

## ON THE $M/G/1$ QUEUE WITH MULTIPLE VACATIONS AND GATED SERVICE DISCIPLINE

Tetsuya Takine    Toshiharu Hasegawa  
*Kyoto University*

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**Abstract** The  $M/G/1$  queue with multiple vacations and gated service discipline is considered. We first provide an alternative approach to derive the Laplace-Stieltjes transform of the limiting waiting time distribution, which reveals the stochastic structure of the gated vacation system. Next we discuss the regenerative cycle of the gated vacation system. Based on these considerations, we carry out a time-dependent analysis, from which various formulas are derived.

### 1. Introduction

In queueing systems with vacations, a server is unavailable for occasional intervals of time called vacations. The vacation systems have been extensively studied for the last two decades, because of the applicability to the performance evaluation of computer communication systems and manufacturing systems. Excellent surveys of the analysis of the vacation systems have been published by Doshi [6], [7].

There are many variants in the vacation systems. Among them, the exhaustive and the gated vacation systems have been considered as the fundamental models. In the exhaustive vacation system, once the service begins, customers are continuously served until the system becomes empty, while, in the gated vacation system, only the customers waiting at the end of a vacation are served continuously and customers arriving to the system during a service period are served in the next service period. For the exhaustive vacation system, several researchers have studied the transient behavior of the system (see, e.g., Takagi [16], [17]).

This paper considers the time-dependent behavior of the gated vacation system. The gated vacation system have been studied by Ali and Neuts [1], Cooper [4], [5], Fuhrmann and Cooper [9], Leibowitz [12], Sumita [13], Takács [14], Takagi [15]. Also multiqueue systems with gated service discipline have been studied by Hashida [10], Ferguson and Aminetzah [8]. To the best of our knowledge, all the works on the gated service system have assumed that the system has been already in equilibrium. On the contrary, in this paper, we assume only that the system is empty at time 0 and the server takes a vacation at that time, and that the offered load is less than one which ensures the existence of the limiting distribution.

In Section 2, we provide an alternative approach to derive the Laplace-Stieltjes transform (LST) of the limiting distribution function (DF) of the waiting time. This approach reveals the underlying structure of the stochastic behavior of the gated vacation system, though the result has been known in the literature. In Section 3, we consider the regenerative cycle of the gated vacation system. By a straightforward approach, the LST of the DF of the regenerative cycle is derived. Based on these considerations, in Section 4, we provide the time-dependent analysis of the queue length distribution, and derive various new formulas for the queue length, the workload and the waiting time. Finally, in Section 5, we analyze

the depletion time distribution for the gated vacation system.

The mathematical model is described in detail. Customers are served according to the gated service discipline. The server takes a vacation after the services of customers waiting at the end of the last vacation. When there are no customers at the end of a vacation, the server takes another vacation. This vacation policy is called multiple vacations. Customers arrive to the system in accordance with a Poisson process of density  $\lambda$ , and they are accommodated in the system and never lost. We assume that customers are served in order of arrival unless state otherwise. The service times of customers are independent and identically distributed in accordance with a general DF which has a positive finite mean  $E[H]$ . Let  $H^*(s)$  ( $\text{Re}(s) > 0$ ) denote the LST of the DF of a service time. The lengths of the vacation periods are independent and identically distributed in accordance with a general DF which has a positive finite mean  $E[V]$ . Let  $V^*(s)$  ( $\text{Re}(s) > 0$ ) denote the LST of the DF of a vacation period. For brevity, let  $\rho$  denote  $\lambda E[H]$ .

Throughout this paper, we assume  $\rho < 1$ , so that customers who arrive to the system are eventually served. Moreover, for avoidance of complexity, it is assumed that there are no customers at time 0 and the server takes the first vacation at that time.

## 2. Limiting Distribution of Waiting Time

In this section, we consider the limiting distribution of the waiting time of a customer. The result is identical to that in equilibrium, which is well-known in the literature (equation (6) in [9]). However, the analysis presented below is quite different from the previous works and reveals the stochastic structure of the gated  $M/G/1$  vacation system.

### 2.1 Basic observation

Observing the state of the server, we note the alternating sequence of vacation periods and service periods. Note that the length of a service period takes zero when there are no customers at the end of the last vacation. Let  $V_n$  and  $S_n$  ( $n = 1, 2, \dots$ ) be the random variables for the lengths of the  $n$ th vacation period and the  $n$ th service period, respectively.

