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OPTIMAL SECOND REPLENISHMENT POLICY IN TWO-PHASED PUSH CONTROL SYSTEM

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Abstract A two-echelon push control system with one central warehouse and m branch warehouses is considered. The system is replenished once every cycle from an outside supplier and each cycle is divided into two different phases. While retaining the safety stock at the central warehouse, the bulk of inventories is directly shipped to the branch warehouses at the beginning of the first phase, and at the beginning of the second phase, a second replenishment is coordinated by the central warehouse through monitoring all the inventory levels in the branch warehouses. We call this kind of system a ***two-phased push control system***. The objective of the system is to minimize the expected number of system backorders. Based on an extended model from Jönsson and Silver, an optimal allocation policy is developed for the non-transshipments condition, and the existence of an optimal second replenishment period which minimizes the expected number of system backorders is also shown. The relationship between the optimal second replenishment period and system conditions is examined through numerical examples.

1 Introduction

Inventory systems can be broadly classified into "*pull control systems*" and "*push control systems*"[1]. In pull control systems, inventory replenishments are decided by each warehouse in each echelon independently, and the warehouses in successive echelons 'pull' inventories from preceding echelons(*suppliers*) when the inventory level drops to reorder point. However, in push control systems, all inventory replenishments are decided by the central warehouse from a system-wide viewpoint through monitoring inventory levels and demand information in the system. The central warehouse 'pushes' inventories from preceding echelons to the successive echelons, which need more inventories to prevent backorders. It is known that this kind of push control system is able to reduce the inventories under a given customer service level[3][7].

In this paper, we consider the optimal allocation rule and timing on the second replenishment of inventories for a single item in a (1, m) type inventory system(one central warehouse and m branch warehouses). The system is replenished once every cycle from an outside supplier, and each cycle consists of a fixed number of periods. Every cycle is divided into two different phases. The first one is from the first period to the second replenishment period, and the second one consists of the remaining periods in each cycle.

Jönsson and Silver[5] deal with this same model, which they call a *particular push inventory control system* under several constraints such as non-transshipments, identical demand distribution and a fixed second replenishment period. And they concluded that their *particular push control system* has better performance than a simple 'ship-all' policy. Based on a complete redistribution model in their prior paper[4], they found that almost all backorders occurred in the last period in each branch warehouse. From this, they proposed fixing the second replenishment period to the beginning of the last period. This assumption is also used in their later paper[5] no matter what the system backorders in the second phase(last period) are significantly reduced by organizing a second replenishment hence its value may be less than those in the first phase.

However, for example, when the coefficient of variation of demand is relatively large, or the system replenishment cycle length is relatively long, or the quantity of direct shipment(first replenishment quantity) is relatively small, the second replenishment period should be expedited to prevent from a further increase in backorders in the first phase. Hence, from the viewpoint of minimizing the system backorders in both the first and second phases, the second replenishment period should be adjusted according to the system conditions, and an optimal second replenishment period may exist for given system conditions. Furthermore, the identical demand distribution is a particular case of demand distribution, the Jönsson-Silver model[5] should be generalized and extended to non-identical condition.

After providing a more detailed explanation of our model, we first formulate the model and derive an optimal allocation policy for the second replenishment. Through the computer simulation(SLAM II) we indicate an existence of optimal second replenishment period. Finally, using an orthogonal array experiment to vary the system condition systematically, we present a relationship between the optimal second replenishment period and system conditions, such as coefficient of variation of demand, ratio of retained stock divided by initial system stock, system replenishment cycle length, and the total number of branch warehouses.

2 Model and Formulation

2.1 Nomenclature

CW : Central warehouse
BW_i : i-th branch warehouse
B : Set of branch warehouses receiving the second replenishment of stock
m : Total number of branch warehouses
n_b : Number of branch warehouses belonging to set B
N_b : Set of subscript of branch warehouses belonging to set B, $N_b = \{1, 2, \cdots, n_b\}$
H : Cycle length of system replenishment
t_1 : Second replenishment period (length of the first phase, $0 < t_1 < H$)
t_1^* : Optimal second replenishment period
τ : Remained periods after the second replenishment in each cycle (length of the second phase, $\tau = H - t_1$)
S_i : Stock level in BW_i at the beginning of each cycle
I_d : Sum of S_i, $I_d = \sum_{i=1}^{m} S_i$
I_c : Inventories initially retained at the central warehouse
I_0 : Initial system inventories at the beginning of each cycle($I_0 = I_d + I_c$)
I_i : On hand stock in BW_i at the end of the first phase (inventory level before receiving second replenishment of stock)
Z_i : Standard I_i, $Z_i = (I_i - \tau\mu_i)/\sqrt{\tau}\sigma_i$
I_i' : Optimal ship-up-to-level in $BW_i \in B$ at the beginning of the second phase. (inventory level after receiving second replenishment of stock)
Z_i' : Standard I_i', $Z_i' = (I_i' - \tau\mu_i)/\sqrt{\tau}\sigma_i$
I_c' : Standard I_c, $I_c' = \sum_{i=1}^{n_b}(Z_i' - Z_i)$

$\triangle_i$	:	Second replenishment quantity of stock shipped to $BW_i \in B$, $\triangle_i = I_i' - I_i$
μ_i	:	Mean demand per period at BW_i
σ_i^2	:	Variance of demand per period at BW_i
$f(\cdot)$	:	p.d.f. of normal distribution
$\phi(\cdot)$	:	p.d.f. of standard normal distribution
$\psi(k)$	:	Upper probability of normal distribution ($\psi(k) = \int_k^\infty \phi(x)dx$)
$G(k)$	:	Unit normal loss function ($G(k) = \int_k^\infty (x-k)\phi(x)dx$)
$R(\cdot)$	:	Expected number of system backorders

2.2 Model

We consider a $(1, m)$ type inventory system as shown in Figure 1. The CW has two main roles: to keep system safety stock so as to permit benefits from the portfolio effect[3]; and to coordinate its second allocation to the branch warehouses so as to minimize the system backorders. Each branch warehouse has two opportunities to be replenished in each cycle, and demands from outside customers are only occurred in the branch warehouses.

