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ON THE QUEUE WITH PH-MARKOV RENEWAL PREEMPTIONS

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Abstract Analysis of the completion time of a service unit with PH-Markov renewal preemptions are presented for several preemptive disciplines. Applying the results of these analysis and the theory of queues with semi-Markov service times, a $PH\text{-}MR, M/G_1, G_2/1$ queue is analyzed and the effect of preemptions is discussed. A non-preemptive queue is also analyzed.

1. Introduction

The completion time of a service unit is defined as the duration of the period that begins from the instant its service starts and ends at the instant the server becomes free to take the next service unit of that class [6]. This time is important in the analysis of preemptive priority queueing models. Gaver [3], Keilson [7] and Avi-Itzhak-Naor [1] have studied completion time for the case of service times interrupted by Poisson customers with general service times. In this paper, their results are extended to an $M/G/1$ type queue whose services are interrupted by a stream of PH-Markov renewal ($PH\text{-}MR$) customers. In the PH-Markov renewal process proposed by this author [8, 9], each element of the semi-Markov kernel of this process has a phase structure.

The fundamental period (or busy period) of the $PH\text{-}MR/G/1$ queue plays an essential role in analyzing the completion time. The fundamental period was proposed by Neuts [11, 12, 13] and has been analyzed by several authors for various queueing models. The busy period can be represented easily through the fundamental period [8, 9]. Completion time distributions for both the preemptive resume and preemptive repeat disciplines are represented by the busy period distribution. In particular, the fundamental period is a special case of the completion time for the preemptive resume discipline. The completion time distributions have matrix structures, and the Laplace-Stieltjes transforms of these distribution matrices are explicitly provided.

These results are applied to the analysis of a $PH\text{-}MR, M/G_1, G_2/1$ queue with preemptive discipline, in which priority and non-priority customer arrival processes are PH-Markov renewal and Poissonian, respectively. Distribution of the number of customers in the system at the service completion epoch of a non-priority customer is analyzed by the embedded Markov chain method. In addition, sojourn times, waiting times and output processes for priority and non-priority customers are derived. Note that the waiting time of non-priority customers in the preemptive-resume queue has an essential role in the analysis of the non-preemptive priority queue, because the waiting time of the preemptive-resume queue is identical to that of the non-preemptive one [17, 18]. The preemptive $PH\text{-}MR, M/G_1, G_2/1$ queueing process is very similar to the $M/SM/1$ queueing process with semi-Markovian services [10], when the service time process is considered as a completion time process. The main difference between the two queueing processes is in the idle period. The idle period process of the former model is far more complex than that of the latter model in which the idle time distribution is exponential. New results for stationary state distribution and waiting time distribution of the preemptive queue are derived.

The models discussed in this paper are important, especially in the telecommunications field, because their results make it possible to analyze statistical multiplexers for packetized voice and data [5, 16]. It is well known that voice packet arrival processes are often modeled as Markov modulated Poisson processes [4, 5], which are special cases of the PH-Markov renewal process [8, 9]. Several service disciplines such as voice-packet preemptive priority may be considered. These systems are generalized models of the PH-$MR, M/G/1$ with the $FIFO$ discipline analyzed by Heffes et al.[5] and Sriram et al.[16]. Specifically, since voice traffic should suffer only very short and infrequent delays to avoid clipping [2], a system in which voice traffic has some priority over data traffic will become more dominant in the near future.

2. Completion time

Consider the service time of a non-priority customer whose service time distribution is denoted by $H_2(x)$. We will study the completion time of a service which is preempted by priority customers whose arrival process is a PH-Markov renewal one with representation $(\boldsymbol{\alpha}, \mathbf{T}, \mathbf{T}^0)$ [8, 9] and service time distribution $H(x)$ is general. Suppose that the service process of a non-priority customer begins at 0 and ends at epoch S in $(x, x+dx)$. Completion time density $\mathbf{C}(x)$ consists of elements

$$c_{ij}(x)dx := \mathrm{P}[x < S < x + dx, J(S) = j \mid J(0) = i], \quad i, j = 1, \cdots, m,$$

where $J(x)$ is the arrival phase state at time x.

Theorem 2.1 Laplace transform $\mathbf{C}^*(s)$ for $Re(s) \geq 0$ of completion time density

$$\mathbf{C}(t) := (c_{ij}(t))(i, j = 1, \cdots, m)$$

is represented by

$$\mathbf{C}^*(s) = H_2^*(s\mathbf{I} - \mathbf{T} - \mathbf{T}^0\mathbf{B}_1^*(s, 1)) \tag{2.1}$$

for Preemptive Resume Discipline; by

$$\begin{aligned}\mathbf{C}^*(s) = \int_0^\infty &[\mathbf{I} - \{\mathbf{I} - \exp(-x(s\mathbf{I} - \mathbf{T}))\}(s\mathbf{I} - \mathbf{T})^{-1}\mathbf{T}^0\mathbf{B}_1^*(s, 1)]^{-1} \\ &\cdot \exp(-x(s\mathbf{I} - \mathbf{T}))dH_2(x)\end{aligned} \tag{2.2}$$

for Preemptive Repeat Identical Discipline; and by

$$\mathbf{C}^*(s) = [\mathbf{I} - \{\mathbf{I} - H_2^*(s\mathbf{I} - \mathbf{T})\}(s\mathbf{I} - \mathbf{T})^{-1}\mathbf{T}^0\mathbf{B}_1^*(s, 1)]^{-1}H_2^*(s\mathbf{I} - \mathbf{T}) \tag{2.3}$$

for Preemptive Repeat Different Discipline.

Here $\mathbf{B}_1^*(s, z)$ for $Re(s) \geq 0$ and $|z| \leq 1$ is the double transform of the joint density of the busy period length and the number of customers served during this period for a PH-$MR/G/1$ with arrival process representation $(\boldsymbol{\alpha}, \mathbf{T}, \mathbf{T}^0)$ and service time distribution $H(x)$, and it satisfies the following functional equation [8, 9, 15].

$$\mathbf{B}_1^*(s, z) = z\boldsymbol{\alpha}H^*(s\mathbf{I} - \mathbf{T} - \mathbf{T}^0\mathbf{B}_1^*(s, z)). \tag{2.4}$$

Proof: Equation (2.1) for the preemptive resume discipline can be proved in a similar manner as Theorem 3.1 in Machihara [9], because we only need to change the preempted service time distribution from $H(x)$ to $H_2(x)$ and set

$$\mathbf{B}_1^*(s,z) = \boldsymbol{\alpha}\mathbf{G}^*(s,z).$$

In order to prove (2.2), we consider a completion time having service time requirement Z. Under the preemptive repeat identical discipline, each time a non-priority customer enters service, it will require service time Z. The completion time process, therefore, ends when this customer finds an uninterrupted time duration of length Z. We introduce density

$$\mathbf{C}(t \mid Z = z) := (c_{ij}(t \mid Z = z))(i, j = 1, \cdots, m)$$

of completion time length S, that is,

$$c_{ij}(t \mid Z = z)dt := P[t < S < t + dt, J(S) = j \mid J(0) = i, Z = z].$$

and define the Laplace transform $\mathbf{C}^*(s \mid Z = z)$ of $\mathbf{C}(t \mid Z = z)$.

To get $\mathbf{C}^*(s \mid Z = z)$, there are two possibilities: either there are no arrivals during $(0, z]$ or there is at least one arrival during $(0, z]$. Routine conditioning arguments lead to the equation

$$\begin{aligned}\mathbf{C}^*(s \mid Z = z) &= \exp(-z(s\mathbf{I} - \mathbf{T})) \\ &+ (\int_0^z e^{-sx}\exp(\mathbf{T}x)\mathbf{T}^0 dx)\mathbf{B}_1^*(s,1)\mathbf{C}^*(s \mid Z = z) \\ &= \exp(-z(s\mathbf{I} - \mathbf{T})) + \{\mathbf{I} - \exp(-z(s\mathbf{I} - \mathbf{T}))\} \\ &\cdot (s\mathbf{I} - \mathbf{T})^{-1}\mathbf{T}^0\mathbf{B}_1^*(s,1)\mathbf{C}^*(s \mid Z = z)\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}\mathbf{C}^*(s \mid Z = z) = &[\mathbf{I} - \{\mathbf{I} - \exp(-z(s\mathbf{I} - \mathbf{T}))\}(s\mathbf{I} - \mathbf{T})^{-1}\mathbf{T}^0\mathbf{B}_1^*(s,1)]^{-1} \\ &\cdot \exp(-z(s\mathbf{I} - \mathbf{T})).\end{aligned}$$

Since the distribution of Z is $H_2(z)$, we finally get (2.2).