Customers are called class  $n$  ( $n = 1, 2, \dots$ ) if they arrive to the system during the  $n$ th vacation period. Also, customers that arrive to the system while a class  $n$  customer is being served are called class  $n$ . Then class  $n$  customers form the ancestral line of customers that arrive to the system during the  $n$ th vacation period (see p.1122 in [9] for the definition of the ancestral line). Thus, only class 1 customers are served in  $S_1$ , while, in  $S_2$ , class 1 customers (that arrived during  $S_1$ ) are first served and then class 2 customers are served. In general, class 1 customers (if any) are first served in  $S_n$  and class  $l$  customers ( $l = 2, \dots, n$ ) are served after the services of class  $k$  customers ( $k = 1, \dots, l - 1$ ). In a word, customers are served in an ascending order of classes in each service period. Thus  $S_n$  is given by

$$(2.1) \quad S_n = \sum_{l=1}^n S_n^{(l)}, \quad n = 1, 2, \dots,$$

where  $S_n^{(l)}$  denotes a random variable for the sum of service times of class  $l$  customers in the  $n$ th service period. Note that the random variables  $S_n^{(l)}$ 's are mutually independent.

We consider the stochastic processes associated with class  $l$  customers. It is readily understood that  $S_n^{(l)}$ 's ( $n = l, l + 1, \dots$ ) form the delayed busy period with the initial delay  $V_l$ . Further  $S_n^{(l)}$  corresponds to the sum of the service times of  $L_n^{(l)}$  customers, where  $L_n^{(l)}$  ( $l \leq n$ ) denotes the number of class  $l$  customers waiting at the end of the  $n$ th vacation period. Note that  $L_n^{(l)}$  ( $l < n$ ) can be viewed as the number of the  $(n - l)$ th generation of offsprings in the branching process with the progenitors (whose number is given by  $L_l^{(l)}$ ) that arrive to the system during the  $l$ th vacation period. Thus the LST  $\theta_k(\omega)$  ( $k = 0, 1, \dots$ )

of the DF of the  $S_{i+k}^{(l)}$  satisfies

$$(2.2) \quad \theta_0(\omega) = V^*(\lambda - \lambda H^*(\omega)),$$

$$(2.3) \quad \theta_k(\omega) = \theta_{k-1}(\lambda - \lambda H^*(\omega)), \quad k \geq 1,$$

for  $Re(\omega) > 0$ . Note that the DF of  $S_{i+k}^{(l)}$  is independent of  $l$ .

### 2.2 Limiting waiting time

We prove the following theorem by a quite different approach from the previous works.

*Theorem 2.1.* Let  $W^*(\omega)$  denote the LST of the limiting DF of the waiting time of a customer. Then

$$(2.4) \quad W^*(\omega) = \frac{(1 - \rho)\{1 - V^*(\omega)\}}{E[V]\{\omega - \lambda + \lambda H^*(\omega)\}} \alpha(\omega), \quad Re(\omega) > 0,$$

where

$$(2.5) \quad \alpha(\omega) = \prod_{k=0}^{\infty} \theta_k(\omega), \quad Re(\omega) > 0.$$

*Proof.* Let us consider the waiting time of a customer that arrives to the system in the  $n$ th ( $n \geq 1$ ) vacation period. This customer should wait for his service during the following periods: the forward recurrence time of the  $n$ th vacation period,  $S_n^{(k)}$ 's ( $k = 1, \dots, n - 1$ ) and the service times of customers that arrive to the system during the backward recurrence time of the  $n$ th vacation period. Let  $N(\tilde{V}_n)$  denote the number of customers that arrive to the system in the backward recurrence time  $\tilde{V}_n$  of the  $n$ th vacation period. The joint transform of  $N(\tilde{V}_n)$  and the forward recurrence time  $\hat{V}_n$  of the  $n$ th vacation is given by (p.783 in Lee [11])

$$(2.6) \quad E [z^{N(\tilde{V}_n)} e^{-\omega \hat{V}_n}] = \frac{V^*(\lambda - \lambda z) - V^*(\omega)}{E[V](\omega - \lambda + \lambda z)}.$$