We have following assumptions on our model.

(1) Demand in each BW_i per period is non-negative and independent, and normally distributed with mean μ_i and variance σ_i^2.

(2) Lead time from CW to each BW is negligible compared with the lead time from outside supplier to CW.

(3) Excess demand in each period in each BW_i is backordered and recovered at the next replenishment time.

(4) Transshipments among the branch warehouses are not permitted.

In the model, S_i, I_c and H are predetermined or given. The system order quantity of stock in each cycle is assumed to be sufficient to restore the stock levels in each BW_i to a fixed level S_i. In the following numerical examinations in section **4**, we determined each S_i, $i = 1, 2, \cdots, m$, to be equivalent to mean demand in a cycle. The cycle length of system replenishment H, for example, can be chosen to minimize the system inventory holding cost

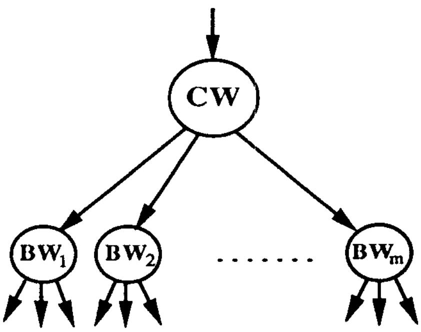
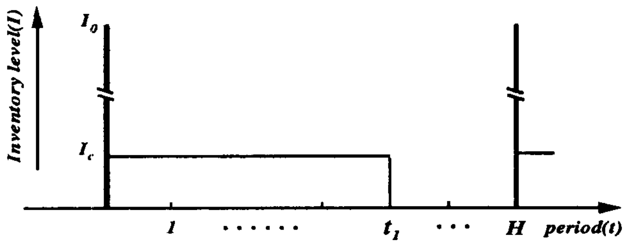
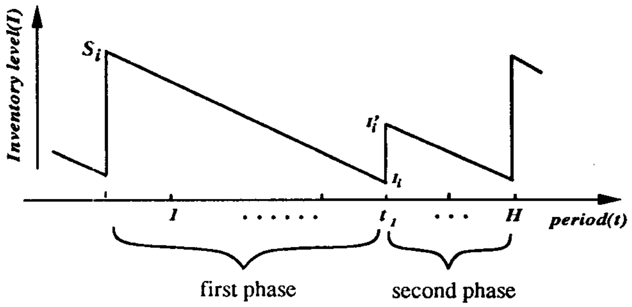


Figure 1: A $(1, m)$ type inventory system



(a) Changes in mean inventory at CW over a cycle



(b) Changes in mean inventory at a BW_i over a cycle

Figure 2: Changes in mean inventory at CW and BW_i

and the set-up cost for the system replenishment from outside supplier. The inventories initially retained at CW, I_c, can be considered as the equivalent of system safety stock, and under the same service level, its value is supposed to be normally less than the sum of safety stocks which is calculated independently and held individually in the branch warehouses[3].

At the beginning of each cycle, while retaining system safety stock I_c in CW, the remainder is directly shipped to the branch warehouses, and the stock level in each branch warehouse is restored to S_i, $i = 1, 2, \cdots, m$. At the end of the first phase, the CW ships out all of the retained stock to selected branch warehouses through monitoring inventory levels in all the branch warehouses.

Figure 2 shows the changes in mean inventory at CW and a BW_i over a cycle.

Our objective in this model is to minimize the expected number of system backorders in each cycle. Letting $R_i(1 \leq t \leq H)$ denote the expected number of backorders in BW_i per cycle, the objective function can be written,

(1) $$min. \quad \sum_{i=1}^{m} R_i(1 \leq t \leq H)$$

2.3 Formulation

For convenience, we consider the system backorders in each phase separately and suppose that $R_i(1 \leq t \leq t_1)$ and $R_i(t_1 < t \leq H)$ are the expected number of backorders in the BW_i in the first phase and second phase respectively. Hence, the expected number of system backorders in a cycle can be written,

$$\sum_{i=1}^{m} R_i(1 \le t \le H) = \sum_{i=1}^{m} R_i(1 \le t \le t_1) + \sum_{i=1}^{m} R_i(t_1 < t \le H) \tag{2}$$

First, $R_i(1 \le t \le t_1)$ can be formulated using the well known formula developed by Roger[6],

$$\sum_{i=1}^{m} R_i(1 \le t \le t_1) = \sum_{i=1}^{m} \sqrt{t_1}\sigma_i G(\frac{S_i - t_1\mu_i}{\sqrt{t_1}\sigma_i}) \tag{3}$$

Letting $\triangle_i$ denote the second replenishment quantity of stock which is shipped to selected BW_i, $i = 1, 2, \cdots, n_b$, the number of backorders in the second phase can be easily written as follows from analogy with equation (3),

$$\sum_{i=1}^{m} R_i(t_1 < t \le H) = \sum_{i=1}^{n_b} \sqrt{\tau}\sigma_i G(\frac{I_i + \triangle_i - \tau\mu_i}{\sqrt{\tau}\sigma_i}) + \sum_{i=n_b+1}^{m} \sqrt{\tau}\sigma_i G(\frac{I_i - \tau\mu_i}{\sqrt{\tau}\sigma_i}) \tag{4}$$

Since I_i is a normally distributed independent random variable with mean $S_i - t_1\mu_i$ and variance $t_1\sigma_i^2$, and $\triangle_i$ is a control variable which depends on I_i and I_c, the expected number of backorders in the second phase can be rewritten as,

$$\sum_{i=1}^{m} R_i(t_1 < t \le H) = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} (\sum_{i=1}^{n_b} \sqrt{\tau}\sigma_i G(\frac{I_i + \triangle_i - \tau\mu_i}{\sqrt{\tau}\sigma_i}) + \sum_{i=n_b+1}^{m} \sqrt{\tau}\sigma_i G(\frac{I_i - \tau\mu_i}{\sqrt{\tau}\sigma_i})) f(I_1) \cdots f(I_m) dI_1 \cdots dI_m \tag{5}$$

