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THE PROBABILISTIC NETWORK DESIGN PROBLEM

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Abstract We analyze the probabilistic variation of the multicommodity discrete network design problem named the probabilistic network design problem in which the commodities are generated probabilistically and the objective is to calculate the expected value of all possible network design instances.

We extend the a priori strategy which has been successfully applied to the probabilistic variations of the traveling salesman and minimum spanning tree problems. We use the a priori network strategy, which can be seen as an extension of the a priori strategy, to calculated the upper bounds of the expectations of the network design costs, and analyze several heuristic methods for constructing the a priori network both in the worse case model and in the probabilistic model. We also derive the lower bounding procedure using the probabilistic extensions of valid inequalities which, by combining the linear programming technique and the cutting plane procedure, induce the lower bounds of the expected network design costs.

1 Introduction

Designing a minimum cost network is a fundamental combinatorial optimization problem which arises in a wide variety applications in distribution, transportation, communication, energy, water network systems, etc. The mathematical abstract of this model is called the network design problem (NDP) which has been extensively analyzed by many researchers (see survey papers: Magnanti and Wong [12], Minoux [13], [14], Wong [21]). Although much previous work mainly concentrates on the deterministic network design problem, the information about the future events is not known in a deterministic sense in several practical situations; thus inserting probabilistic elements to the problem setting is important for the strategic planning in the middle or long term decision models. Such situations frequently occur in distribution, transportation and communication network design environments.

In this paper, we consider the probabilistic variation of the NDP in which the flow requirements occur probabilistically. The formal description of the model named the probabilistic network design problem (PNDP) is as follows:

Definition 1 *Given a directed graph $G(N,A)$, where N is a set of nodes $\{1,\cdots,n\}$ and A is a set of arcs, a set of distinct commodities $K \subseteq N \times N$ with its origin $O(k) \in N$ and its destination $D(k) \in N - \{O(k)\}$ for each $k \in K$, cost functions $F : A \rightarrow \Re_+$ and $d : A \times K \rightarrow \Re_+$, where $\Re_+$ is the set of nonnegative real numbers, a probability function $p : 2^K \rightarrow [0,1]$, the objective is to calculate the expected cost*

$$E[c] = \sum_{S \subseteq K} p(S)c(S) \quad (1)$$

where summation is taken over all subsets of K and $c(S)$ can be computed by solving the following deterministic network design problem.

$$c(S) = \min \quad \sum_{(i,j)\in A} \sum_{k\in S} d^k_{ij} f^k_{ij} + \sum_{(i,j)\in A} F_{ij} y_{ij} \quad (2)$$

subject to

$$(3) \qquad \sum_{j\in N-\{i\}} f_{ij}^k - \sum_{j\in N-\{i\}} f_{ji}^k = \begin{cases} 1 & i = O(k) \\ 0 & i \in V - \{O(k), D(k)\} \\ -1 & i = D(k) \end{cases} \qquad k \in S$$

$$(4) \qquad f_{ij}^k \le y_{ij} \quad (i,j) \in A, k \in S$$

$$(5) \qquad f_{ij}^k \ge 0 \quad (i,j) \in A, k \in S$$

$$(6) \qquad y_{ij} \in \{0,1\} \quad (i,j) \in A$$

In the formulation above, variable f_{ij}^k represents the flow volume of commodity k through arc (i,j), while variable y_{ij} means whether arc (i,j) is constructed. The objective function (2) minimizes the total cost consisting of the sum of the fixed and variable costs. Constraints (3) are the flow conservation equations. Constraints (4) represent that the flow volume thorough arc (i,j) must be 0 if the arc is not constructed.

Remark that function $p(S)$ represents that the probability with which all commodities in set S are present and other commodities are not present.

Since the NDP is known to be an NP-hard combinatorial optimization problem [7] and the number of possible instances is exponentially large, calculating the expectation according to the problem definition is infeasible in practice. In fact we must solve $2^{|K|}$ NP-hard combinatorial optimization problems ! It is natural to induce the lower and upper bounds instead of calculating the exact expectation; thus our objective is to induce tight lower and upper bounds of the expectations without enumerating all possible instances.

After reviewing the related works in Section 2, we induce the upper bound of the PNDP in Section 3. The upper bound is obtained using the concept of the a priori strategy introduced by Jaillet [8]. In Section 4, we derive the lower bound of the PNDP using the the concept of the probabilistic valid inequality. In Section 5, we give a small numerical example to illustrate our lower and upper bounding procedures. Section 6 contains conclusions.

2 Previous Works

The deterministic network design problem (NDP) has been extensively analyzed in the literature. The survey papers for the NDP (Magnanti and Wong [12], Minoux [13], [14], Wong [21]) contain more than 100 references. Readers may refer to these excellent survey papers.

Soroush and Mirchandani [17] formulate several classes of the multicommodity network flow problem called the stochastic network flow problem in which the arc attributes are not necessarily deterministic but random variables. Their problem settings are different from ours.

Several researchers (Yaged [23], Zadeh [24], Minoux [13]) observe that the classical static model of the NDP is not appropriate for the long term decision model. Though the model they propose are taking account of the dynamic nature of the NDP, the probabilistic nature of the real world model is not incorporated. In this content, their model can be seen as the multi-period network design problem.

