Derivation of (3.14) and (3.15)

In (3.9), for $r \leq s_i$, the sum is essentially up to r, and since there holds

(A2.1)
$$\sum_{j=1}^{r} {\binom{r-1}{j-1} \binom{n-r}{i-j}} = {\binom{n-1}{i-1}},$$

(3.9) becomes

(A2.2)
$$f(i,r) = Q(i-1) \cdot \frac{1}{i} \binom{n-1}{i-1} / \binom{n}{i} \\ = Q(i-1)/n \qquad (1 \le r \le s_i)$$

which is (3.14).

Next, for $s_i \leq r \leq n-1$, we get from (3.9)

(A2.3)
$$g(i,r) = f(i,r) - f(i,r+1) \\ = \frac{Q(i-1)}{i} \sum_{j=1}^{s_i} \left[\binom{r-1}{j-1} \binom{n-r}{i-j} - \binom{r}{j-1} \binom{n-r-1}{i-j} \right] / \binom{n}{i}.$$

 But

$$\begin{aligned} &(A2.4) \\ &\sum_{j=1}^{s_i} \left[\binom{r-1}{j-1} \binom{n-r}{i-j} - \binom{r}{j-1} \binom{n-r-1}{i-j} \right] \\ &= \binom{n-r}{i-1} - \binom{n-r-1}{i-1} + \sum_{j=2}^{s_i} \left[\binom{r-1}{j-1} \binom{n-r}{i-j} - \left\{ \binom{r-1}{j-1} + \binom{r-1}{j-2} \right\} \binom{n-r-1}{i-j} \right] \\ &= \binom{n-r-1}{i-2} + \sum_{j=2}^{s_i} \left[\binom{r-1}{j-1} \left\{ \binom{n-r}{i-j} - \binom{n-r-1}{i-j} \right\} - \binom{r-1}{j-2} \binom{n-r-1}{i-j} \right] \\ &= \binom{n-r-1}{i-2} + \sum_{j=2}^{s_i} \left[\binom{r-1}{j-1} \binom{n-r-1}{i-j-1} - \binom{r-1}{j-2} \binom{n-r-1}{i-j} \right] \\ &= \binom{r-1}{s_i-1} \binom{n-r-1}{i-s_i-1}. \end{aligned}$$

Therefore, (A2.3) becomes

(A2.5)
$$g(i,r) = \frac{Q(i-1)}{i} {\binom{r-1}{s_i-1} {\binom{n-r-1}{i-s_i-1}}} / {\binom{n}{i}} \quad (s_i \le r \le n-i+s_i),$$

which is (3.15).

Appendix 3. Another Set of Formulae On the Distribution of the Absolute Rank

(3.11) can also be written as

(A3.1)
$$f(r) = \sum_{s=1}^{n} \sum_{i \in I_s} f(i, r)$$

(cf. (2.7)).

Accordingly, its backward difference becomes

(A3.2)
$$g(r) \equiv f(r) - f(r+1) = \sum_{s=1}^{n} \sum_{i \in I_s} \{f(i,r) - f(i,r+1)\} = \sum_{s=1}^{n} \sum_{i \in I_s} g(i,r).$$

Since, for $i \in I_s$, g(i,r) is not zero only for r which satisfies $s \leq r \leq n-i+s$, the outer summation in (A3.2) is to be done only for s which satisfies

(A3.3)
$$1 \le s \le r \text{ and } i_s - s \le n - r.$$

For such *s*, we get

$$\begin{split} \sum_{i \in I_s} g(i,r) &= \sum_{i \in I_s} \frac{Q(i-1)}{i} \binom{r-1}{s-1} \binom{n-r-1}{i-s-1} / \binom{n}{i} \\ &= \left\{ \prod_{l=1}^s (i_l-l) \right\} \binom{r-1}{s-1} \sum_{i \in I_s} \frac{1}{i^{(s+1)}} \frac{(n-r-1)!}{(i-s-1)!(n-r-i+s)!} \frac{i!(n-i)!}{n!} \\ &= \left\{ \prod_{l=1}^s (i_l-l) \right\} \binom{r-1}{s-1} \frac{(n-r-1)!}{n!} \sum_{i \in I_s} (n-i)^{(r-s)}. \end{split}$$

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But

(A3.5)
$$\sum_{i \in I_s} (n-i)^{(r-s)} = \sum_{i=i_s}^{i_{s+1}-1} \frac{(n-i+1)^{(r-s+1)} - (n-i)^{(r-s+1)}}{r-s+1}$$
$$= \frac{(n-i_s+1)^{(r-s+1)} - (n-i_{s+1}+1)^{(r-s+1)}}{r-s+1}.$$

Hence (A3.4) becomes

(A3.6)
$$\sum_{i \in I_s} g(i,r) = \left\{ \prod_{l=1}^s (i_l - l) \right\} \binom{r-1}{s-1} \frac{(n-i_s+1)^{(r-s+1)} - (n-i_{s+1}+1)^{(r-s+1)}}{n^{(r+1)}(r-s+1)}.$$

These formulae will be useful for very large n, and also when discussing the limit as $n \to \infty$.

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Sigeiti Moriguti Syoan 2-16-10 Suginami-ku Tokyo 167, JAPAN