Combining equation (3) and equation (5), we can obtain the function for the total expected number of system backorders as below.

$$\sum_{i=1}^{m} R_i(1 \le t \le H) = \sum_{i=1}^{m} \sqrt{t_1}\sigma_i G(\frac{S_i - t_1\mu_i}{\sqrt{t_1}\sigma_i}) + \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} (\sum_{i=1}^{n_b} \sqrt{\tau}\sigma_i G(\frac{I_i + \triangle_i - \tau\mu_i}{\sqrt{\tau}\sigma_i}) + \sum_{i=n_b+1}^{m} \sqrt{\tau}\sigma_i G(\frac{I_i - \tau\mu_i}{\sqrt{\tau}\sigma_i})) f(I_1) \cdots f(I_m) dI_1 \cdots dI_m \tag{6}$$

3 Optimal Allocation Policy

3.1 Standard On-Hand Stock and Optimal Allocation

The optimal allocation policy in our model is used to decide the following two problems.

- Selection of the branch warehouses for the allocation.
- Determination of the allocation quantity $\triangle_i$, $i = 1, 2, \cdots, n_b$, or ship-up-to-level I_i', $i = 1, 2, \cdots, n_b$, for the selected branch warehouses.

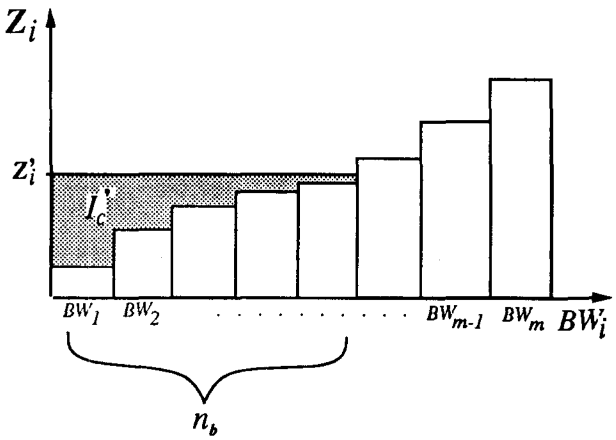


Figure 3: Ordered Z_i and the proposed allocation policy

For example, under the condition of identical demand distribution, we can minimize the expected number of system backorders using a simple optimal allocation policy, e.g. select the BW_i which holds less on-hand stock I_i for the allocation and determine all the ship-up-to-level I_i', $i = 1, 2, \cdots, n_b$, to be equal.

However, under the non-identical demand distribution, such an allocation policy may no longer apply. This is because some branch warehouses may hold little but sufficient inventories for the second phase, and hence need no more inventories to be allocated for the second phase. Therefore, we introduce a standard on-hand stock level Z_i which can be calculated by equation(7), and we arrange all branch warehouses in ascending order of Z_i as shown in Figure 3.

(7) $$Z_i = \frac{I_i - \tau\mu_i}{\sqrt{\tau}\sigma_i}$$

(8) $$Z_i' = \frac{I_i' - \tau\mu_i}{\sqrt{\tau}\sigma_i}$$

Using the Z_i, we extend the allocation policy under the identical demand distribution, and propose an optimal allocation policy for our non-identical demand distribution model as below.

Optimal allocation policy:

- Select BW_i which results in the lowest standard inventories Z_i, $i = 1, 2, \cdots, n_b$, for the allocation(see Figure 3).
- Determine optimal ship-up-to-levels I_i', $i = 1, 2, \cdots, n_b$, and its standard levels Z_i' to be equal.

We derive the optimal ship-up-to-level in section **3.2**, and develop an algorithm for finding the set B or selecting $BW_i \in B$ in section **3.3**. We also prove the optimality of the proposed allocation policy in section **3.3**.

Figure 3 illustrates an ordered Z_i and the proposed allocation policy. I'_c denotes the standard retained stock, and its value can be determined by equation (9). Also, n_b denotes the number of selected branch warehouses. The BW_i, $i = 1, 2, \cdots, n_b$, form set B, and the subscriptions of the BW_i form set $N_b(N_b = \{1, 2, \cdots, n_b\})$.

$$I'_c = \sum_{i \in N_b} (Z'_i - Z_i) \tag{9}$$

3.2 Optimal Ship-Up-To-Level

Under the condition of predetermined I_c, we find that the set B(or n_b) and Z'_i are interdependent(see Figure 3). Therefore, we temporarily assume that B is given, and derive the optimal ship-up-to-level in this section.

Our objective here is to determine the Δ_i at the end of period t_1 which minimizes $\sum_{i=1}^{m} R_i(1 \le t \le H)$. Since Δ_i is unrelated to $\sum_{i=1}^{m} R_i(1 \le t \le t_1)$ and $\sum_{i=n_b+1}^{m} R_i(t_1 < t \le H)$, the problem becomes one of minimizing $\sum_{i=1}^{n_b} R_i(t_1 < t \le H)$ under the constraint I_c is given. Therefore, the problem can be formulated as follows.

Objective

$$min. \quad \sum_{i \in N_b} \sqrt{\tau}\sigma_i G\left(\frac{I_i + \Delta_i - \tau\mu_i}{\sqrt{\tau}\sigma_i}\right) \tag{10}$$

Subject to

$$\sum_{i \in N_b} \Delta_i = I_c$$

The objective function (10) reflects the expected number of backorders in those branch warehouses($BW \in B$) whose standard stock level is relatively low and should be replenished at the end of period t_1. The constraint implies that no additional stock may be demanded in the second phase.