To prove (2.3), we observe that under the preemptive repeat different discipline, each time a non-priority customer enters service, it will require a service time underlying distribution $H_2(x)$. Let us introduce joint density $\mathbf{C}(t)$, as in the case of preemptive repeat identical discipline, that is,

$$\mathbf{C}(t)dt = (c_{ij}(t)dt) = (P[t < S < t + dt, J(S) = j \mid J(0) = i])(i, j = 1, \cdots, m).$$

To get the Laplace transform $\mathbf{C}^*(s)$ of $\mathbf{C}(t)$, there are two possibilities: either there are no arrivals during the first customer's service period or there is at least one arrival during this period. Then, we obtain

$$\mathbf{C}^*(s) = H_2^*(s\mathbf{I} - \mathbf{T}) + H_2^{c*}(s\mathbf{I} - \mathbf{T})\mathbf{T}^0\mathbf{B}_1^*(s,1)\mathbf{C}^*(s),$$

where $H_2^{c*}(s)$ is the Laplace-Stieltjes transform of $1 - H_2(x)$ which is the complement of $H_2(x)$. $H_2^{c*}(s\mathbf{I} - \mathbf{T})\mathbf{T}^0$ corresponds to the case in which a new priority customer arrives before the first customer's service ends.

Since

$$H_2^{c*}(s\mathbf{I} - \mathbf{T}) = (\mathbf{I} - H_2^*(s\mathbf{I} - \mathbf{T}))(s\mathbf{I} - \mathbf{T})^{-1},$$

we obtain (2.3). □

Remark: It can easily be proved that (2.1) is identical to(2.3) when $H_2(t)$ is an exponential distribution, and that (2.2) is identical to (2.3) when $H_2(t)$ is the unit distribution.

3. Application: The Priority Queue with PH-Markov Preemption and its Related Models

A queueing model $PH\text{-}MR, M/G_1, G_2/1$ will now be studied in which the priority PH-Markov renewal customers preempt non-priority Poissonian customer services. Let $(\boldsymbol{\alpha}, \mathbf{T}, \mathbf{T}^0)$ and $f_2(X) = \lambda_2 \exp(-\lambda_2 x)$ denote the representation of the preemptive PH-Markov renewal arrival process and interarrival time density of the non-priority Poissonian customers, respectively. Let $H(x)$ and $H_2(x)$ denote the service time distribution of the PH-Markov renewal customers and that of the Poissonian customers, respectively.

Since the arrival and service processes for the priority customers are not influenced by those for the non-priority customers, the characteristics of the priority customers can be analyzed by the $N/G/1$ queue theory [14].

Now let us study the characteristics of non-priority customers. Let $Z_1, Z_2, \cdots$ denote successive service completion epochs of the Poissonian customers. We define N_k^* as the number of Poissonian customers at epoch $Z_k + 0$ immediately after $Z_k (k = 1, 2, \cdots)$. The utilization factor ρ is given by

$$\rho = \rho_1 + \rho_2 \text{ for } \rho_1 = (\mathbf{q}\boldsymbol{\alpha}(-\mathbf{T})^{-2}\mathbf{T}^0\mathbf{e})^{-1}\mu_1^{-1} \text{ and } \rho_2 = \lambda_2 \mu_2^{-1},$$

where $\mathbf{e}$ is a column vector with all elements equal to 1, $\mathbf{q}$ is the invariant probability vector for $\boldsymbol{\alpha}(-\mathbf{T})^{-1}\mathbf{T}^0, \mu_1$ is the service rate of the priority customers, and μ_2 is the service rate of the non-priority customers. For $\rho < 1$, we define the stationary probabilities and the associated generating function as

$$\boldsymbol{\pi}_i := (\pi_i(1), \cdots, \pi_i(m)) = \lim_{k\to\infty} (P[N_k^* = i, J_k^* = 1], \cdots, P[N_k^* = i, J_k^* = m])$$

and

$$\mathbf{g}(z) := \sum_{i=0}^{\infty} z^i \boldsymbol{\pi}_i \text{ for } |z| \leq 1,$$

where J_k^* is the arrival phase state at $Z_k + 0$.

In particular, in order to get $\boldsymbol{\pi}_0$, we will introduce the following random variable T_f :

Suppose that 0 is the service completion epoch for a Poisson customer who leaves the system empty. Consider the first passage time

$$T_f = \inf\{Z_k : N^*(Z_k + 0) = 0 \mid N^*(0) = 0\}, \tag{3.1}$$

where $N^*(x)$ denotes the number of Poisson customers at x.

Let us consider the joint density $\mathbf{A}(x, n)$ of T_f and the number N_q of Poissonian customers served during $(0, T_f)$. That is,

$$\mathbf{A}(x,n)dx := (P\{x < T_f < x + dx, N_q = n, J(T_f) = j \mid J(0) = i\}), (1 \leq i, j \leq m). \tag{3.2}$$

The double transform is written as

$$\mathbf{A}^*(s, z) = \int_0^{\infty} e^{-sx} \sum_{n=0}^{\infty} z^n \mathbf{A}(x, n)dx, \text{ for } Re(s) \geq 0 \text{ and } |z| \leq 1. \tag{3.3}$$

In addition, we introduce the joint density of the busy period length and the number of customers served during this period for a semi-Markovian service system $M/SM/1$ [10] in which the Poissonian arrival is λ_2 and service time density is $\mathbf{C}(x)$, and its double transform $\mathbf{B}^*_{\mathbf{SM}}(s,z)$ for $Re(s) \geq 0$ and $|z| \leq 1$ given by

$$\begin{aligned}\mathbf{B}^*_{\mathbf{SM}}(s,z) &= z\sum_{n=0}^{\infty}\{(\int_0^{\infty} e^{-sx}\exp(-\lambda_2 x)\frac{(\lambda_2 x)^n}{n!}\mathbf{C}(x)dx)\mathbf{B}^{*n}_{\mathbf{SM}}(s,z)\} \\ &= z\int_0^{\infty}\mathbf{C}(x)\exp\{(-(s+\lambda_2)\mathbf{I}+\lambda_2\mathbf{B}^*_{\mathbf{SM}}(s,z))x\}dx. \end{aligned} \tag{3.4}$$

Theorem 3.1 The generating function $\mathbf{g}(z)$ for $|z| \leq 1$ is given by

$$\begin{aligned}\mathbf{g}(z)(z\mathbf{I}-\mathbf{C}^*(\lambda_2(1-z))) &= \boldsymbol{\pi}_0[\lambda_2,\mathbf{I}-\mathbf{T}-\mathbf{T}^0\mathbf{B}_1^*(\lambda_2,1)]^{-1} \\ &\cdot [\lambda_2 z\mathbf{I}-\lambda_2\mathbf{I}+\mathbf{T}+\mathbf{T}^0\mathbf{B}_1^*(\lambda_2(1-z),1))]\mathbf{C}^*(\lambda_2(1-z)),\end{aligned} \tag{3.5}$$

where $\boldsymbol{\pi}_0$ is given by

$$\boldsymbol{\pi}_0 = \frac{\mathbf{q}_0}{\mathbf{q}_0(d\mathbf{A}^*(0,z)/dz\,|_{z=1})\mathbf{e}}. \tag{3.6}$$