In this paper we use the following asymptotic notations.

$$O(g(n)) = \{f(n) \mid \text{ there exists positive constants } c \text{ and } n_0 \text{ such that } 0 \le f(n) \le cg(n) \text{ for all } n \ge n_0.\}$$

$$\Theta(g(n)) = \{f(n) \mid \text{ there exists positive constants } c_1,\ c_2 \text{ and } n_0 \text{ such that } 0 \le c_1 g(n) \le f(n) \le c_2 g(n) \text{ for all } n \ge n_0.\}$$

3 Upper Bound of Probabilistic Network Design Problem

Most probabilistic combinatorial optimization problems concentrate on the calculation of the expected cost of finitely many combinatorial optimization instances which are generated probabilistically. Although the number of deterministic instances to be solved is finite, it may be exponentially large number. Solving all instances causes the computational explosion even if the corresponding deterministic problem is polynomial solvable. Further in our network design model, the problem which must be solved for each instance is an NP-hard combinatorial optimization problem. Thus we must calculate the expected cost without enumerating all possible instances.

In this section, we extend the concept of the a priori strategy introduced by Jaillet [8] for the probabilistic traveling salesman problem and extended to the probabilistic minimum spanning tree problem by Bertsimas [2], [3]. We call our strategy *the a priori network strategy.* Before describing our strategy, we need few definitions.

Definition 2 *We say that a graph $G'(N',A')$ is a subgraph of $G(N,A)$ if $N' \subseteq N$ and $A' \subseteq A$. Especially we say that a subgraph $G'(N',A')$ is spanning if $N' = N$ and there exists a path from $i \in N$ to $j \in N - \{i\}$ using only arcs in A'.*

The a priori network is a special purpose spanning subgraph of $G(N,A)$ for the a priori network strategy. Using the a priori network, we now can describe our strategy for the PNDP.

(The A Priori Network Strategy)
Given the a priori network, we construct the subgraph for all present commodities using only parts of the a priori network. We call the subgraph constructed from the a priori network *the subnetwork.* ▮

Since the traveling salesman problem and the minimum spanning tree problem are special cases of the network design problem, the a priori network strategy is an extension of the a priori strategy analyzed in [2], [8].

The major problems which we will concern in this section are summarized as follows.

1. How to construct the subnetwork when we are given present commodities.

   In the operational decision level, we are given the a priori network and the commodities which have been generated probabilistically. Then the problem is to get an actual subnetwork through which the present commodities pass. The daily network must be obtained using cheap computational requirement. We derive an algorithm which runs in $O(n^2 \mid S \mid)$ time where n and S are the number of of nodes and the subset of commodities present, respectively.

2. How to compute the expected cost for all possible subsets of commodities.

   In the strategic decision level, we must evaluate effectiveness of given a priori networks. Calculating the expected cost by enumerating all possible instances is computationally infeasible. We present explicit formulas to compute the expectation when we are given an a priori network.

3. How to construct the a priori network which attains the minimum expected cost.

   The final problem which also occurs in the strategic planning level is to construct the a priori network which attends good expected costs. We analyze special cases and design several algorithms for constructing the a priori network. The presented algorithms are evaluated in the worst case and in the probabilistic model.

3.1 How to Construct Subnetwork

First we consider the problem of constructing the subnetwork when we are given the a priori network and the subset of commodities which are generated probabilistically.

Let us denote the arc set of the a priori network by $\alpha \subseteq A$. Then the set of commodities $K_{ij}^{\alpha} \subseteq K$ through arc $(i,j) \in \alpha$ in the a priori network can be obtained by solving shortest path problems between $O(k)$ and $D(k)$ for all $k \in K$. Remark that each commodity passes on a single path from its origin to its destination since the capacity of the network is infinity under the assumption of our model. Once K_{ij}^{α} is obtained, the node set and the arc set of the subnetwork for the present commodities, denoted by N_S and A_S respectively, can be obtained by deleting unnecessary nodes and arcs.

The formal description of the algorithm for computing the subnetwork is as follows:

Algorithm: Construct Subnetwork for commodity set S
Input: the subset of commodities $S \subseteq K$, the arc set of the a priori network α
Output: the node set N_S and the arc set A_S of the subnetwork
begin

{ Initialization }
{ Remark: K_{ij}^{α} represents the set of commodities through arc (i,j) }
$K_{ij}^{\alpha} = \emptyset$ for all $(i,j) \in \alpha$;
for all $k \in S$ **do**
begin

compute the shortest path between $O(k)$ and $D(k)$;
for all (i,j) on the path from $O(k)$ to $D(k)$ **do**
$K_{ij}^{\alpha} := K_{ij}^{\alpha} \cup \{k\}$;

end ;
{ delete unnecessary arcs }

$$A_S := \alpha - \{(i,j) \in A \mid K_{ij}^{\alpha} = \emptyset\};$$

{ delete isolated nodes }

$$N_S := N - \{i \in N \mid (i,j) \notin A_S \text{ and } (j,i) \notin A_S \text{ for all } j \in N - \{i\}\}$$

end.

Since the dominant task of the above procedure is the part of computing shortest paths for all present commodities S, and each shortest path can be found is $O(n^2)$ operations (see for example [1], [19]); the total computational requirement of the above algorithm is $O(n^2 \mid S \mid)$.