Applying Lagrange Multipliers and utilizing properties such as the monotonicity of the distribution function, we can derive the following computation function of the optimal ship-up-to-level(see Appendix).

$$I'_i = \tau\mu_i + \frac{\sigma_i}{\sum_{j \in N_b} \sigma_j}(I_c + \sum_{j \in N_b} (I_j - \tau\mu_j)) \,, \qquad i \in N_b \tag{11}$$

where

$$I'_i = I_i + \Delta_i$$

Hence, the second replenishment quantities for shipment to the BW_i can be given by equation (12), and the Z'_i by equation (13), (14). Needless to say that all the optimal standard ship-up-to-levels Z'_i, $i = 1, 2, \cdots, n_b$, are equal.

$$\Delta_i = I'_i - I_i \qquad i \in N_b \tag{12}$$

$$Z'_1 = Z'_2 = \cdots = Z'_m = Z'_0 \tag{13}$$

$$Z'_0 = \frac{I_c + \sum_{i \in N_b}(I_i - \tau\mu_i)}{\sqrt{\tau}\sum_{i \in N_b} \sigma_i} \tag{14}$$

The branch warehouses not belonging to set B are not replenished, and there are no changes in their stock levels I_i, $i = n_b + 1, \cdots, m$.

3.3 Algorithm for Finding an Optimal Set B

Based on the optimal ship-up-to-level derived in section **3.2**, we propose an algorithm for finding an optimal set $B(BW_i|I_i' \geq I_i)$ which minimizes the expected number of system backorders. We have two constraints on the problem, $I_i' \geq I_i$(same as $Z_i' \geq Z_i$), and predetermined I_c. Furthermore, all the on-hand stocks $I_i, i = 1, 2, \cdots, m$, are given, since all branch warehouse inventory levels in each period are totally monitored by the central warehouse in the system.

The algorithm for finding the optimal set B, is given briefly as follows:

Step 1. Letting I_c=0, n_b=m, compute all the ship-up-to-levels I_i' for BW_i, $i = 1, 2, \cdots, m$, using equation (11). Comparing I_i' and I_i, find an initial set $B(BW_i|I_i' \geq I_i)$, and its supplementary set $B'(BW_i|I_i' < I_i)$. The number of $BW \in B$ is chosen to be the initial n_b. Go to next step.

Step 2. Substitute I_c with the actual value of retained stock, and recompute I_i' for $BW_i \in B$ using the initial n_b. Go to next step.

Step 3. If there exists any BW_i violating $I_i' \geq I_i$ in the set B, then go to the next step. Otherwise, go to step 6.

Step 4. Delete the BW_i which violates the constraint $I_i' \geq I_i$ and has maximum value of Z_i from set B, and let $n_b = n_b - 1$. Recompute I_i' for $BW_i \in B$ using the equation (11).

Step 5. Check the constraint. If all the branch warehouses which belong to set B satisfy the constraint, then record the results, B, n_b, and I_i', and finish. Otherwise, return to step 4.

Step 6. Find the BW_j which has the minimum value of Z_i among set B', and add it to set B. Let $n_b = n_b + 1$, and recompute I_i' for $BW_i \in B$ using equation (11). Go to the next step.

Step 7. Check the constraint. If all the branch warehouses which belong to set B still satisfy the constraint, then return to step 6. Otherwise, record the results, B, n_b and I_i'(computed before the constraint violation) and finish.

Figure 4 shows the algorithm and the relationships between the steps. In this way, we can find an optimal set B, and its optimal ship-up-to level I_i' which can minimize the expected number of system backorders. Furthermore, the proposed algorithm can guarantees optimality of the set B and I_i', as verified below.

Proposition : For the I_i' and set B which are obtained by the proposed algorithm, the expected number of system backorders is increased or remains unchanged, if any one or more $BW \in B$ is exchanged for any $BW \in B'$ or eliminated from set B.