Here $\mathbf{q}_0$ is the invariant probability vector of

$$\begin{aligned}\mathbf{A}^*(0,1) &= \{\lambda_2\mathbf{I}-\mathbf{T}-\mathbf{T}^0\mathbf{B}_1^*(\lambda_2,1)\}^{-1}[\lambda_2\mathbf{B}^*_{\mathbf{SM}}(0,1) \\ &+\mathbf{T}^0\{\int_0^{\infty}\mathbf{B}_1(x,1)\exp\{(-\lambda_2\mathbf{I}+\lambda_2\mathbf{B}^*_{\mathbf{SM}}(0,1))\}dx-\mathbf{B}_1^*(\lambda_2,1)\}],\end{aligned} \tag{3.7}$$

where

$$\mathbf{B}^*_{\mathbf{SM}}(0,1) = \int_0^{\infty}\mathbf{C}(x)\exp\{(-\lambda_2\mathbf{I}+\lambda_2\mathbf{B}^*_{\mathbf{SM}}(0,1))x\}dx \tag{3.8}$$

and $\mathbf{B}_1(x,1)$ is the Laplace inverse transform of $\mathbf{B}_1^*(s,1)$.
Also, for the invariant probability vector $\mathbf{b}_{\mathbf{SM}}$ of $\mathbf{B}^*_{\mathbf{SM}}(0,1)$,

$$\begin{aligned}&\frac{d\mathbf{A}^*(0,z)}{dz}\,|_{z=1}\,\mathbf{e} = \{\lambda_2\mathbf{I}-\mathbf{T}-\mathbf{T}^0\mathbf{B}_1^*(\lambda_2,1)\}^{-1} \\ &\cdot[\lambda_2(\mathbf{I}-\mathbf{B}^*_{\mathbf{SM}}(0,1)+\mathbf{e}\mathbf{b}_{\mathbf{SM}})\{\mathbf{I}-\mathbf{C}^*(0)+(\mathbf{e}+\lambda_2\mathbf{C}^{*\prime}(0)\mathbf{e})\mathbf{b}_{\mathbf{SM}}\}^{-1} \\ &+\mathbf{T}^0\{\mathbf{B}_1^*(0,1)-\mathbf{B}_1^*(\lambda_2,1) \\ &-\int_0^{\infty}\mathbf{B}_1(x,1)\exp\{(-\lambda_2\mathbf{I}+\lambda_2\mathbf{B}^*_{\mathbf{SM}}(0,1))x\}dx+\mathbf{B}_1^*(\lambda_2,1) \\ &+\lambda_2(\int_0^{\infty}x\mathbf{B}_1(x,1)dx)\mathbf{e}\mathbf{b}_{\mathbf{SM}}\}\{\mathbf{I}-\mathbf{C}^*(0)+(\mathbf{e}+\lambda_2\mathbf{C}^{*\prime}(0)\mathbf{e})\mathbf{b}_{\mathbf{SM}}\}^{-1}]\mathbf{e}.\end{aligned} \tag{3.9}$$

Proof: Let us consider the transition probability between epochs Z_k and Z_{k+1}, that is,

$$\mathbf{P}_{ij} := (p_{ij}(l,l')) = (P[N^*_{k+1}=j, J^*_{k+1}=l' \mid N^*_k = i, J^*_k = l]) \qquad (1 \leq l,l' \leq m), \qquad 0 \leq i,j < \infty.$$

When $N^*_k = i > 0$, the completion time process begins at $Z_k + 0$. Thus,

$$\mathbf{P}_{ij} := \begin{cases}\displaystyle\int_0^{\infty}\frac{\exp(-\lambda_2 x)(\lambda_2 x)^{j-i+1}}{(j-i+1)!}\mathbf{C}(x)dx, & j=i-1,i,\cdots, \\ 0, & \text{otherwise.}\end{cases} \tag{3.10}$$

When $N_k^* = 0$, the random variable I_b is the number of successive busy periods for the priority customers before the arrival of non-priority customers after Z_k. Then, we obtain the equation

$$\begin{aligned}
&P\{N_{k+1}^* = j, I_b = n \mid N_k^* = 0\} \\
&= \{(\lambda_2\mathbf{I} - \mathbf{T})^{-1}\mathbf{T}^0 \int_0^\infty \exp(-\lambda_2 x)\mathbf{B_1}(x,1)dx\}^n(\lambda_2\mathbf{I} - \mathbf{T})^{-1} \\
&\cdot (\lambda_2 \int_0^\infty \exp(-\lambda_2 x)\frac{(\lambda_2 x)^j}{j!}\mathbf{C}(x)dx) \\
&+ \{(\lambda_2\mathbf{I} - \mathbf{T})^{-1}\mathbf{T}^0 \int_0^\infty \exp(-\lambda_2 x)\mathbf{B_1}(x,1)dx\}^n(\lambda_2\mathbf{I} - \mathbf{T})^{-1}\mathbf{T}^0 \\
&\cdot \sum_{j_1=1}^{j+1}(\int_0^\infty \exp(-\lambda_2 x)\frac{(\lambda_2 x)^{j_1}}{j_1!}\mathbf{B_1}(x,1)dx)) \\
&\cdot (\int_0^\infty \exp(-\lambda_2 x)\frac{(\lambda_2 x)^{j-j_1+1}}{(j-j_1+1)!}\mathbf{C}(x)dx). \qquad (3.11)
\end{aligned}$$

The first term on the right hand side of (3.11) implies that the first non-priority customer after Z_k arrives during the idle period. The second term implies that this customer arrives during the busy period of the priority customers. See Figure 3.1. Summing (3.11) over all n yields

$$\begin{aligned}
\mathbf{P}_{0j} &= \sum_{n=0}^{\infty} P\{N_{k+1}^* = j, I_b = n \mid N_k^* = 0\} \\
&= (\lambda_2\mathbf{I} - \mathbf{T} - \mathbf{T}^0\mathbf{B}_1^*(\lambda_2, 1))^{-1}[\lambda_2 \int_0^\infty \exp(-\lambda_2 x)\frac{(\lambda_2 x)^j}{j!}\mathbf{C}(x)dx \\
&+ \mathbf{T}^0 \sum_{j_1=1}^{j+1}(\int_0^\infty \exp(-\lambda_2 x)\frac{(\lambda_2 x)^{j_1}}{j_1!}\mathbf{B_1}(x,1)dx)) \\
&\cdot (\int_0^\infty \exp(-\lambda_2 x)\frac{(\lambda_2 x)^{j-j_1+1}}{(j-j_1+1)!}\mathbf{C}(x)dx)]. \qquad (3.12)
\end{aligned}$$

From their definitions, $\boldsymbol{\pi}_i$ and $\mathbf{P}_{ij}$ satisfiy the equation

$$\boldsymbol{\pi}_j = \sum_{i=0}^{\infty} \boldsymbol{\pi}_i \mathbf{P}_{ij}.$$

Thus, we obtain

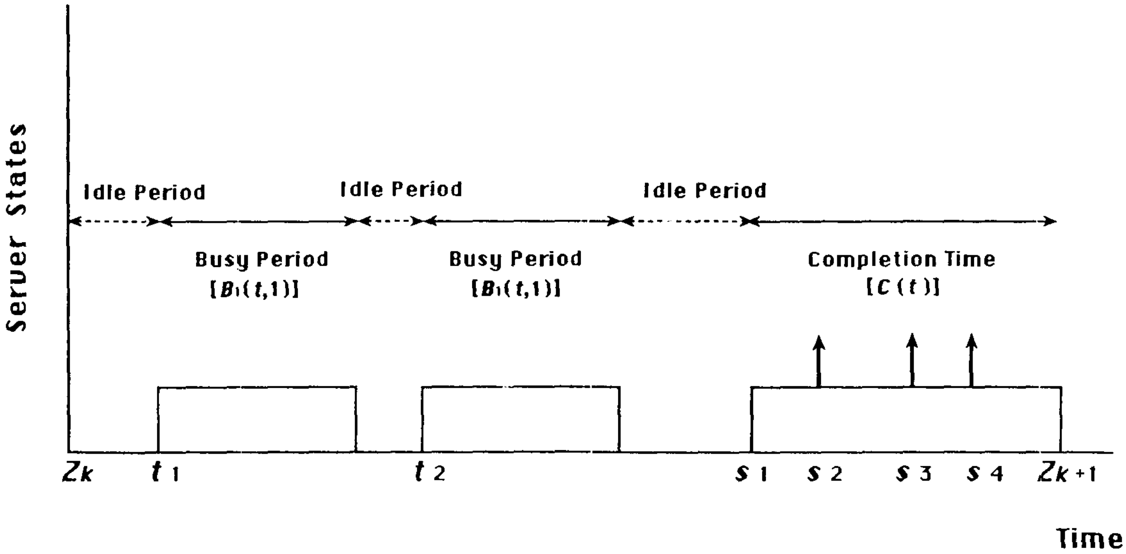
$$\mathbf{g}(z) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} z^j \boldsymbol{\pi}_i \mathbf{P}_{ij}. \qquad (3.13)$$