3.2 How to Compute Expected Cost

Next we consider the expected cost derived by the a priori network strategy. Let us denote the cost of instance S by $c_{\alpha}(S)$ and the expected cost of the a priori network strategy by $E[c_{\alpha}]$ which can be defined as follows:

$$E[c_{\alpha}] = \sum_{S \subseteq K} p(S) c_{\alpha}(S) \tag{7}$$

where the summation is taken over all subsets of K. If we denote the set of arcs from $O(k)$ to $D(k)$ by $PATH_k$, the cost $c_{\alpha}(S)$ of instance S is calculated by

$$c_{\alpha}(S) = \sum_{(i,j) \in A_S} F_{ij} + \sum_{k \in S} \sum_{(i,j) \in PATH_k} d_{ij}^{k}. \tag{8}$$

We want to calculate $E[c_\alpha]$ without enumerating all possible instances. Below we give simple explicit formulas of the expectation $E[c_\alpha]$. Remark that we can model the dependency of the presence probability of the commodities in this level of generality. For notational convenience, we introduce the following probability functions: let $h(S)$ be the probability with which no commodity in $S(\subseteq K)$ is present, i.e.,

$$h(S) = \sum_{L \subseteq K-S} p(L) \tag{9}$$

and p_k is the probability with which the commodity k is present, i.e.,

$$p_k = \sum_{k \in S} p(S). \tag{10}$$

Then we can get the expected cost $E[c_\alpha]$ of the a priori network strategy:

$$E[c_\alpha] = \sum_{(i,j)\in\alpha} F_{ij}\left\{1 - h(K_{ij}^\alpha)\right\} + \sum_{(i,j)\in\alpha}\sum_{k\in K_{ij}^\alpha} d_{ij}^k p_k. \tag{11}$$

The above formula indicates that if we are given probability function $h(S)$ for each $S \subseteq K$ and p_k for each $k \in K$, we can compute the expectation $E[c_\alpha]$ of the a priori strategy in $O(|K||A|)$ time.

The practically important special case is that each commodity, say k, is generated with probability p_k independently. In this case the expected cost $E[c_\alpha]$ of the a priori network strategy is

$$E[c_\alpha] = \sum_{(i,j)\in\alpha} F_{ij}\left\{1 - \prod_{k\in K_{ij}^\alpha}(1-p_k)\right\} + \sum_{(i,j)\in\alpha}\sum_{k\in K_{ij}^\alpha} d_{ij}^k p_k. \tag{12}$$

When $p_k = p$ for all k, we get the following simple formula of the expected cost.

$$E[c_\alpha] = \sum_{(i,j)\in\alpha} F_{ij}\left\{1 - (1-p)^{|K_{ij}^\alpha|}\right\} + p\sum_{(i,j)\in\alpha}\sum_{k\in K_{ij}^\alpha} d_{ij}^k \tag{13}$$

We consider this special case to analyze the properties of $E[c_\alpha]$ in the next subsection.

3.3 How to Construct A Priori Network

The problem we consider in this subsection is how to construct the a priori network which attains the minimum expected cost. Since the network design problem itself is a very hard combinatorial optimization problem, constructing the a priori network is also difficult; we need some heuristic procedures for this problem. The most practical heuristic method for such hard problems is the local search heuristic which searches the neighbor of the current solution until no better neighbor can be found. The typical local search procedure for the NDP is called the add-drop heuristic (see for example [11]) in which several arcs are dropped or added to find a better solution. We can easily construct the local search procedure using the explicit representation (11) of the expected cost .

We must remark that the local search procedure has no theoretical guarantee; sometimes it gives good solutions, but sometimes bad solutions. Below we consider several well-solved special cases and heuristic procedures for constructing the a priori network with theoretical guarantees under the assumption that all present probabilities of commodities are identical, i.e., $p_k = p$.

First we analyze the properties of the following function.

$$f(p) = \min_{\alpha} E[c_\alpha] = \min_{\alpha}\left[\sum_{(i,j)\in\alpha} F_{ij}\left\{1-(1-p)^{|K^\alpha_{ij}|}\right\} + p\sum_{(i,j)\in\alpha}\sum_{k\in K^\alpha_{ij}} d^k_{ij}\right] \tag{14}$$

Lemma 1 *The function $f(p)$ defined in (14) is continuous, nondecreasing, piecewise differentiable and concave.*

Proof: If we fix the a priori network, then $f(p)$ becomes a polynomial function of p. Since we select the a priori network which gives the minimum value of the expected cost, $f(p)$ is a lower envelope of some polynomial functions; this leads that $f(p)$ is continuous and piecewise differentiable.

Then let us consider the function $\phi(p) = 1-(1-p)^k$. For $p \in [0,1]$ and an integer $k(>0)$, we get

$$\phi'(p) = k(1-p)^{k-1} \geq 0 \tag{15}$$

and

$$\phi''(p) = k(k-1)(1-p)^{k-2} \leq 0. \tag{16}$$

Since $\phi'(p) \geq 0$, $f(p)$ is a nondecreasing function of p. We observe that $f(p)$ can be decomposed into the following two components:

1. Convex combinations of F_{ij} and $\phi(p)$,
2. Sum of linear functions $d^k_{ij}p$.

Since the linear function is a special case of the general concave function, the sum of the above two components is also concave. Since the minimum of the concave functions is also concave, we get the concavity of function $f(p)$. Combining above results gives the desired result. ▌