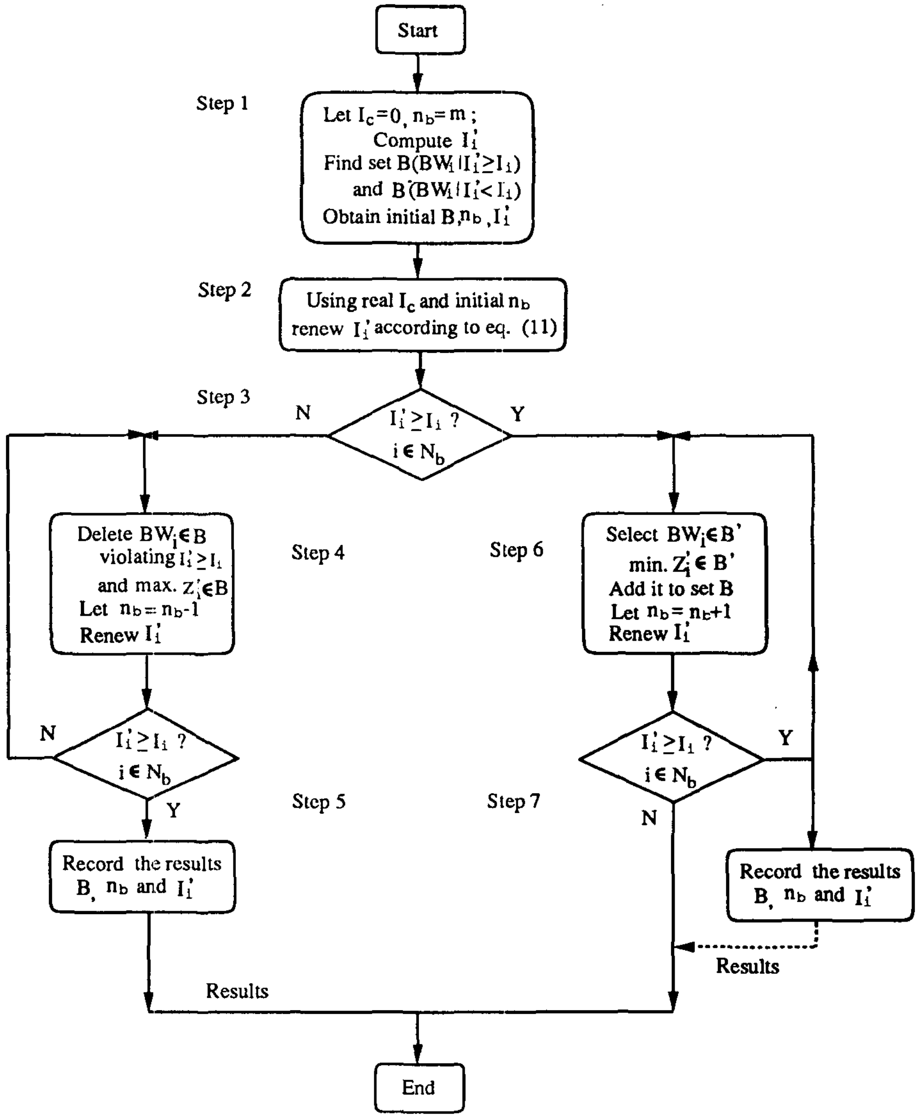


Figure 4: Algorithm for finding the optimal set B

Proof : Deleting an arbitrarily selected BW_k from set B we form a set B_1, and deleting an arbitrarily selected BW_p from set B' we form a set B'_1(see Figure 5). Letting R_B and $R_{B'}$ denote the sum of expected backorders in set B and set B' respectively, the R_B and $R_{B'}$ can be written as follows:

$$R_B = \sum_{BW_i \in B_1} \sqrt{\tau}\sigma_i G(Z'_0) + \sqrt{\tau}\sigma_k G(Z'_0)$$

$$R_{B'} = \sum_{BW_j \in B'_1} \sqrt{\tau}\sigma_j G(Z_j) + \sqrt{\tau}\sigma_p G(Z_p)$$

Exchanging $BW_k \in B$ for $BW_p \in B'$, we construct a new set B_n including BW_p and a new set B'_n including BW_k. Letting R_{B_n} and $R_{B'_n}$ represent the sum of expected number of backorders in the new set B_n and B'_n respectively, the R_{B_n} and $R_{B'_n}$ can be written as follows:

$$R_{B_n} = \sum_{BW_i \in B_1} \sqrt{\tau}\sigma_i G(Z''_0) + \sqrt{\tau}\sigma_p G(Z''_p)$$

$$R_{B'_n} = \sum_{BW_j \in B'_1} \sqrt{\tau}\sigma_j G(Z_j) + \sqrt{\tau}\sigma_k G(Z_k)$$

where

$$Z''_0 = \frac{I''_i - \tau\mu_i}{\sqrt{\tau}\sigma_i} \qquad BW_i \in B_n$$

$$Z''_p = max.\{Z''_0, Z_p\} \qquad BW_p \in B_n$$

and I''_i represents the optimal ship-up-to-level of $BW_i \in B_n$.

Our objective is to prove that for any $BW_k \in B$ and $BW_p \in B'$, the expected number of system backorders, $R_{B_n} + R_{B'_n}$, is always greater than or equal to $R_B + R_{B'}$. As the Z'_0 and Z''_0 are unrelated to i, the difference between $R_{B_n} + R_{B'_n}$ and $R_B + R_{B'}$ is given as below.

$$(R_{B_n} + R_{B'_n}) - (R_B + R_{B'}) = (\sum_{BW_i \in B_1} \sqrt{\tau}\sigma_i)(G(Z''_0) - G(Z'_0)) + \sqrt{\tau}\sigma_p(G(Z''_p) - G(Z_p)) + \sqrt{\tau}\sigma_k(G(Z_k) - G(Z'_0)) \tag{15}$$

Since $\frac{dG(Z_i)}{dZ_i} = -\psi(Z_i) < 0$, $G(Z_i)$ is a monotonously decreasing continuous function. Utilizing the properties of monotonous and continuous function, we can find a Z_{00} between Z'_0 and Z''_0 which satisfies

$$G(Z''_0) = G(Z'_0) - (Z''_0 - Z'_0)\psi(Z_{00}) \;, \tag{16}$$

find a Z_{k0} between Z_k and Z'_0 which satisfies

$$G(Z_k) = G(Z'_0) + (Z'_0 - Z_k)\psi(Z_{k0}) \tag{17}$$

and also find a Z_{pp} between Z_p and Z''_p which satisfies

$$G(Z''_p) = G(Z_p) - (Z''_p - Z_p)\psi(Z_{pp}) \tag{18}$$

Substituting $G(Z''_0)$, $G(Z_k)$, $G(Z''_p)$ in equation (15) by equations (16), (17), (18), we obtain

$$(R_{B_n} + R_{B'_n}) - (R_B + R_{B'}) = -(\sum_{BW_i \in B_1} \sqrt{\tau}\sigma_i)(Z''_0 - Z'_0)\psi(Z_{00}) \tag{19}$$

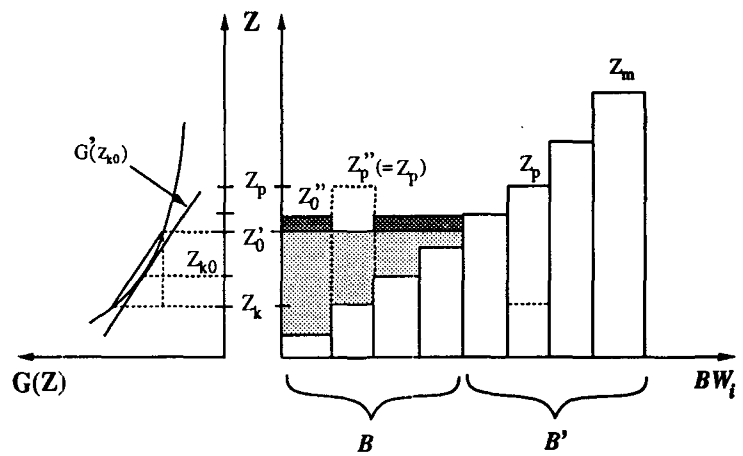


Figure 5: Optimality of the proposed allocation policy

$$-\sqrt{\tau}\sigma_p(Z_p'' - Z_p)\psi(Z_{pp}) + (\sqrt{\tau}\sigma_k)(Z_0' - Z_k)\psi(Z_{k0})$$

Noting that $Z_p \geq Z_0'$ and that there is an increase of $Z_0' - Z_k$ available standard stock on the determination of I_i'', we can say that Z_0'' must be greater than or equal to Z_0', and thus the equation below holds.