Equation (3.10) gives

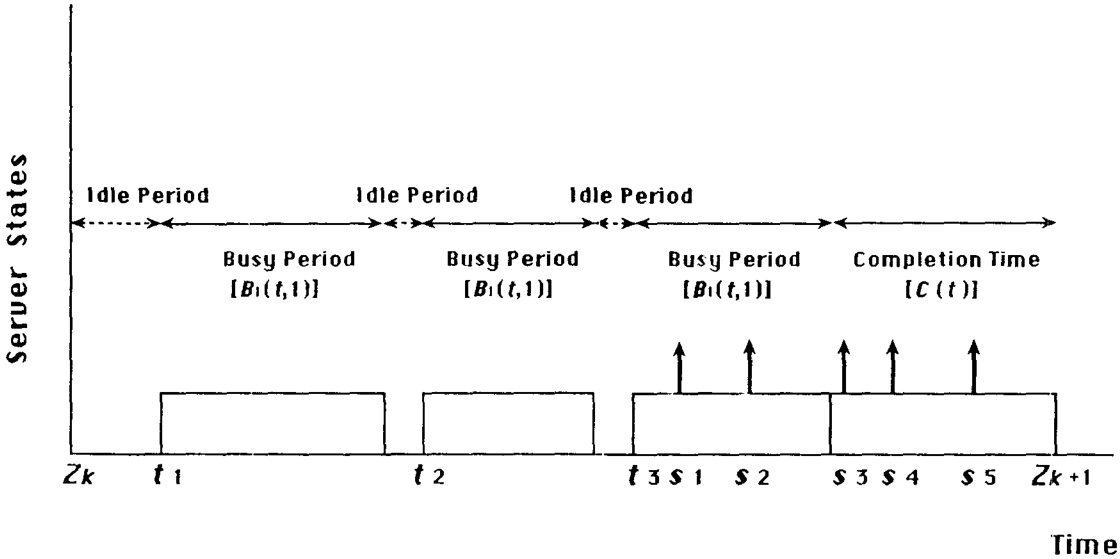
$$\sum_{j=0}^{\infty} z^j \mathbf{P}_{ij} = z^{i-1}\mathbf{C}^*(\lambda_2(1-z)) \text{ for } i > 0. \qquad (3.14)$$

Equation (3.12) gives

$$\begin{aligned}
&\sum_{j=0}^{\infty} z^j \mathbf{P}_{0j} = (\lambda_2\mathbf{I} - \mathbf{T} - \mathbf{T}^0\mathbf{B}_1^*(\lambda_2, 1))^{-1} \\
&\cdot \{\lambda_2\mathbf{I} + z^{-1}\mathbf{T}^0(\mathbf{B}_1^*(\lambda_2(1-z), 1) - \mathbf{B}_1^*(\lambda_2, 1))\}\mathbf{C}^*(\lambda_2(1-z)). \qquad (3.15)
\end{aligned}$$



(a) Case in which Poisson customer arrives during idle period



(b) Case in which Poisson customer arrives during busy period

ti: Arrival time of PH-Markov renewal customer
si: Arrival time of Poisson customer

Fig. 3.1 Sample paths for server states

Substituting (3.14) and (3.15) into (3.13), we obtain (3.5).

Now, let us consider $\boldsymbol{\pi}_0$. $\mathbf{A}^*(s,z)$ is given by

$$\begin{aligned}
&\mathbf{A}^*(s,z) = \{\lambda_2\mathbf{I} - \mathbf{T} - \mathbf{T}^0\mathbf{B}_1^*(\lambda_2,1)\}^{-1}[\lambda_2\mathbf{B}_{\mathbf{SM}}^*(s,z) \\
&+ \mathbf{T}^0\sum_{n=1}^{\infty}\{(\int_0^{\infty} e^{-sx}\exp(-\lambda_2 x)\frac{(\lambda_2 x)^n}{n!}\mathbf{B}_1(x,1)dx)\mathbf{B}_{\mathbf{SM}}^{*\,n}(s,z)\}] \\
&= \{\lambda_2\mathbf{I} - \mathbf{T} - \mathbf{T}^0\mathbf{B}_1^*(\lambda_2,1)\}^{-1}[\lambda_2\mathbf{B}_{\mathbf{SM}}^*(s,z) \\
&+ \mathbf{T}^0(\int_0^{\infty}\mathbf{B}_1(x,1)\exp\{(-(s+\lambda_2)\mathbf{I} + \lambda_2\mathbf{B}_{\mathbf{SM}}^*(s,z))x\}dx \\
&- \mathbf{B}_1^*(s+\lambda_2,1))].
\end{aligned} \tag{3.16}$$

Equation (3.16) can be derived in a manner similar to (3.12).

Equation (3.6) can be obtained in the following manner:

$\mathbf{q}_0$ is equivalent to the stationary arrival phase state probability vector at the service completion epoch for a non-priority customer who leaves the system empty. Therefore, we can write

$$\boldsymbol{\pi}_0 = K\mathbf{q}_0 \text{ for some } K > 0.$$

Let $Z_1(0), Z_2(0), \cdots$ denote the successive service completion epochs for non-priority customers who leave the server idle. The ratio of 1 to $\boldsymbol{\pi}_0\mathbf{e}$ is identical to the mean number of non-priority customers who complete their services between $Z_k(0)$ and $Z_{k+1}(0)$, that is,

$$K = \boldsymbol{\pi}_0\mathbf{e} = \frac{1}{\mathbf{q}_0(d\mathbf{A}^*(0,z)/dz\mid_{z=1})\mathbf{e}},$$

thus we obtain (3.6).

Equations (3.7) and (3.8) can be directly derived by (3.16) and (3.4).

Let us consider $(d\mathbf{A}^*(0,z)dz\mid_{z=1})\mathbf{e}$. It is clear that

$$\begin{aligned}
&\frac{d\mathbf{A}^*(0,z)}{dz}\mid_{z=1}\mathbf{e} = \{\lambda_2\mathbf{I} - \mathbf{T} - \mathbf{T}^0\mathbf{B}_1^*(\lambda_2,1)\}^{-1} \\
&\quad\cdot[\lambda_2\frac{d\mathbf{B}_{\mathbf{SM}}^*(0,z)}{dz}\mid_{z=1}\mathbf{e} + \mathbf{T}^0\sum_{n=1}^{\infty}(\int_0^{\infty}\exp(-\lambda_2 x)\frac{(\lambda_2 x)^n}{n!}\mathbf{B}_1(x,1)dx) \\
&\quad\cdot(\sum_{k=0}^{n-1}\mathbf{B}_{\mathbf{SM}}^{*\,k}(0,1))(\frac{d\mathbf{B}_{\mathbf{SM}}^*(0,z)}{dz}\mid_{z=1}\mathbf{e})].
\end{aligned} \tag{3.17}$$