Lemma 2 *Let Z_{NDP} be the optimal cost of the PNDP with $p=1$, i.e., the optimal cost of the deterministic network design problem, and let $Z_{complete}$ be the total cost of the complete network with $p=1$, then the following relations hold:*

$$pZ_{NDP} \leq f(p) \leq pZ_{complete}. \tag{17}$$

Proof: Since $f(p)$ is a concave function and $f(1) = Z_{NDP}$, we get $f(p) \geq pf(1) + 0p(0)$; this implies the lower bound in (17). It is easily observed that $(1-p)^k \geq 1-pk$ by the mathematical induction. This leads

$$\begin{aligned} f(p) &\leq p\min_{\alpha}\left[\sum_{(i,j)\in\alpha} F_{ij} \mid K^\alpha_{ij} \mid + \sum_{(i,j)\in\alpha}\sum_{k\in K^\alpha_{ij}} d^k_{ij}\right] \\ &= p\sum_{k\in K}\{\text{Shortest path length between } O(k) \text{ and } D(k) \\ &\qquad\qquad \text{with distance matrix } [F_{ij}+d^k_{ij}]\} \\ &\leq Z_{complete}. \end{aligned}$$

Thus we get the upper bound in (17). ∎

Using the properties of $f(p)$ derived above, we get the following results.

Proposition 1 *If the optimal solution of the PNDP with $p = 1$, i.e., the optimal solution of the network design problem, is the complete network with the optimal value $Z_{complete}$, then it is also the optimal a priori network for all $0 \le p \le 1$ and the optimal value of the PNDP is $pZ_{complete}$.*

Proof: Since the optimal solution is a complete network, we observe that $| K_{ij}^{\alpha} |= 1$ holds. This implies the lower and upper bounds in (17) are identical; we get the desired result. ∎

Proposition 2 *If the number of commodities on the arcs are bounded by the value D in the optimal solution of the PNDP with $p = 1$, i.e., the optimal solution of the network design problem and denote its value by Z_{NDP}, then the optimal value of the PNDP is less or equal to pDZ_{NDP} for all $0 \le p \le 1$.*

Proof: Since $| K_{ij}^{\alpha} |\le D$, the first term in (14) is bounded by $pD\sum_{(i,j)\in\alpha} F_{ij}$, we get the following relation.

$$\begin{aligned} f(p) &\le p\min_{\alpha}\left[\sum_{(i,j)\in\alpha} F_{ij}D + \sum_{(i,j)\in\alpha}\sum_{k\in K_{ij}^{\alpha}} d_{ij}^{k}\right] \\ &= pD\min_{\alpha}\left[\sum_{(i,j)\in\alpha} F_{ij} + \sum_{(i,j)\in\alpha}\sum_{k\in K_{ij}^{\alpha}} d_{ij}^{k}/D\right] \\ &\le pDZ_{NDP} \end{aligned}$$

This implies the desired result. ∎

We then investigate several heuristic algorithms for constructing the a priori network and analyze their worst and average case behaviors. In the following argument, we divide the total expected cost $E[c_{\alpha}]$ into the expected construction cost $E[CC]$ and the expected routing cost $E[RC]$, and analyze them separately.

The first heuristic is a direct extension of the heuristic for the deterministic network design problem due to Wong [20]. The algorithm which we call the star tree heuristic can be described as follows.

(Star Tree Heuristic)

For all $i \in N$, construct the star tree network rooted at i, i.e., the network consisting of the arcs between i and every other nodes, and find a root i^* with the minimum routing cost. ∎

For a general class of the network design problem, which is a deterministic variant of the PNDP, Wong [20] proved that finding an approximate solution whose worst case ratio is $n^{1-\epsilon}$ for some $\epsilon > 0$ is NP-hard. This implies that the worst case behavior of the star tree heuristic must be poor under the general model. Below we analyze the worst case model of the star tree network under the assumptions that $p_k = p, F_{ij} = F$ and $d_{ij}^{k} = d_{ij}$. Under this assumption, the construction costs of all star tree networks are identical as shown in the following theorem.

Theorem 1 *If we assume $p_k = p$ and $F_{ij} = F$, then the expected construction cost $E[CC]$ of the star tree network is*

$$E[CC] = 2F(n-1)\{1-(1-p)^{n-1}\}. \tag{18}$$

Proof: The arc (i,j) is constructed when at least one commodity through the arc is present. Since each commodity is generated with probability p, the probability with which at least one of $n-1$ commodities is present becomes $1-(1-p)^{n-1}$. Since the number of arcs in the star tree network is $2(n-1)$ and the construction cost of every arc is F, we get the expectation $E[CC]$ of the star tree heuristic. ∎

Since Theorem 1 indicates that every star tree network has the same construction cost under the assumption, we must choose the star tree network with the minimum routing cost; this is the motivation of the star tree heuristic. We then analyze the expected routing cost $E[RC]$ of the star tree heuristic. Let us denote the routing cost from node i to every other node by $\overrightarrow{COST}(i)$ and the routing cost from every other node to node i by $\overleftarrow{COST}(i)$, i.e.,

$$\overrightarrow{COST}(i) = \sum_{j\neq i} d_{ij} \tag{19}$$

and

$$\overleftarrow{COST}(i) = \sum_{j\neq i} d_{ji}. \tag{20}$$

In fact the routing cost between node i and every other node is

$$\begin{aligned} COST(i) &= \sum_{j\neq i} d_{ij} + \sum_{j\neq i} d_{ji} \\ &= \overrightarrow{COST}(i) + \overleftarrow{COST}(i). \end{aligned} \tag{21}$$

Let i^* be the node which has the minimum value of $COST(i)$. Then the solution obtained by the star tree heuristic is the star tree rooted at i^*.