(20)
$$\sqrt{\tau}\sigma_k(Z_0' - Z_k) = I_k' - I_k = \sum_{BW_i \in B_1} (I_i'' - I_i') + I_p'' - I_p$$
$$= (\sum_{BW_i \in B_1} \sqrt{\tau}\sigma_i)(Z_0'' - Z_0') + \sqrt{\tau}\sigma_p(Z_p'' - Z_p)$$

Since $Z_p'' \geq \{Z_p, Z_0''\} \geq Z_0' \geq Z_k$, Z_{00} and Z_{pp} are greater than or equal to Z_{k0}. Therefore, $\psi(Z_{k0})$ is always greater than or equal to $\psi(Z_{00})$ and $\psi(Z_{pp})$. Substituting equation (20) for equation (19), we obtain the inequality function as below.

(21)
$$(R_{B_n} + R_{B_n'}) - (R_B + R_{B'})$$
$$\geq \sqrt{\tau}\sigma_k(Z_0' - Z_k) \times (\psi(Z_{k0}) - max.\{\psi(Z_{00}), \psi(Z_{pp})\}) \geq 0$$

In summary, for any $BW_k \in B$ and $BW_p \in B'$, the expected number of system backorders is increased or remains unchanged, when the BW_k is exchanged for BW_p.

Applying a similar process using two or more branch warehouses instead of one, we can also prove that the expected number of system backorders is increased or remains unchanged, when two or more $BW \in B$ are exchanged for two or more $BW \in B'$. For example, exchanging arbitrarily selected $BW_{k_1}, BW_{k_2} \in B$ for arbitrarily selected $BW_{p_1}, BW_{p_2} \in B'$, we can obtain the inequality function as below.

(22)
$$(R_{B_n} + R_{B_n'}) - (R_B + R_{B'}) \geq (\sqrt{\tau}\sigma_{k_1}(Z_0' - Z_{k_1}) + \sqrt{\tau}\sigma_{k_2}(Z_0' - Z_{k_2}))$$
$$\times(min.\{\psi(Z_{k_1 0}), \psi(Z_{k_2 0})\} - max.\{\psi(Z_{00}), \psi(Z_{p_1 p_1}), \psi(Z_{p_2 p_2})\}) \geq 0$$

where

$$\{Z_{00}, Z_{p_1p_1}, Z_{p_2p_2}\} \geq Z'_0 \geq \{Z_{k_10}, Z_{k_20}, Z_{k_1}, Z_{k_2}\}$$

Deleting two or more branch warehouses, we can also prove the optimality of set B and ship-up-to-level I'_i. For example, deleting arbitrarily selected BW_{k_1} and BW_{k_2} from set B, we can obtain the inequality function as below.

$$(23) \qquad (R_{B_n} + R_{B'_n}) - (R_B + R_{B'}) \geq (\sqrt{\tau}\sigma_{k_1}(Z'_0 - Z_{k_1}) + \sqrt{\tau}\sigma_{k_2}(Z'_0 - Z_{k_2})) \times(min.\{\psi(Z_{k_10}), \psi(Z_{k_20})\} - \psi(Z_{00})) \geq 0$$

where

$$Z_{00} \geq Z'_0 \geq \{Z_{k_10}, Z_{k_20}, Z_{k_1}, Z_{k_2}\}$$

Therefore the set B and I'_i, which were obtained by the proposed algorithm, can guarantee the minimum expected number of system backorders.

4 Second Replenishment Period

As the demand in each branch warehouse is random variable, inventory level in each branch warehouse at the end of period t_1 is randomly varied. Hence, the B in each replenishment cycle is different, and the number of $BW \in B$, i.e., n_b is a random variable. So, an analytical computation of the expected number of system backorders($R(t_1)$) is very troublesome under the non-transshipments condition.

Based on the mean n_b, the analytical computation of the $R(t_1)$ could be simplified under the identical demand distribution[5]. But the mean n_b may no longer apply under the non-identical demand distribution. Therefore we employ computer simulation for the computation of $R(t_1)$, based on the optimal ship-up-to-level and the algorithm for finding optimal B proposed in section **3**.

In the simulation, mean demand in BW_i is assumed to be i multiplied by μ_1, and the value of μ_1 is assumed to be chosen randomly. For convenience and to concentrate our attention on finding the optimal second replenishment period, we also assume that the coefficient of variation of demand in each BW is equal. We examined the trends of $R(t_1)$ along with t_1 under the several conditions, and found an existence of the optimal second replenishment period t_1^*, though the value of t_1^* depends on the system conditions. Here, we show only one result under the conditions, $\mu_1 = 40$, $\sigma_i/\mu_i = 0.3$, $I_c/I_0 = 0.15$, $H = 20$, and $m = 5$. The other simulation conditions are shown in Table 1.

The occurrence of system backorders is recorded in each cycle, and through 3600 simulation runs for each given t_1, the expected number of system backorders $R(t_1)$ is computed.

Table 1: Conditions of Simulation

	BW_1	BW_2	BW_3	BW_4	BW_5
μ_i	40	80	120	160	200
σ_i	12	24	36	48	60
$H = 20,\ m = 5,\ S_i = H\mu_i$					
$I_0 = \sum_{i=1}^m H\mu_i + 2\sqrt{H\sum_{i=1}^m \sigma_i^2}$					

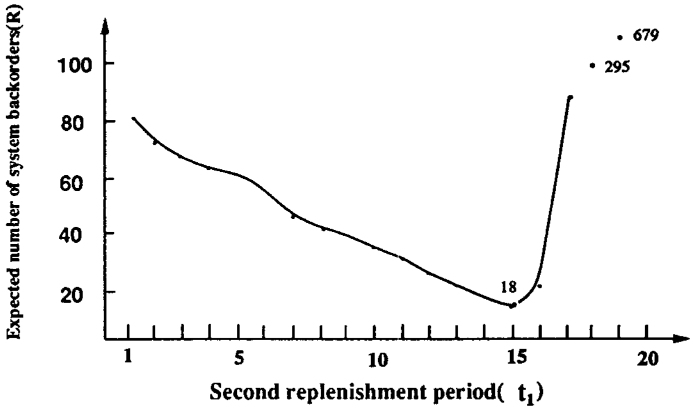


Figure 6: Trends of $R(t_1)$ along with t_1 under five system conditions.