Substituting

$$\begin{aligned}
&\sum_{n=1}^{\infty}(\int_0^{\infty}\exp(-\lambda_2 x)\frac{(\lambda_2 x)^n}{n!}\mathbf{B}_1(x,1)dx)(\sum_{k=0}^{n-1}\mathbf{B}_{\mathbf{SM}}^{*\,k}(0,1)) \\
&\quad\cdot(\mathbf{I} - \mathbf{B}_{\mathbf{SM}}^*(0,1) + \mathbf{e}\mathbf{b}_{\mathbf{SM}}) \\
&\quad= \mathbf{B}_1^*(0,1) - \mathbf{B}_1^*(\lambda_2,1) \\
&\quad-\sum_{n=1}^{\infty}(\int_0^{\infty}\exp(-\lambda_2 x)\frac{(\lambda_2 x)^n}{n!}\mathbf{B}_1(x,1)dx)\mathbf{B}_{\mathbf{SM}}^{*\,n}(0,1) \\
&\quad+\lambda_2(\int_0^{\infty}x\mathbf{B}_1(x,1)dx)\mathbf{e}\mathbf{b}_{\mathbf{SM}}
\end{aligned} \tag{3.18}$$

and

$$\begin{aligned}
\frac{d\mathbf{B}_{\mathbf{SM}}^*(0,z)}{dz}\mid_{z=1}\mathbf{e} &= (\mathbf{I} - \mathbf{B}_{\mathbf{SM}}^*(0,1) + \mathbf{e}\mathbf{b}_{\mathbf{SM}}) \\
&\quad\cdot\{\mathbf{I} - \mathbf{C}^*(0) + (\mathbf{e} + \lambda_2\mathbf{C}^{*\prime}(0)\mathbf{e})\mathbf{b}_{\mathbf{SM}}\}^{-1}\mathbf{e},
\end{aligned} \tag{3.19}$$

we obtain (3.9). See Neuts [12] for (3.19). □

The $\boldsymbol{\pi}_0$ obtained in Theorem 3.1 also gives interdeparture time distribution for nonpriority customers. That is, the LST $d^*(s)$ of the interdeparture time distribution is represented by

$$d^*(s) = \boldsymbol{\pi}_0\{(s+\lambda_2)\mathbf{I} - \mathbf{T} - \mathbf{T}^0\mathbf{B}_1^*(s+\lambda_2,1)\}^{-1} \\ \cdot [\lambda_2\mathbf{I} + \mathbf{T}^0\{\mathbf{B}_1^*(s,1) - \mathbf{B}_1^*(\lambda_2,1)\}]\mathbf{C}^*(s)\mathbf{e} + (\mathbf{g}(1) - \boldsymbol{\pi}_0)\mathbf{C}^*(s)\mathbf{e}. \tag{3.20}$$

See (3.28) for the realized form of $\mathbf{g}(1)$.

As mentioned by Neuts [11], $\mathbf{A}^*(s,z)$ and $[d\mathbf{A}^*(s,z)/dz]\mathbf{e}$ may be computed by successive substitutions. Under preemptive resume discipline, $\mathbf{A}^*(s,z)$, in particular, can be transformed into a more suitable computational form. Under this work-conserving service discipline, the server is indifferent as to whether or not the PH-Markov renewal customers have priority. Therefore, joint distribution of the busy period length and the number of service completions during this period are independent of the service order. This implies that $\mathbf{B}_{\mathbf{SM}}^*(s,z)$ and $\sum_{n=1}^{\infty}\{(\int_0^{\infty} e^{-sx}\exp(-\lambda_2 x)\frac{(\lambda_2 x)^n}{n!}\mathbf{B}_1(x,1)dx)\mathbf{B}_{\mathbf{SM}}^{*}{}^n(s,z)\}$ are independent of their service orders, and the following lemma is satisfied.

Lemma 3.2 For the preemptive resume discipline, one obtains

$$\mathbf{A}^*(0,1) = [\lambda_2\mathbf{I} - \mathbf{T} - \mathbf{T}^0\mathbf{B}_1^*(\lambda_2,1)]^{-1}[\lambda_2 B_2^*(-\mathbf{T} - \mathbf{T}^0\mathbf{B}_\mathbf{c}^*(0,1)) \\ + \mathbf{T}^0(\mathbf{B}_\mathbf{c}^*(0,1) - \mathbf{B}_1^*(\lambda_2,1))] \tag{3.21}$$

and

$$\frac{d\mathbf{A}^*(0,z)}{dz}\Big|_{z=1}\mathbf{e} = \frac{\lambda_2}{(1-\rho_2)}[\lambda_2\mathbf{I} - \mathbf{T} - \mathbf{T}^0\mathbf{B}_1^*(\lambda_2,1)]^{-1} \\ \cdot [\mathbf{I} + \lambda_2\mu_1^{-1}\frac{dB_2^*(-\mathbf{T} - \mathbf{T}^0\mathbf{B}_\mathbf{c}^*(0,z))}{dz}\Big|_{z=1} + \mu_1^{-1}\mathbf{T}^0\frac{d\mathbf{B}_\mathbf{c}^*(0,z)}{dz}\Big|_{z=1}]\mathbf{e}, \tag{3.22}$$

where $B_2^*(s)$ for $Re(s) \geq 0$ is the Laplace-Stieltjes transform of the $M(\lambda_2)/G(H_2)/1$ busy period length (arrival rate is λ_2 and service time distribution is $H_2(x)$) and satisfies

$$B_2^*(s) = H_2^*(s + \lambda_2 - \lambda_2 B_2^*(s)).$$

Also, $\mathbf{B}_\mathbf{c}^*(s,z)$ is the double transform of the length of the PH-$MR/G/1$ busy period, and the number of customers served during this period, where the arrival process is a PH-Markov renewal one with representation $(\boldsymbol{\alpha}, \mathbf{T}, \mathbf{T}^0)$ and service time distribution is general one whose LST for $Re(s) \geq 0$ is

$$H_c^*(s) := H^*(s + \lambda_2 - \lambda_2 B_2^*(s,1)).$$

$\mathbf{B}_\mathbf{c}^*(s,z)$ for $Re(s) \geq 0$ and $|z| \leq 1$ satisfies

$$\mathbf{B}_\mathbf{c}^*(s,z) = z\boldsymbol{\alpha} H_c^*(s\mathbf{I} - \mathbf{T} - \mathbf{T}^0\mathbf{B}_\mathbf{c}^*(s,z)). \tag{3.23}$$

Proof: Let us consider the other expression for $\mathbf{B}_{\mathbf{SM}}^*(s,z)$ defined in (3.4). Note that this $M/SM/1$ busy period is equivalent to the busy period of the PH-$MR, M/G_1, G_2/1$ considered in this section, when the busy period starts by the Poisson customer's arrival. Since this $M/SM/1$ busy period distribution is independent of the service order, we may consider the Poisson customers as having preemptive priority. Under this discipline, after the $M(\lambda_2)/G(H_2)/1$ busy period process, the PH-Markov renewal customers who have arrived during this busy period are served. Here, the services of the PH-Markov renewal customers are preempted by the Poisson customers again. Therefore, the busy period of the $M/SM/1$ can be decomposed into two parts. One is the $M(\lambda_2)/G(H_2)/1$ busy period T_b, whose LST is given by $B_2^*(s) = H_2^*(s+\lambda_2-\lambda_2 B_2^*(s))$. The other is the n-th busy period of the PH-$MR/G(H_c)/1$, where n is the number of PH-Markov renewal customers who arrive during the $M(\lambda_2)/G(H_2)/1$ busy period, and H_c means that the service time distribution is identical to the completion time distribution $H_c(t)$ whose LST satisfies $H_c^*(s) = H^*(s+\lambda_2-\lambda_2 B_2^*(s))$. Therefore, we can assume that the $M/SM/1$ busy period is identical to the completion time of service time T_b, (underlying the distribution in which LST is $B_2^*(s)$), which is preempted by priority is customers whose arrival representation is $(\boldsymbol{\alpha}, \mathbf{T}, \mathbf{T}^0)$, and service time density is $H_c(x)$. Therefore, from (2.1) we obtain

$$\mathbf{B}_{\mathbf{SM}}^*(s,z) = z^a B_2^*(s\mathbf{I} - \mathbf{T} - \mathbf{T}^0\mathbf{B}_{\mathbf{c}}^*(s,z^b)), \tag{3.24}$$

where $a = (1-\rho_2)^{-1}$, $b = \lambda_2\{\mu_1(1-\rho_2)\}^{-1}$
and

$$\mathbf{B}_{\mathbf{c}}^*(s,z) = z\boldsymbol{\alpha} H_c^*(s\mathbf{I} - \mathbf{T} - \mathbf{T}^0\mathbf{B}_{\mathbf{c}}^*(s,z)).$$

Here $(1-\rho_2)^{-1}$ and $\lambda_2\{\mu_1(1-\rho_2)\}^{-1}$ are the mean number of Poisson customers served during the $M(\lambda_2)/G(H_2)/1$ busy period and the mean number of Poisson customers served during the service time underlying the distribution $H_c(x)$, respectively.