Therefore the LST of the DF of the waiting time  $W_n^V$  of a customer that arrives to the system during the  $n$ th vacation period is found to be

$$(2.7) \quad \begin{aligned} E [e^{-\omega W_n^V}] &= E \left[ H^*(\omega)^{N(\tilde{V}_n)} e^{-\omega \sum_{k=1}^{n-1} S_n^{(k)}} e^{-\omega \hat{V}_n} \right] \\ &= \frac{V^*(\lambda - \lambda H^*(\omega)) - V^*(\omega)}{E[V]\{\omega - \lambda + \lambda H^*(\omega)\}} \prod_{k=1}^{n-1} S_n^{(k)}(\omega) \\ &= \frac{\theta_0(\omega) - V^*(\omega)}{E[V]\{\omega - \lambda + \lambda H^*(\omega)\}} \prod_{k=1}^{n-1} \theta_k(\omega), \quad Re(\omega) > 0, \end{aligned}$$

where

$$(2.8) \quad S_n^{(k)}(\omega) = E \left[ e^{-\omega S_n^{(k)}} \right], \quad Re(\omega) > 0.$$

In the above and the following equations, the product taken in decreasing order is defined to be one. We denote by  $W^V$  the waiting time of a customer that arrives to the system during a vacation period in the limit  $n \rightarrow \infty$ . It follows from (2.7) that

$$(2.9) \quad E [e^{-\omega W^V}] = \frac{\theta_0(\omega) - V^*(\omega)}{E[V]\{\omega - \lambda + \lambda H^*(\omega)\}} \prod_{k=1}^{\infty} \theta_k(\omega), \quad Re(\omega) > 0.$$

Next we consider a class  $l$  customer that arrives to the system during the  $n$ th service period. Recall that this customer does not served in the  $n$ th service period, but is served in the  $(n + 1)$ th service period because of the gated service discipline. Thus this customer should wait for his service during the following periods: the forward recurrence time of  $S_n^{(l)}$ ,  $S_n^{(k)}$ 's ( $k = l + 1, \dots, n$ ), the  $(n + 1)$ th vacation period,  $S_{n+1}^{(k)}$ 's ( $k = 1, \dots, l - 1$ ) and service times of customers who arrived to the system during the backward recurrence time of  $S_n^{(l)}$ . Thus we get for  $Re(\omega) > 0$

$$(2.10) \quad E \left[ e^{-\omega W_n^{S(l)}} \right] = \frac{S_n^{(l)}(\lambda - \lambda H^*(\omega)) - S_n^{(l)}(\omega)}{E[S_n^{(l)}] \{ \omega - \lambda + \lambda H^*(\omega) \}} \prod_{k=l+1}^n S_n^{(k)}(\omega) V^*(\omega) \prod_{k=1}^{l-1} S_{n+1}^{(k)}(\omega),$$

where  $W_n^{S(l)}$  denotes the waiting time of a class  $l$  customer that arrives to the system during the  $n$ th service period. Therefore the LST of the DF of the waiting time  $W_n^S$  of a customer that arrives to the system during the  $n$ th service period is found to be

$$(2.11) \quad \begin{aligned} E \left[ e^{-\omega W_n^S} \right] &= \sum_{l=1}^n \frac{E[S_n^{(l)}]}{E[S_n]} E \left[ e^{-\omega W_n^{S(l)}} \right] \\ &= \frac{V^*(\omega)}{E[S_n]} \sum_{l=1}^n \left( \prod_{k=n+2-l}^n \theta_k(\omega) \frac{\theta_{n+1-l}(\omega) - \theta_{n-l}(\omega)}{\omega - \lambda + \lambda H^*(\omega)} \prod_{k=0}^{n-1-l} \theta_k(\omega) \right) \\ &= \frac{V^*(\omega)}{E[S_n] \{ \omega - \lambda + \lambda H^*(\omega) \}} \prod_{k=0}^n \theta_k(\omega) \left( \frac{1}{\theta_0(\omega)} - \frac{1}{\theta_n(\omega)} \right). \end{aligned}$$

We denote by  $W^S$  the waiting time of a customer that arrives to the system during a service period in the limit of  $n \rightarrow \infty$ . The assumption of  $\rho < 1$  asserts that the sequence of service periods  $\{S_n; n = 1, 2, \dots\}$  converges in distribution to a random variable  $S$  whose distribution is given by the limiting distribution of the service period. Further  $\theta_n(\omega)$  approaches to one as  $n$  goes to infinity. Thus we have

$$(2.12) \quad E \left[ e^{-\omega W^S} \right] = \frac{V^*(\omega)}{E[S] \{ \omega - \lambda + \lambda H^*(\omega) \}} \{ 1 - \theta_0(\omega) \} \prod_{k=1}^{\infty} \theta_k(\omega), \quad Re(\omega) > 0.$$

Note that  $E[S]$  can be expressed in terms of  $E[V]$  and  $\rho$  (see (A.9) in Appendix).