Comparing the values of $R(t_1)$, we find a minimum $R(t_1^*)$ whose corresponding t_1^* is the optimal second replenishment period under the non-transshipments model. The term "optimal" maybe not suitable for t_1^*, because simulation was used instead of a precise mathematical proof. However, we did not find any different t_1^* when the simulation run surpassed 100 times. Therefore, we use the term "optimal" approximately here, though simulation is employed.

Figure 6 shows the trend of $R(t_1)$ along with t_1.

As can be easily seen, there definitely exists an optimal second replenishment period t_1^*, and its value in this case is 15.

5 Relationship Between t_1^* and System Conditions

In order to investigate more general relationship between t_1^* and system conditions, we selected four main factors for further examination: σ_i/μ_i, I_c/I_0, H and m, with three levels each. Though the value of retained stock I_c was set to be equivalent to the system safety stock in previous section, it is not necessary to do so. Instead, it is possible to reduce system backorders by adjusting the quantity of retained stock[2]. Hence, the ratio of retained stock I_c divided by initial system stock I_0, was incorporated as a factor for the simulation.

In the simulation, an $L_{27}(3^{13})$ orthogonal array was used incorporating some interaction effects. I_0 is computed by summing mean demands and system safety stock as given by equation (24), and divided into direct shipment quantity I_d and second replenishment quantity I_c. The stock levels in each BW at the beginning of the first phase, S_i, $i = 1, 2, \cdots, m$, are determined by equation (25).

$$I_0 = \sum_{j=1}^{m} H\mu_j + 2\sqrt{H\sum_{j=1}^{m}\sigma_j^2} \quad (24)$$

$$S_i = H\mu_i + \frac{\sigma_i}{\sum_{j=1}^{m}\sigma_j}(I_d - \sum_{j=1}^{m} H\mu_j) \quad (25)$$

Table 2 shows the settings of the levels for the four simulation factors, and Figure 7 ~ 10 shows the relationship between the t_1^* and system conditions. The axis of ordinates

Table 2: System factors and their levels

Levels	System Factors			
	σ_i/μ_i	I_c/I_0	H	m
0	0.1	0.03	10	3
1	0.3	0.15	20	5
2	0.6	0.3	30	10

Table 3: ANOVA Summary Table

Factors	SS	df	MS	F
A(σ_i/μ_i)	20.23	2	10.115	10.88
B(I_c/I_0)	114	2	57	61.29
C(H)	1073.56	2	536.78	577.18
D(m)	8.23	2	4.115	4.43
A× B	1.12	4	0.28	0.30
A× C	2.23	4	0.558	0.60
Error	9.3	10	0.93	

represents the values of the t_1^* in Figure 7 and represents the values of the ratio t_1^*/H in Figure 8 ~ 10. The axis of abscissas in figure 7 ~ 10 represents the levels of each simulation factor. The average values of the t_1^* in figure 7 and the average value of the ratio t_1^*/H in figure 8 ~ 10 are connected by broken lines respectively.

ANOVA was applied to the data t_1^* obtained from the simulations. All main effects of the four factors were statistically significant with 5% risk each, while the interaction effects were negligible. From the ANOVA summary table, H was found to have the greatest effect on the second replenishment period, followed by I_c/I_0, σ_i/μ_i, and m. Table 3 shows the summarized results of ANOVA.

Explanations for the relationship between t_1^* and system conditions may be summarized as follows:

(1) From Figure 7, t_1^* is apparently delayed by increases in H. However, the value of the ratio t_1^*/H is quite stable. For example, under the conditions shown in Table 1, $H = 10, 20, 30$ may yield $t_1^* = 7, 15, 22$ and hence the ratio $t_1^*/H = 0.7, 0.75, 0.73$. For this reason, we take up the ratio of t_1^*/H for further consideration instead of t_1^* though the variation in t_1^* between levels of the factors are larger than the variation in t_1^*/H.

(2) As is evident from Figure 8, the ratio of t_1^*/H varies inversely to I_c/I_0. The reason may be as follows. Because $I_0 = I_d + I_c$, for a given value of I_0, any increase in I_c results in a comparable decrease in I_d. Also, a smaller I_d(hence bigger I_c) brings about a higher probability of backorders in the first phase. Therefore, to reduce backorders in the first phase, an earlier second replenishment may be necessary.

(3) The probability of backorders increases proportionally with increases in the coefficient of variation of demand. However, as the branch warehouses have a second replenishment opportunity, the number of backorders in the second phase is sufficiently curbed.

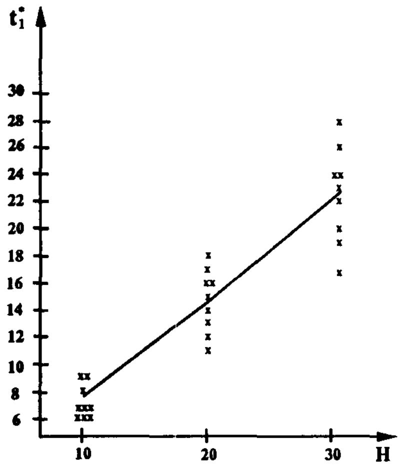


Figure 7: Relationship between t_1^* and H.