Since

$$\sum_{n=0}^{\infty}\{(\int_0^{\infty} e^{-sx}\frac{(\lambda_2 x)^n \exp(-\lambda_2 x)}{n!}\mathbf{B}_1(x,1)dx)\mathbf{B}_{\mathbf{SM}}^{*\,n}(s,1)\}$$

is identical to the LST of the PH-$MR/G(H_c)/1$ busy period length, it follows that

$$\sum_{n=1}^{\infty}\{(\int_0^{\infty} e^{-sx}\frac{(\lambda_2 x)^n \exp(-\lambda_2 x)}{n!}\mathbf{B}_1(x,1)dx)\mathbf{B}_{\mathbf{SM}}^{*\,n}(s,1)\}$$
$$= \mathbf{B}_{\mathbf{c}}^*(s,1) - \mathbf{B}_{\mathbf{1}}^*(s+\lambda_2,1).$$

Now, we obtain (3.21) from (3.16).

From (3.24), we obtain

$$\frac{d\mathbf{B}_{\mathbf{SM}}^*(0,z)}{dz}\Big|_{z=1}\mathbf{e}$$
$$= [\frac{1}{1-\rho_2} + \frac{\lambda_2}{\mu_1(1-\rho_2)}\frac{dB_2^*(-\mathbf{T}-\mathbf{T}^0\mathbf{B}_{\mathbf{c}}^*(0,z))}{dz}\Big|_{z=1}]\mathbf{e}$$

and

$$\sum_{n=1}^{\infty}(\int_0^{\infty} e^{-sx}\frac{(\lambda_2 x)^n \exp(-\lambda_2 x)}{n!}\mathbf{B}_1(x,1)dx)\frac{d\mathbf{B}_{\mathbf{SM}}^{*\,n}(0,z)}{dz}\Big|_{z=1}$$
$$= \frac{\lambda_2}{\mu_1(1-\rho_2)}\frac{d\mathbf{B}_{\mathbf{c}}^*(0,z)}{dz}\Big|_{z=1}.$$

Thus, we obtain (3.22) from (3.16). □

Lemma 3.2 gives a computational method for $\mathbf{A}^*(0,1)$ and $d\mathbf{A}^*(0,z)/dz\,|_{z=1}\,\mathbf{e}$. It is well known that the LST of busy period distribution $\mathbf{B}_{\mathbf{c}}^*(0,1)$ can be computed by the following iterative substitutions [11]:

$$\mathbf{B}_{\mathbf{c}}^{*(0)}(0,1) = \boldsymbol{\alpha} H_c^*(-\mathbf{T}),$$

$$\mathbf{B}_{\mathbf{c}}^{*(n+1)}(0,1) = \boldsymbol{\alpha} H_c^*(-\mathbf{T} - \mathbf{T}^0\mathbf{B}_{\mathbf{c}}^{*(n)}(0,1)), \quad n = 0, 1, \cdots.$$

If $\mathbf{T} + \mathbf{T}^0\mathbf{B}_{\mathbf{c}}^{*(n)}(0,1)$ is diagonizable, then $H_c^*(-\mathbf{T} - \mathbf{T}^0\mathbf{B}_{\mathbf{c}}^{*(n)}(0,1))$ can be transformed by

$$\begin{aligned} &H_c^*(-\mathbf{T} - \mathbf{T}^0\mathbf{B}_{\mathbf{c}}^{*(n)}(0,1)) \\ &= \int_0^\infty \mathbf{P}^{(n)}\mathrm{diag}[\exp(-\gamma_1^{(n)}x), \cdots, \exp(-\gamma_m^{(n)}x)]\mathbf{P}^{(n)-1}dH_c(x) \\ &= \mathbf{P}^{(n)}\mathrm{diag}[H_c^*(\gamma_1^{(n)}), \cdots, H_c^*(\gamma_m^{(n)})]\mathbf{P}^{(n)-1} \end{aligned}$$

for $\mathbf{P}^{(n)-1}(-\mathbf{T} - \mathbf{T}^0\mathbf{B}_{\mathbf{c}}^{*(n)}(0,1))\mathbf{P}^{(n)} = \mathrm{diag}[\gamma_1^{(n)}, \cdots, \gamma_m^{(n)}]$.

For a sufficiently large $n = N$, $\mathbf{B}_{\mathbf{c}}^*(0,1)$ can be determined as

$$\mathbf{B}_{\mathbf{c}}^*(0,1) = \mathbf{B}_{\mathbf{c}}^{*(N+1)}(0,1) = \boldsymbol{\alpha}\mathbf{P}^{(N)}\mathrm{diag}[H_c^*(\gamma_1^{(N)}, \cdots, H_c^*(\gamma_m^{(N)})]\mathbf{P}^{(N)-1}$$

Without loss of generality, write $\gamma_i = \gamma_i^{(N)}$ and $\mathbf{P} = \mathbf{P}^{(N)}$. Then, the following lemma holds.

Lemma 3.3 For the preemptive resume discipline, if $\mathbf{T} + \mathbf{T}^0\mathbf{B}_{\mathbf{c}}^{*(n)}(0,1)$ is diagonizable for each n,

$$B_2^*(-\mathbf{T} - \mathbf{T}^0\mathbf{B}_{\mathbf{c}}^*(0,1)) = \mathbf{P}\mathrm{diag}(1, B_2^*(\gamma_2), \cdots, B_2^*(\gamma_m))\mathbf{P}^{-1}, \tag{3.25}$$

$$\begin{aligned} &\frac{d\mathbf{B}_{\mathbf{c}}^*(0,z)}{dz}\Big|_{z=1} \mathbf{e} \\ &= \{\mathbf{I} - \boldsymbol{\alpha}\mathbf{P}\mathrm{diag}(-H_c^{*\prime}(0), \frac{1 - H_c^*(\gamma_2)}{\gamma_2}, \cdots, \frac{1 - H_c^*(\gamma_m)}{\gamma_m})\mathbf{P}^{-1}\mathbf{T}^0\}^{-1}\mathbf{e} \end{aligned} \tag{3.26}$$

and

$$\begin{aligned} &\frac{dB_2^*(-\mathbf{T} - \mathbf{T}^0\mathbf{B}_{\mathbf{c}}^*(0,z))}{dz}\Big|_{z=1} \mathbf{e} \\ &= \mathbf{P}\mathrm{diag}[-B_2^{*\prime}(0), \frac{1 - B_2^*(\gamma_2)}{\gamma_2}, \cdots, \frac{1 - B_2^*(\gamma_m)}{\gamma_m}]\mathbf{P}^{-1}\mathbf{T}^0 \\ &\cdot \frac{d\mathbf{B}_{\mathbf{c}}^*(0,z)}{dz}\Big|_{z=1} \mathbf{e}, \end{aligned} \tag{3.27}$$

where

$$-H_c^*(0) = \frac{1}{\mu_1(1-\rho_2)} \text{ and } -B_2^{*\prime}(0) = \frac{1}{\mu_2(1-\rho_2)}.$$