From (2.9) and (2.12), the LST  $W^*(\omega)$  of the limiting DF of the waiting time is given by

$$(2.13) \quad \begin{aligned} W^*(\omega) &= \lim_{t \rightarrow \infty} \Pr\{ \text{on vacation at time } t \} E \left[ e^{-\omega W^V} \right] \\ &\quad + \lim_{t \rightarrow \infty} \Pr\{ \text{on service at time } t \} E \left[ e^{-\omega W^S} \right]. \end{aligned}$$

because of Poisson arrivals. To complete the proof, we need the following lemma.

*Lemma 2.2.*

$$(2.14) \quad \lim_{t \rightarrow \infty} \Pr\{ \text{on vacation at time } t \} = 1 - \rho.$$

The formal proof of Lemma 2.2 is provided in Appendix. Note that Lemma 2.2 is also derived by Little's formula. Theorem 2.1 immediately follows from (2.13) and Lemma 2.2. □

*Remark.* By the definition of  $\theta_k(\omega)$ ,  $\alpha(\omega)$  in (2.5) corresponds to the LST of the limiting DF of a service period. Further  $\alpha(\omega)$  satisfies the equation

$$(2.15) \quad \alpha(\omega) = V^*(\lambda - \lambda H^*(\omega))\alpha(\lambda - \lambda H^*(\omega)), \quad Re(\omega) > 0.$$

### 3. Regenerative Cycle

In this section, we consider the regenerative cycle of the gated vacation system. As we stated in the last section, we have the alternating sequence of  $V_n$  and  $S_n$  ( $n = 1, 2, \dots$ ). Recall that there are no customers at time 0 and the server takes the first vacation at that time. Suppose that there are no customers at the end of the  $m$ th service period for some  $m$  ( $m \geq 1$ ). Clearly the stochastic behavior of the alternating sequence of  $V_n$  and  $S_n$  ( $n = m+1, m+2, \dots$ ) is a probabilistic replica of the alternating sequence beginning at time 0. Thus we define the regenerative cycle as an interval between successive instants that the server takes vacations and there are no customers at the beginnings of those vacations.

We define  $\psi_k(\omega_1, \omega_2)$  ( $k = 1, 2, \dots$ ) associated with the delayed cycle generated by customers of each class by

$$(3.16) \quad \psi_1(\omega_1, \omega_2) = V^*(\omega_1 + \lambda - \lambda H^*(\omega_2)),$$

$$(3.17) \quad \psi_k(\omega_1, \omega_2) = \psi_{k-1}(\omega_1, \omega_1 + \lambda - \lambda H^*(\omega_2)), \quad k \geq 2,$$

$$(3.18) \quad \phi_k(\omega_1) = \psi_k(\omega_1, \omega_1 + \lambda), \quad k \geq 1,$$

for  $Re(\omega_j) > 0$  ( $j = 1, 2$ ). Note that  $\psi_k(\omega_1, \omega_2)$  ( $k \geq 2$ ) is the joint transform of the delay cycle generated by first  $k - 1$  generations of offsprings and the sum of service times of the  $k$ th generation of offsprings. We now prove the following theorem.

*Theorem 3.3.* Let  $R^*(\omega)$  denote the LST of the DF of the length of the regenerative cycle. Then

$$(3.19) \quad R^*(\omega) = \frac{\phi(\omega)}{1 + \phi(\omega)}, \quad Re(\omega) > 0,$$

where

$$(3.20) \quad \phi(\omega) = \sum_{j=1}^{\infty} \prod_{k=1}^j \phi_k(\omega).$$

*Proof.* We consider the first regenerative cycle beginning at time 0. We define  $R_n(\omega_1, \omega_2)$  by

$$(3.21) \quad R_n(\omega_1, \omega_2) = E \left[ e^{-\omega_1 T_{n-1}} e^{-\omega_1 V_n} e^{-\omega_2 S_n} \mid n \leq n^* \right] \Pr\{n \leq n^*\},$$

$$Re(\omega_j) > 0 \quad (j = 1, 2),$$

where

$$(3.22) \quad T_n = \sum_{j=1}^n (V_j + S_j), \quad n \geq 0,$$

$$(3.23) \quad n^* = \inf\{n \mid \text{no customers at the end of the } n\text{th } (n \geq 1) \text{ service period}\}.$$

In the above and the following equations, the summation taken in decreasing order is defined to be zero. By the definition, we have

$$(3.24) \quad R_1(\omega_1, \omega_2) = V^*(\omega_1 + \lambda - \lambda H^*(\omega_2)).$$