Theorem 2 *If we assume $p_k = p$, $F_{ij} = F$ and $d^k_{ij} = d_{ij}$, then the expected routing cost $E[RC]$ of the star tree network is given by*

$$E[RC] = p(n-1)COST(i^*). \tag{22}$$

Proof: Consider the routing cost of the commodities originated from node $j(\neq i^*)$. The routing cost can be decomposed into two pars; the cost from j to i^* and the cost from i^* to every other node except j. Since the number of commodities originated from j is $n-1$ and each of them is present independently with probability p, the expected routing cost of the former part is $p(n-1)d_{ji^*}$. The routing cost from i^* to every other node except j is given by $\overrightarrow{COST}(i^*) - d_{i^*j}$. By summing up the routing costs of all nodes $j \in N - \{i^*\}$, we get $p(n-1)\overleftarrow{COST}(i^*) + p(n-1)\overrightarrow{COST}(i^*) - p\sum_{j\in N-\{i^*\}} d_{i^*j}$. Finally, by adding the routing cost $p\overrightarrow{COST}(i^*) = p\sum_{j\in N-\{i^*\}} d_{i^*j}$ of commodities originated from i^*, we get $E[RC] = p(n-1)COST(i^*)$. ∎

Table 1: Construction and Routing Costs for Complete and Star Tree Networks

Type of Network	Construction Cost	Routing Cost
Complete Network	$Fn(n-1)\{1-(1-p)^2\}$	$p\sum_{i=1}^{n} COST(i)/2$
Star Tree Network	$2F(n-1)\{1-(1-p)^{n-1}\}$	$p(n-1)COST(i^*)$

We then analyze the worst case behavior of $E[RC]$ of the star tree heuristic by comparing with the lower bound of the optimal expected routing cost. Obviously the lower bound is given by $p\sum_{i=1}^{n} COST(i)/2$ because the complete network has the minimum routing cost. The expected values of the construction and routing costs are compared in Table 1.

Corollary 1 *If $p_k = p$, $F_{ij} = F$ and $d_{ij}^k = d_{ij}$, then*

(23) $$\frac{E[RC]}{E[RC^*]} \leq 2$$

where $E[RC]$ is the expected routing cost of the star tree network and $E[RC^*]$ is the optimal expected value of the routing cost.

Proof: Since $(n-1)COST(i^*) \leq \sum_{i=1}^{n} COST(i)$, we get the result. ∎

Next we consider the probabilistic model in which the nodes are distributed uniformly and independently on the two dimensional Euclidean plane in the unit circle, and one unit of commodities must be shipped between all pairs of nodes with probability p independently each other. We must remark that if the arc (i,j) is constructed, the commodity can pass from i to j and from j to i, i.e., the routing is symmetric. The second heuristic called the threshold heuristic [22] can be described as follows:

(Threshold Heuristic)

We add a dummy center (node 0) at the center of the circle. Start with a star tree network rooted at the dummy center (node 0) and add arcs whose length is less or equal to L where L is a predetermined parameter such that $L = An^{-1/3}$ for some constant $0 < A \leq 0$. In the sequel we call the network obtained by this heuristic procedure *the threshold network.* ∎

We first analyze the expected construction cost of the threshold network. We use the following result from the geometric probability (see for example Eilon, Watson-Gandy and Christofides [5]).

Lemma 3 *Let x be a uniformly chosen point and x^* be the center of the unit circle, then*

(24) $$E[\| x^* - x \|] = 2/3.$$

Lemma 4 *Given n uniformly and independently distributed points $x_1, \cdots, x_n$ in the unit circle, then*

(25) $$E[\| x_i - x_j \|] \approx 0.905.$$

Theorem 3 *If we denote the expected construction cost of the threshold heuristic by $E[CC]$, then*

(26) $$E[CC] \leq \frac{2}{3}n(n-1)L^3(1-q^2) + \frac{4}{3}n\left\{1-q^{2(n-1)}\right\}$$

where $q = 1-p$ and $L = A^{-1/3}$ for some constant $0 < A \leq 1$.

Proof: The expected construction cost $E[CC]$ of the threshold network is the sum of E_1 (the expected cost of added arcs) and E_2 (the expected cost of the star tree centered at 0). We first derive E_1. Consider the node i and the circle C_i centered at i with radius L. Then the probability with which the node $j(\neq i)$ lies in C_i is $\frac{\pi L^2}{\pi} = L^2$. Since Lemma 3 shows that the expected cost (arc length) between i and j is $\frac{2}{3}L$, we obtain $E_1 = n(n-1) \times \frac{2}{3}L^3 = 2n(n-1)L^3/3$.