Proof: Equation (3.25) is clear. Since $(\mathbf{T} + \mathbf{T}^0\mathbf{B}_{\mathbf{c}}^*(0,1))\mathbf{e} = \mathbf{0}$, we have that

$$\begin{aligned} \frac{d\mathbf{B}_{\mathbf{c}}^*(0,z)}{dz}\Big|_{z=1} \mathbf{e} &= \boldsymbol{\alpha} H_c^*(-\mathbf{T} - \mathbf{T}^0\mathbf{B}_{\mathbf{c}}^*(0,1))\mathbf{e} \\ &+ \boldsymbol{\alpha}\{\int_0^\infty \sum_{n=1}^\infty (x^n \frac{(\mathbf{T} + \mathbf{T}^0\mathbf{B}_{\mathbf{c}}^*(0,1))^{n-1}}{n!})dH_c(x)\}\mathbf{T}^0 \frac{d\mathbf{B}_{\mathbf{c}}^*(0,z)}{dz}\Big|_{z=1} \mathbf{e}. \end{aligned}$$

Using $-\mathbf{T} - \mathbf{T}^0\mathbf{B}_{\mathbf{c}}^*(0,1) = \mathbf{P}\mathrm{diag}(\gamma_1, \cdots, \gamma_m)\mathbf{P}^{-1}(\gamma_1 = 0)$, simple calculus shows

$$\begin{aligned} &\sum_{n=1}^\infty x^n \frac{(\mathbf{T} + \mathbf{T}^0\mathbf{B}_{\mathbf{c}}^*(0,1))^{n-1}}{n!} \\ &= \mathbf{P}\mathrm{diag}\{x, \frac{1 - \exp(-\gamma_2 x)}{\gamma_2}, \cdots, \frac{1 - \exp(-\gamma_m x)}{\gamma_m}\}\mathbf{P}^{-1}. \end{aligned}$$

We, therefore, obtain

$$\frac{d\mathbf{B}_c^*(0,z)}{dz}\,|_{z=1}\,\mathbf{e}=\mathbf{e}+\boldsymbol{\alpha}\mathbf{P}$$
$$\cdot\,\mathrm{diag}\{-H_c^*(0),\frac{1-H_c^*(\gamma_2)}{\gamma_2},\cdots,\frac{1-H_c^*(\gamma_m)}{\gamma_m}\}\mathbf{P}^{-1}\mathbf{T}^0\frac{d\mathbf{B}_c^*(0,z)}{dz}\,|_{z=1}\,\mathbf{e}.$$

Since the inverse of

$$\mathbf{I}-\boldsymbol{\alpha}\mathbf{P}\mathrm{diag}\{-H_c^*(0),\frac{1-H_c^*(\gamma_2)}{\gamma_2},\cdots,\frac{1-H_c^*(\gamma_m)}{\gamma_m}\}\mathbf{P}^{-1}\mathbf{T}^0$$

exists from stationary condition, $\rho<1$, (3.26) is proved.

Equation (3.27) can be similarly proved. □

If $\boldsymbol{\pi}_0$ can be determined, then the n-th derivative $\mathbf{g}^{(n)}(1)$ of the generating fuction $\mathbf{g}(z)$ at $z=1$ is given by the following theorem.

Theorem 3.4 $\mathbf{g}^{(n)}(1)(n=0,1,2,\cdots)$ is obtained for the invariant probability vector $\boldsymbol{\pi}$ of $\mathbf{C}^*(0)$ as

$$\mathbf{g}(1)=\mathbf{g}^{(0)}(1)=\boldsymbol{\pi}+\mathbf{U}(1)(\mathbf{I}-\mathbf{C}^*(0)+\mathbf{e}\boldsymbol{\pi})^{-1} \tag{3.28}$$

and

$$\theta_n=[\mathbf{U}^{(n)}(1)+\sum_{k=1}^{n}\binom{n}{k}(-\lambda_2)^k\mathbf{g}^{(n-k)}(1)\mathbf{C}^{*(k)}(0)-n\mathbf{g}^{(n-1)}(1)]$$
$$\cdot\,(\mathbf{I}-\mathbf{C}^*(0)+\mathbf{e}\boldsymbol{\pi})^{-1}(-\lambda_2)\mathbf{C}^{*\prime}(0)\mathbf{e}, \tag{3.29}$$

$$L_n=[(n+1)(1+\lambda_2\boldsymbol{\pi}\mathbf{C}^{*\prime}(0)\mathbf{e})]^{-1}$$
$$\cdot\,[(n+1)\theta_n+\mathbf{U}^{(n+1)}(1)\mathbf{e}+\sum_{k=2}^{n+1}(\binom{n+1}{k})\mathbf{g}^{(n+1-k)}(1)(-\lambda_2)^k\mathbf{C}^{*(k)}(0)\mathbf{e}], \tag{3.30}$$

$$\mathbf{g}^{(n)}(1)=L_n\boldsymbol{\pi}+[\mathbf{U}^{(n)}(1)+\sum_{k=1}^{n}\binom{n}{k}\mathbf{g}^{(n-k)}(1)(-\lambda_2)^k\mathbf{C}^{*(k)}(0)-n\mathbf{g}^{(n-1)}(1)]$$
$$\cdot\,(\mathbf{I}-\mathbf{C}^*(0)+\mathbf{e}\boldsymbol{\pi})^{-1},$$
$$n=1,2,\cdots, \tag{3.31}$$

where

$$\mathbf{U}(1)=\mathbf{U}^{(0)}(1)=\boldsymbol{\pi}_0\{\lambda_2\mathbf{I}-\mathbf{T}-\mathbf{T}^0\mathbf{B}_1^*(\lambda_2,1)\}^{-1}(\mathbf{T}+\mathbf{T}^0\mathbf{B}_1^*(0,1))\mathbf{C}^*(0), \tag{3.32}$$

$$\mathbf{U}^{(n)}(1)=\boldsymbol{\pi}_0\{\lambda_2\mathbf{I}-\mathbf{T}-\mathbf{T}^0\mathbf{B}_1^*(\lambda_2,1)\}^{-1}$$
$$\cdot\,\frac{d^n}{dz^n}\{\lambda_2z\mathbf{I}-\lambda_2\mathbf{I}+\mathbf{T}+\mathbf{T}^0\mathbf{B}_1^*(\lambda_2(1-z),1)\}\mathbf{C}^*(\lambda_2(1-z))\,|_{z=1}\quad,$$
$$n=1,2,\cdots. \tag{3.33}$$

Proof: Using similar discussions to those in Neuts [12] (pp. 143–148), we have the theorem. □

Next, we will consider the waiting times for non-priority Poisson customers. Let $\mathbf{w}_N^*(s)$ for $Re(s) \geq 0$ denote the LST of the distribution vector $\mathbf{w}_N(t)$ of the waiting time for an arbitrary non-priority Poisson customer waiting in the queue for first-time service. The LST of the sojourn time distribution of this arbitrary customer is given by $\mathbf{h}_N^*(s) = \mathbf{w}_N^*(s)\mathbf{C}^*(s)$.

Theorem 3.5 We have

$$\begin{aligned}\mathbf{w}_N{}^*(s)[(s-\lambda_2)I + \lambda_2\mathbf{C}^*(s)] \\ = \lambda_2\boldsymbol{\pi}_0[\lambda_2\mathbf{I} - \mathbf{T} - \mathbf{T}^0\mathbf{B}_1^*(\lambda_2,1)]^{-1}[s\mathbf{I} - \mathbf{T} - \mathbf{T}^0\mathbf{B}_1^*(s,1))]\end{aligned} \tag{3.34}$$

and

$$\mathbf{g}(z) = \mathbf{h}_N{}^*(\lambda_2(1-z)) = \mathbf{w}_N{}^*(\lambda_2(1-z))\mathbf{C}^*(\lambda_2(1-z)). \tag{3.35}$$

Proof: Using similar discussions to those in Neuts [12] (pp. 184–189), we have (3.34) and (3.35). □

Using (3.34) and (3.35), we obtain the n-th moment of waiting times for non-priority Poisson customers as follows:

$$E[\mathbf{w}_N{}^0] = \mathbf{w}_N{}^*(0) \text{ for } n = 0,$$

where $\mathbf{w}_N{}^*(0)$ is the probability vector satisfying

$$\mathbf{g}(1) = \mathbf{w}_N{}^*(0)\mathbf{C}^*(0) \text{ and } \mathbf{w}_N{}^*(0)\mathbf{e} = 1,$$

and

$$E[\mathbf{w}_N{}^n] = (-1)^n \frac{d^n \mathbf{w}_N{}^*(s)}{ds^n}\Big|_{s=0} \text{ for } n = 1, 2, \cdots.$$