Further, by noting that

$$(3.25) \quad R_n(\omega_1, \omega_1 + \lambda - \lambda H^*(\omega_2)) - R_n(\omega_1, \omega_1 + \lambda) \\ = E \left[ e^{-\omega_1 T_n} e^{-\omega_2 \sum_{k=1}^n S_{n+1}^{(k)}} \mid n < n^* \right] \Pr\{n < n^*\},$$

we have the following equation for  $n \geq 2$ :

$$(3.26) \quad R_n(\omega_1, \omega_2) = \{R_{n-1}(\omega_1, \omega_1 + \lambda - \lambda H^*(\omega_2)) - R_{n-1}(\omega_1, \omega_1 + \lambda)\} \\ \cdot V^*(\omega_1 + \lambda - \lambda H^*(\omega_2)).$$

From (3.24) and (3.26), the  $R_n(\omega_1, \omega_2)$  can be expressed as

$$(3.27) \quad R_n(\omega_1, \omega_2) = \sum_{j=1}^n a_{n,j}(\omega_1) \prod_{k=1}^j \psi_k(\omega_1, \omega_2), \quad n \geq 1,$$

where

$$(3.28) \quad a_{1,1}(\omega_1) = 1,$$

$$(3.29) \quad a_{n,j}(\omega_1) = a_{n-1,j-1}(\omega_1), \quad 2 \leq j \leq n,$$

$$(3.30) \quad a_{n,1}(\omega_1) = - \sum_{j=1}^{n-1} a_{n-1,j}(\omega_1) \prod_{k=1}^j \phi_k(\omega_1), \quad n \geq 2.$$

The expression (3.27) can be readily verified by the direct substitution in (3.24) and (3.26). We now define  $R_n^*(\omega)$  by

$$(3.31) \quad R_n^*(\omega) = E \left[ e^{-\omega T_n} \mid n^* = n \right] \Pr\{n^* = n\}.$$

We then have

$$(3.32) \quad R_n^*(\omega) = R_n(\omega, \omega + \lambda) \\ = \sum_{j=1}^n a_{n,j}(\omega) \prod_{k=1}^j \phi_k(\omega), \quad n \geq 1, \operatorname{Re}(\omega) > 0.$$

Therefore the LST  $R^*(\omega)$  of the DF of the length of the regenerative cycle is given by

$$(3.33) \quad R^*(\omega) = \sum_{n=1}^{\infty} R_n^*(\omega) \\ = \sum_{j=1}^{\infty} \sum_{n=j}^{\infty} a_{n,j}(\omega) \prod_{k=1}^j \phi_k(\omega).$$

Note here that

$$(3.34) \quad \sum_{n=j}^{\infty} a_{n,j}(\omega) = \sum_{n=1}^{\infty} a_{n,1}(\omega) \\ = 1 - R^*(\omega),$$

because  $a_{n,1}(\omega) = -R_{n-1}^*(\omega)$  for  $n \geq 2$ . Thus we get

$$(3.35) \quad R^*(\omega) = \{1 - R^*(\omega)\} \sum_{j=1}^{\infty} \prod_{k=1}^j \phi_k(\omega).$$

Theorem 3.1 immediately follows from (3.35). □

*Remarks.* 1. Since  $\phi_{j+1-k}(\omega)$  ( $j = 1, 2, \dots$ , and  $k = 1, \dots, j$ ) can be interpreted to be

$$(3.36) \quad \phi_{j+1-k}(\omega) = E \left[ e^{-\omega \left( V_k + \sum_{m=k}^j S_m^{(k)} \right)} \right. \\ \left. \begin{array}{l} | \text{ no class } k \text{ customers at the end of the } j\text{th service period} \\ \cdot \Pr\{ \text{ no class } k \text{ customers at the end of the } j\text{th service period} \}, \end{array} \right]$$

the last factor in (3.35) means

$$(3.37) \quad \prod_{k=1}^j \phi_k(\omega) = E \left[ e^{-\omega T_j} | \text{ no customers at the end of the } j\text{th service period} \right] \\ \cdot \Pr\{ \text{ no customers at the end of the } j\text{th service period} \}.$$

Thus  $R_n^*(s)$  can be expressed as

$$(3.38) \quad R_n^*(\omega) = \prod_{k=1}^n \phi_k(\omega) - \sum_{j=1}^{n-1} R_{n-j}^*(\omega) \prod_{k=1}^j \phi_k(\omega),$$

which can be verified by (3.32).