Then we derive E_2. Again consider the node i and the circle C_i. Then the construction cost F_{0i} is incurred when at least one of the commodities whose destination (or origin) is not in C_i is present. The probability with which no such commodity is present is $1-(1-L^2)p$. Since the number of commodities originated from i or destined to i is $2(n-1)$, the expected length of arc $(0,i)$ becomes $F_{0i}[1-\{1-(1-L^2)p\}^{2(n-1)}]$. Since the expected value of $\sum_{i=1}^{n}(F_{0i}+F_{i0})$ is $\frac{4}{3}n$ (from Lemma 3), we get $E_2 = \frac{4}{3}n\left[1-\{1-(1-L^2)p\}^{2(n-1)}\right]$. Combining E_1 and E_2 yields

$$E[CC] = \frac{2}{3}n(n-1)L^3\left\{1-(1-p)^2\right\} + \frac{4}{3}n\left[1-\left\{1-(1-L^2)p\right\}^{2(n-1)}\right]. \tag{27}$$

Since $L^2 \geq 0$, we get $1-(1-L^2)p \geq 1-p$; this leads the upper bound of $E[CC]$ by setting $q = 1-p$. ∎

The expected values of the construction costs of the threshold network and the complete network are compared in Table 2. Then we compare the expected construction cost of the threshold network and the complete network when p approaches to 0 or 1.

Table 2: Construction Cost under the Random Model (Complete and Threshold Networks)

Type of Network	Construction Cost
Complete Network	$0.905n(n-1)\{1-(1-p)^2\}$
Threshold Network	$\frac{2}{3}n(n-1)L^3\{1-(1-p)^2\} + \frac{4}{3}n\left[1-\{1-(1-L^2)p\}^{2(n-1)}\right]$

Corollary 2 *Let $E[CC_n]$ and $E[CC_n^*]$ be the expected construction costs of the threshold and complete networks with n nodes, respectively. Then*

$$\lim_{n\to\infty, p\to 1} \frac{E[CC_n]}{E[CC_n^*]} = 1/\Theta(n) \tag{28}$$

and

$$\lim_{n\to\infty, p\to 0} \frac{E[CC_n]}{E[CC_n^*]} \approx 1.473 \tag{29}$$

hold.

Proof: When p approaches to 1, the expected construction cost $E[CC_n]$ of the threshold network approaches $\Theta(n)$ because we set $L = An^{-1/3}$, while the expected construction $E[CC_n^*]$ cost of the complete network approaches $\Theta(n^2)$; so the relation (28) holds.

When p approaches to 0, using L'Hospital Theorem, the ratio $E[CC_n]/E[CC_n^*]$ approaches to $4/(3 \times 0.905) \approx 1.473$; we get (29). ∎

The asymptotic construction cost of the threshold network is approximately 1.473 times as large as the cost of the complete network when p approaches to 0; this implies that if p is very small, we should use the complete network as the a priori network. On the contrary, when p approaches to 1, i.e., all commodities are almost present, the threshold network gives much smaller expected construction cost than the complete network. This fact corresponds to the result due to Wong [22].

Then we turn to analyze the expected routing cost.

Proposition 3 *If we denote the expected routing costs of the threshold and complete networks with n nodes by $E[RC_n]$ and $E[RC_n^*]$, respectively, then*

$$\lim_{n\to\infty}\frac{E[RC_n]}{E[RC_n^*]}=1 \tag{30}$$

holds.

Proof: We get the result as in the deterministic case (see [22]); so the proof is omitted. ∎

4 Lower Bound of the Probabilistic Network Design Problem

In this section, we derive the lower bounds of the expected cost of the PNDP. We first introduce several concepts for deriving the lower bounds of the general probabilistic combinatorial optimization problem. Then we derive several types of probabilistic valid inequalities for the PNDP. The inequalities derived below and the cutting plane technique are essential ingredients of the lower bounding procedure for the PNDP.

4.1 Some Results on the General Probabilistic Combinatorial Optimization Problem

We first define the general probabilistic combinatorial optimization problem, and extend several important concepts of polyhedral theory. The results shown in this subsection are used to derive the probabilistic valid inequalities (defined below) for the PNDP.

Let Ψ be the set of all instances which can occur. Suppose that every instance $\psi \in \Psi$ occurs with probability $\wp(\psi)$. Associate with each instance ψ, we define a set of feasible solutions $\Xi(\psi)$. The objective of the probabilistic combinatorial optimization problem (PCOP) is to find the expectation $E(\Gamma)$ over $|\Psi|$ instances of combinatorial optimization problems, i.e.,

$$E(\Gamma)=\sum_{\psi\in\Psi}\wp(\psi)\Gamma(\psi) \tag{31}$$

where $\Gamma(\psi)$ is the optimal solution value of instance ψ. More precisely, $\Gamma(\psi)$ can be determined as follows :

$$\Gamma(\psi)=\min \sum_{j\in J}c_j\xi_j \tag{32}$$

subject to

$$\xi\in\Xi(\psi), \tag{33}$$

$$\xi_j\geq 0 \quad j\in J. \tag{34}$$

We need some definitions, which are extensions of the ordinal concepts of the polyhedral theory, before describing our main result of this subsection.

Definition 3 *Let $\xi_j^*(\psi), j \in J$ be the optimal solution of instance $\psi \in \Psi$. Then the optimal solution of the PCOP is defined by*

(35) $$x_j^* = \sum_{\psi\in\Psi} \wp(\psi)\xi_j(\psi) \quad j \in J.$$

In an ordinary sense, a valid inequality for a feasible set $R \subseteq \Re^{|J|}$ is defined as follows (see for example Nemhauser and Wolsey [15]):

Definition 4 *An inequality $\sum_{j\in J} A_j x_j \geq b$ is called valid for $R \subseteq \Re^{|J|}$, if $R \subseteq \{x \in \Re^{|J|} \mid \sum_{j\in J} A_j x_j \geq b\}$, i.e., R is contained in the halfspace defined by $\sum_{j\in J} A_j x_j \geq b$.*

The definition of a valid inequality for the PCOP is not well-defined because the feasible set R cannot be determined deterministically; we extend the concept of the valid inequalities as follows.