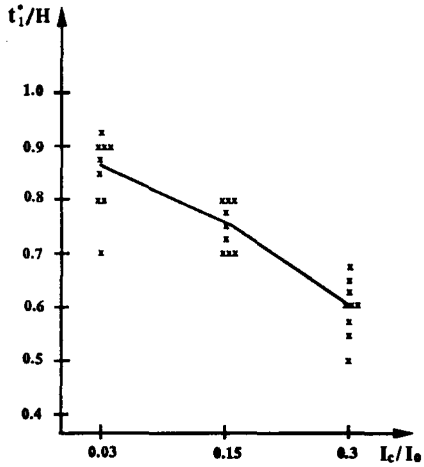


Figure 8: Relationship between t_1^*/H and I_c/I_0

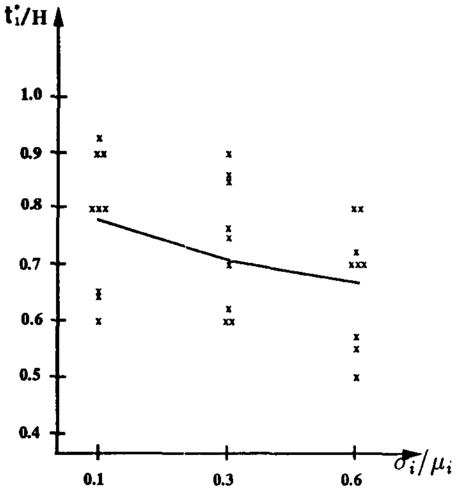


Figure 9: Relationship between t_1^*/H and σ_i/μ_i

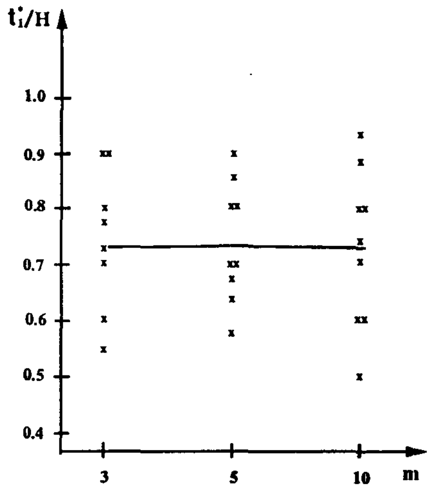


Figure 10: Relationship between t_1^*/H and m

Therefore, to reduce backorders in the first phase, an earlier second replenishment may be necessary. For this reason, the ratio of t_1^*/H may varies inversely to σ_i/μ_i(Figure 9).

(4) As I_0 increases with m and I_c is proportional to I_0, it follows that I_c also increase with an increases in m(*supply*). On the other hand, a large number of branch warehouses(m) would require a large amount of stock in both the first and second phases(***demand***), with the result that this *'demand'* and *'supply'* may maintain a balance. As a consequence, m may have little impact on t_1^*(Figure 10).

In brief, t_1^*/H (or t_1^*) moves up proportionally to increases in σ_i/μ_i and I_c/I_0, while t_1^* is delayed proportionally to increases in H. The ratio of t_1^* to H is quite stable, and t_1^* is little affected by m.

6 Conclusions

Under the extended model from Jönsson and Silver, an optimal allocation policy is proposed. Based on the optimal allocation policy, the existence of an optimal second replenishment period was verified, and shown numerically. The following conclusions were derived regarding the relationships between the proposed t_1^* and system conditions.

(1) t_1^* varies inversely to the coefficient of variation of demand and the ratio of retaining stock divided by the initial system stock level.

(2) t_1^* varies directly to the cycle length of system replenishment, though the ratio of t_1^* to H is quite stable.

(3) t_1^* has little relation to the total number of branch warehouses.

Finally, compared to 'ship-all' policy, the two-phased push control system imposes additional delivery cost while reducing the holding costs and penalty costs through the elimination of overstocks and reduction of system backorders. The evaluation of two-phased push control system in terms of system costs would be a interesting topic for further research.

Appendix

Derivation of the Optimal Ship-Up-To-Level

Objective

$$min. \quad \sum_{i\in N_b} \sqrt{\tau}\sigma_i G(\frac{I_i + \triangle_i - \tau\mu_i}{\sqrt{\tau}\sigma_i})$$

subject to

$$\sum_{i\in N_b} \triangle_i = I_c$$

Applying the Lagrange Multipliers, the optimization function is

$$min. \quad Y = \sum_{i\in N_b} \sqrt{\tau}\sigma_i G(\frac{I_i + \triangle_i - \tau\mu}{\sqrt{\tau}\sigma_i}) + \lambda(\sum_{i\in N_b} \triangle_i - I_c) \tag{26}$$

By differentiating and letting to zero, we obtain,

$$\frac{\partial Y}{\partial \triangle_i} = -\psi(p_i) + \lambda = 0$$

where

$$p_i = \frac{I_i + \triangle_i - \tau\mu_i}{\sqrt{\tau}\sigma_i}, \quad i \in N_b$$

Here, the $\psi(p_i)(=\lambda)$ is obviously a constant. As $\psi(p_i)$ is the upper probability of normal distribution, it varies monotonously with p_i. Consequently, p_i is also a constant, i.e., equation (27) holds for all i and j.

$$\frac{I_i + \triangle_i - \tau\mu_i}{\sqrt{\tau}\sigma_i} = \frac{I_j + \triangle_j - \tau\mu_j}{\sqrt{\tau}\sigma_j} \tag{27}$$

Multiplying both left and right sides of equation by $\sqrt{\tau}\sigma_j$, and summing for $j \in N_b$, we obtain,

$$\sum_{j\in N_b} \frac{\sigma_j}{\sigma_i}(I_i + \triangle_i - \tau\mu_i) = \sum_{j\in N_b} \triangle_j + \sum_{j\in N_b}(I_j - \tau\mu_j)$$

By rearrangement, the optimal ship-up-to-level is given by,

$$I_i' = \tau\mu_i + \frac{\sigma_i}{\sum_{j\in N_b}\sigma_j}(I_c + \sum_{j\in N_b}(I_j - \tau\mu_j)) \tag{28}$$

where $I_i'(= I_i + \triangle_i)$ represents the optimal ship-up-to-level.

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