That is, for L_n given in (3.30),

$$E[\mathbf{w}_N{}^n]\mathbf{e} = \lambda_2^{-n}L_n - \sum_{k=0}^{n-1}\binom{n}{k}E[\mathbf{w}_N{}^k](-1)^{n-k}\mathbf{C}^{*(n-k)}(0)\mathbf{e}, \tag{3.36}$$

$$\begin{aligned}E[\mathbf{w}_N{}^n] &= [-\lambda_2^{-1}nE[\mathbf{w}_N{}^{n-1}] + \sum_{k=0}^{n-1}\binom{n}{k}E[\mathbf{w}_N{}^k](-1)^{n-k}\mathbf{C}^{*(n-k)}(0) \\ &+ (-1)^n\boldsymbol{\pi}_0[\mathbf{I} - \mathbf{T} - \mathbf{T}^0\mathbf{B}_1^*(\lambda_2,1)]^{-1}\mathbf{T}^0\mathbf{B}_1^{*(n)}(0,1)](\mathbf{I} - \mathbf{C}^*(0) + \mathbf{e}\boldsymbol{\pi})^{-1} \\ &+ E[\mathbf{w}_N{}^n]\mathbf{e}\boldsymbol{\pi},\ n = 1, 2, \cdots.\end{aligned} \tag{3.37}$$

We write

$$\mathbf{C}^{*(n)}(0) = \frac{d^n\mathbf{C}^*(s)}{ds^n}\Big|_{s=0} \text{ and } \mathbf{B}_1^{*(n)}(0,1) = \frac{d^n\mathbf{B}_1^*(s,1)}{ds^n}\Big|_{s=0}.$$

4. Numerical Examples

Figure 4.1 shows the mean sojourn time $E[h_N]$ of the non-priority Poissonian customers for a $PH\text{-}MR, M/D_1, D_2/1$ with preemptive resume discipline, where priority and non-priority customer service times have constants, $\mu_1^{-1} = d_1$ and $\mu_2^{-1} = d_2 = 1$, respectively. The arrival process of the priority customers is a two-state Markov modulated Poisson process

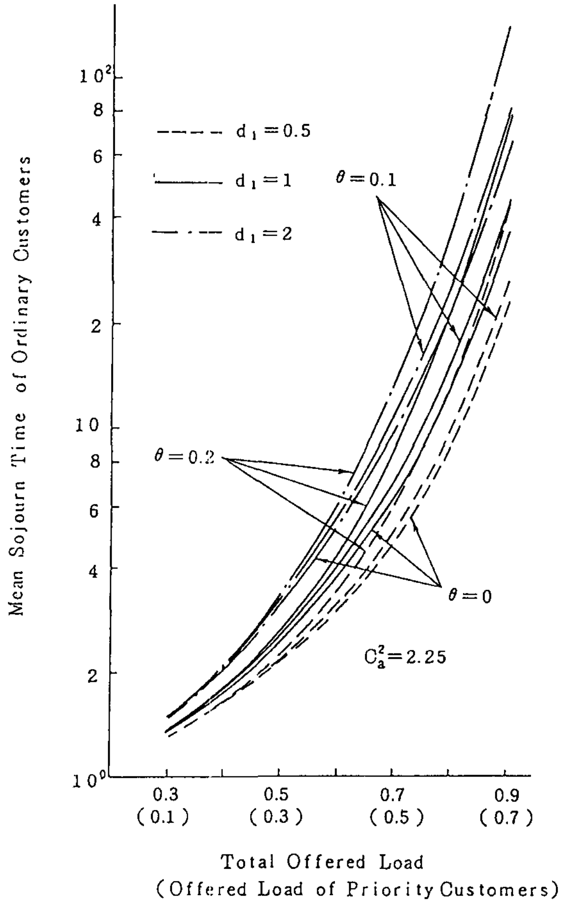


Fig. 4.1 Mean Sojourn Time

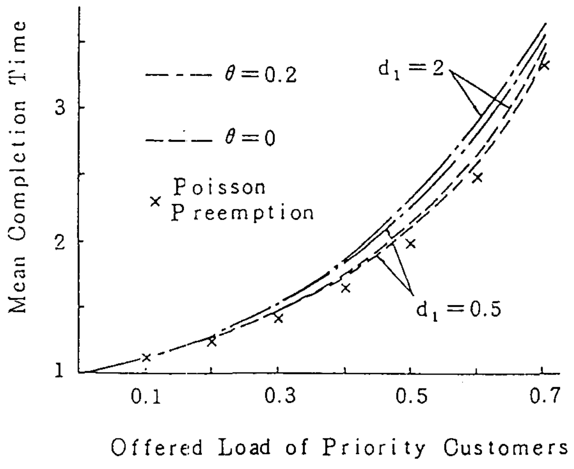


Fig. 4.2 Mean Completion Time

(two-state $MMPP$), often called a switched Poisson process [19]. This is an important model for packet multiplexers handling voice and data [5]. A two-state $MMPP$ is governed by

$$\boldsymbol{\alpha} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{T} = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} \text{ and } \mathbf{T}^0 = \begin{pmatrix} t_{11}^0 & 0 \\ 0 & t_{22}^0 \end{pmatrix},$$

where $t_{11} = -t_{12} - t_{11}^0$ and $t_{22} = -t_{21} - t_{22}^0$.

The four parameters, $t_{ij} (1 \le i, j \le 2)$ are determined by arrival rare λ_1, the squared coefficient of the variation for interarrival times c_a^2, the balanced condition and the autocorrelation coefficient of a sequence of interarrival times θ in the same manner as [19]. Poisson arrival rate $\lambda_2 = 0.2$ and mean service time $d_2 = 1$ are fixed.

The influence of θ is very noticeable, which implies that it is inadvisable to approximate the voice-packet arrival process with a renewal process. The influence of service time d_1 is noticeable. Therefore, it is uncertain whether the priority $PH\text{-}MR, M/G_1, G_2/1$ queue can be closely approximated by the model $PH\text{-}MR, M/G/1$. With this model, the service time distribution is an appropriate mixture of the two customer classes. However, the $FIFO$ queue can be closely approximated [5].

The results mentioned above can be directly applied to a queue where the $PH\text{-}MR$ customers have non-preemptive priority. This is because the waiting time distribution of the non-priority Poisson customers for the non-preemptive queue is identical to that for the preemptive queue [17, 18]. Therefore, the LST, $h_{\mathrm{N}}^{(NP)*}(s)$, of the sojourn tome distribution of the non-priority customers is given by $(\mathbf{w}_{\mathrm{N}}{}^*(s)\mathbf{e})H_2^*(s)$. In particular, mean sojourn time is given by

$$E[h_{\mathrm{N}}^{(NP)}] = E[\mathbf{w}_{\mathrm{N}}]\mathbf{e} + \frac{1}{\mu_2} = E[h_{\mathrm{N}}] - (\mathbf{w}_{\mathrm{N}}{}^*(0)(-\mathbf{C}^{*\prime}(0))\mathbf{e} - \frac{1}{\mu_2}). \tag{4.1}$$

Figure 4.2 shows mean completion time $\mathbf{w}_{\mathrm{N}}{}^*(0)(-\mathbf{C}^{*\prime}(0))\mathbf{e} = E[h_{\mathrm{N}}] - E[h_{\mathrm{N}}^{(NP)}] + \mu_2^{-1}$. The mean completion time with non-Poissonian preemption is generally different from the well-known result for Poissonian preemption, that is,

$$\mathbf{w}_{\mathrm{N}}{}^*(0)(-\mathbf{C}^{*\prime}(0))\mathbf{e} \neq \boldsymbol{\pi}(-\mathbf{C}^{*\prime}(0))\mathbf{e} = \frac{1}{\mu_2(1-\rho_1)}. \tag{4.2}$$

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