Definition 5 *Let x^* be the optimal solution of the PCOP. Then, an inequality $\sum_{j\in J} A_j x_j \geq b$ is called valid for the PCOP if $x^* \in \{x \in \Re^{|J|} \mid \sum_{j\in J} A_j x_j \geq b\}$, i.e., x^* is contained in the halfspace defined by $\sum_{j\in J} A_j x_j \geq b$.*

In th sequel, we call the valid inequality for the PCOP *the probabilistic valid inequality.*

Definition 6 *The inequality $\sum_{j\in J} a_{ij}\xi_j \geq 1$ is called valid with probability p_i, if $p_i = \sum_{\psi\in\hat{\Psi}(i)} \wp(\psi)$ where*

$$\hat{\Psi}(i) = \{\psi \in \Psi \mid \textit{the } i\textit{-th inequality is valid (in ordinary sense) with respect to the feasible set } \Xi(\psi) \textit{ of instance } \psi\},$$

i.e., the summation of the probability over all instances in which the inequality $\sum_{j\in J} a_{ij}\xi_j \geq 1$ is valid is p_i.

Following is the main result for deriving the probabilistic valid inequalities for the PCOP.

Theorem 4 *If the inequalities $\sum_{j\in J} a_{ij}\xi_j \geq 1$ are valid with probability p_i for all $i \in I$, then the following relation holds:*

(36) $$LB \leq E(\Gamma),$$

where LB is the optimal solution of the following linear programming problem:

(37) $$LB = \min \quad \sum_{j\in J} c_j x_j$$

subject to

(38) $$\sum_{j\in J} a_{ij} x_j \geq p_i \quad i \in I,$$

(39) $$x_j \geq 0 \quad j \in J.$$

Proof: Let $\{w_i\}$ be the $|I|$ dimensional non-negative real vector and let $\bar{c}_j = c_j - \sum_{i \in I} a_{ij} w_i$ for all $j \in J$. Consider the problem with the cost $[\bar{c}_j]$ and the same constraints as the original PCOP, and denote the expected cost of the modified problem by $E(P')$. If we denote by $\xi_j^*(\psi)$ an optimal solution of instance $\psi \in \Psi$, then we get

$$(40) \qquad \begin{aligned} E(P') &\leq \sum_{\psi \in \Psi} \wp(\psi) \sum_{j \in J} c_j \xi_j^*(\psi) - \sum_{\psi \in \Psi} \wp(\psi) \sum_{i \in I} \sum_{j \in J} a_{ij} w_i \xi_j^*(\psi) \\ &\leq E(\Gamma) - \sum_{i \in I} p_i w_i. \end{aligned}$$

This leads

$$(41) \qquad E(\Gamma) \geq \sum_{i \in I} p_i w_i + E(P').$$

The maximum lower bound can be obtained by solving the following linear programming problem.

$$(42) \qquad \max \quad \sum_{i \in I} p_i w_i$$

subject to

$$(43) \qquad c_j - \sum_{i \in I} a_{ij} w_i \geq 0 \quad j \in J,$$

$$(44) \qquad w_i \geq 0 \quad i \in I.$$

Since the constraints (43) mean $\bar{c}_j \geq 0$ for all $j \in J$, we get

$$(45) \qquad E(P') \geq 0.$$

Since the dual of the above linear programming problem is identical to the linear programming problem (37),(38) and (39), the strong duality leads

$$(46) \qquad \sum_{i \in I} p_i w_i = LB.$$

Combining (41),(45) and (46), we get

$$(47) \qquad \begin{aligned} E(\Gamma) &\geq \sum_{i \in I} p_i w_i + E(P') \\ &\geq LB. \end{aligned}$$

This leads the theorem. ∎

Each inequality in (38) is a valid inequality for the PCOP, because the optimal solution x^* of the PCOP is contained in the halfspace defined by $\sum_{j \in J} a_{ij} x_j \geq p_i$ for each $i \in I$. Then we get the following corollary which will be sometimes used to derive the probabilistic valid inequalities for the PNDP in the following subsection.

Corollary 3 *If the inequality $\sum_{j \in J} a_{ij} \xi_j \geq 1$ is valid with probability p_i, then $\sum_{j \in J} a_{ij} x_j \geq p_i$ is a valid inequality for the PCOP.*

For equality constraints, we can derive the similar results; hence omitted.

4.2 Probabilistic Valid Inequalities for the Probabilistic Network Design Problem

We use the results proved above to derive several probabilistic valid inequalities for the PNDP, which are used to induce the lower bounds of the expected network design costs for the PNDP by combining the linear programming and cutting plane (row generation) techniques.

First we derive the probabilistic valid equalities for the PNDP. Recall that $p(S)$ is the probability with which the set of commodities in $S(\subseteq K)$ are present, and p_k is the probability with which the commodity k is present.

Theorem 5 *The following constraints are the probabilistic valid equalities for the PNDP.*

$$\sum_{j\in N-\{i\}} f_{ij}^k - \sum_{j\in N-\{i\}} f_{ji}^k = \begin{cases} p_k & i=O(k) \\ 0 & i\in V-\{O(k),D(k)\} \\ -p_k & i=D(k) \end{cases} \qquad k\in K \tag{48}$$

Proof: Since commodity k is present with probability p_k, each constraint in (3) becomes valid with probability p_k. Using Corollary 3, we get the result. ∎

We then show the obvious inequalities which are valid with probability 1; these inequalities are essentially same as the deterministic counterparts.

Theorem 6 *The following constraints are the probabilistic valid inequalities for the PNDP.*

$$\sum_{k\in K} f_{ij}^k \leq |K| \, y_{ij} \qquad (i,j)\in A \tag{49}$$

$$f_{ij}^k \geq 0 \qquad (i,j)\in A, k\in S \tag{50}$$

$$0 \leq y_{ij} \leq 1 \qquad (i,j)\in A \tag{51}$$

The probabilistic valid inequalities derived above have almost same structures as in the deterministic case. Of course the above inequalities give the lower bonds of the expected value $E[c]$ of the PNDP; the bounds are not very tight. We then derive inequalities which exhibit the distinguishing features of the PNDP.

For a cutset $(T, V-T)$, we denote the set of commodities whose the origin is in T and the destination is in $V-T$ by $K_{(T,V-T)}$. Then we get the following probabilistic valid inequalities for the PNDP.

Theorem 7 *The following constraints named the probabilistic cutset inequalities are the probabilistic valid inequalities for the PNDP.*

$$\sum_{(i,j)\in(T,N-T)} y_{ij} \geq 1 - h(K_{(T,V-T)}) \qquad T\subseteq N, T\neq\emptyset, T\neq N \tag{52}$$

where h is the probability function defined in (9).

Table 3: Exact Design Costs and Upper Bounds of All Possible Instances

Set of Commodities S	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
Design Cost $c(S)$	0	1.1	1.1	1.1	2.2	2.2	2.2	2.4
Upper Bound $c_\alpha(S)$	0	1.1	1.1	2.2	2.2	2.3	2.3	2.4

Proof: Consider a cutset $(T, V-T)$ and the commodities $K_{(T,V-T)}$ which pass the cutset. If at least one commodity in $K_{(T,V-T)}$ is present, the following inequality becomes valid (in an ordinal sense).

$$\sum_{(i,j)\in(T,N-T)} y_{ij} \geq 1 \tag{53}$$

Since the probability with which at least one commodity is present is $1 - h(K_{(T,V-T)})$, we conclude the constraints (52) are probabilistic valid inequalities for the PNDP. ∎

Following is a practically important special case of the probabilistic cutset inequalities.

Corollary 4 *If each commodity, say k, is present independently with probability p_k, the following constraints are the probabilistic valid inequalities for the PNDP.*

$$\sum_{(i,j)\in(T,N-T)} y_{ij} \geq 1 - \prod_{k\in K_{(T,V-T)}} (1-p_k) \qquad T \subseteq N, T \neq \emptyset, T \neq N \tag{54}$$

5 Numerical Example

Consider the following small example with three nodes to illustrate the procedures described in the previous two sections. Three nodes have complete and symmetric origin-destination requirements each of which has a present probability 1/2. Since the number of commodities is 3, there are $2^3 = 8$ instances each has occurrence probability 1/8. The fixed costs are all 1, while the variable costs are all 0.1.

By enumerating all possible cases, which are shown in Table 3, we get the optimal expected cost $E[c] = 1.53$. The upper bound $E[c_\alpha]$ obtained by the a priori network strategy with $\alpha = \{(1,2),(1,3)\}$ is 1.7. If we use the complete network as the a priori network, i.e., $\alpha = \{(1,2),(1,3),(2,3)\}$, the upper bound is much worse; we get $E[c_\alpha] = 2.3$. The linear programming relaxation without the probabilistic cutset inequalities yields the initial lower bound 0.65, which can be improved by adding the following probabilistic cutset inequalities

$$y_{12} + y_{13} \geq 3/4$$
$$y_{12} + y_{23} \geq 3/4$$
$$y_{23} + y_{13} \geq 3/4$$

and by solving the linear programming relaxation; we get the improved lower bound $E[\bar{c}] = 1.275$.

6 Conclusion

We analyzed the probabilistic network design problem which has wide applicability to several network design problems under the uncertainty condition of the demands. The model analyzed in this paper can be applied to the distribution network design, the communication network design, the transportation network design, or other related application fields in which the uncertainty is the major factor of the problem.

We presented the methodologies for deriving the upper and lower bounds of the expected network design costs over all possible instances generated probabilistically. The upper bounds can be obtained using the concepts of the a priori network strategy. This approach is useful both in the operational (short term) model and in the strategic (middle or long term) model of decision planning. The daily network can be constructed using the a priori network, and the expected cost of the daily network costs can be also evaluated using the formulas derived in Section 3. Further we analyzed two heuristics for constructing the a priori network both in the worst case and in the probabilistic model.

The lower bounds are obtained using the concepts of the probabilistic valid inequalities. We extended several concepts of the polyhedral theory to the probabilistic variants, and then derived several types of the probabilistic valid inequalities which lead the lower bounds of the expectations.

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