2.  $\phi(\omega)$  can be viewed as the LST of the renewal function for the regenerative cycle. That is, from Theorem 3.1

$$(3.39) \quad \phi(\omega) = \frac{R^*(\omega)}{1 - R^*(\omega)} \\ = \sum_{k=1}^{\infty} \{R^*(\omega)\}^k.$$

3. The average length  $\bar{R}$  of the regenerative cycle is equivalent to the average recurrence time of state 0 in the imbedded Markov renewal process formulated in Appendix. Thus, with (A.7) and (A.8), we have (equation (4.5) in Çinlar [3])

$$(3.40) \quad \bar{R} = \frac{E[V]}{(1 - \rho)\alpha(\lambda)},$$

where  $\alpha(\omega)$  is defined in (2.5).

#### 4. Time-dependent Analysis

In this section, we provide the time-dependent analysis of the gated vacation system under the assumption that there are no customers at time 0 and the server takes the first vacation at that time. Let  $L(t)$  denote the number of customers in the system at time  $t \geq 0$ . Note that, during a service period, we have two types of customers in the system. One consists of customers that is being or will be served in the current service period and the other consists of customers that will be served in the next service period (i.e., those arrived to the system in the current service period). Let  $L_1(t)$  and  $L_2(t)$  denote the numbers of customers of the former type and of the latter type, respectively, in the system at time  $t$  when the server is busy. Note that  $L(t) = L_1(t) + L_2(t)$ . Let  $\hat{S}(t)$  denote the forward

recurrence time of the current service at time  $t$  when the server is busy. Similarly let  $\hat{V}(t)$  denote the forward recurrence time of the current vacation at time  $t$  when the server is on vacation.

**4.3 Time-dependent analysis of joint distributions**

We define the joint transforms  $\Omega(s, z, \omega)$  and  $\Pi(s, z_1, z_2, \omega)$  by

$$(4.41) \quad \Omega(s, z, \omega) = E \left[ e^{-st} z^{L(t)} e^{-\omega \hat{V}(t)} \mid \text{on vacation at time } t \right] \cdot \Pr\{ \text{on vacation at } t \}, \quad \text{Re}(s) > 0, |z| \leq 1, \text{Re}(\omega) > 0,$$

$$(4.42) \quad \Pi(s, z_1, z_2, \omega) = E \left[ e^{-st} z_1^{L_1(t)} z_2^{L_2(t)} e^{-\omega \hat{S}(t)} \mid \text{on service at time } t \right] \cdot \Pr\{ \text{on service at time } t \}, \\ \text{Re}(s) > 0, |z_1| \leq 1, |z_2| \leq 1, \text{Re}(\omega) > 0.$$

We prove the following theorem.

*Theorem 4.4.* The joint transforms  $\Omega(s, z, \omega)$  and  $\Pi(s, z_1, z_2, \omega)$  of the time-dependent DF's are given by

$$(4.43) \quad \Omega(s, z, \omega) = \{1 + \psi(s, s + \lambda - \lambda z)\} \frac{V^*(s + \lambda - \lambda z) - V^*(\omega)}{\omega - s - \lambda + \lambda z},$$

$$(4.44) \quad \Pi(s, z_1, z_2, \omega) = \frac{\psi(s, s + \lambda - \lambda z_2) - \{1 + \psi(s, s + \lambda - \lambda z_1)\} V^*(s + \lambda - \lambda z_1)}{H^*(s + \lambda - \lambda z_2) - z_1} \cdot \frac{H^*(s + \lambda - \lambda z_2) - H^*(\omega)}{\omega - s - \lambda + \lambda z_2} \cdot z_1,$$

where

$$(4.45) \quad \psi(\omega_1, \omega_2) = \sum_{j=1}^{\infty} \prod_{k=1}^j \psi_k(\omega_1, \omega_2), \quad \text{Re}(\omega_j) > 0 \quad (j = 1, 2).$$

*Proof.* We first derive the joint transform  $\Omega_n^{(1)}(s, z, \omega)$  and  $\Pi_n^{(1)}(s, z_1, z_2, \omega)$  of the time-dependent DF's associated with the first regenerative cycle, which are defined by

$$(4.46) \quad \Omega_n^{(1)}(s, z, \omega) = E \left[ e^{-st} z^{L(t)} e^{-\omega \hat{V}(t)} \mid T_{n-1} \leq t < T_{n-1} + V_n \text{ and } n \leq n^* \right] \cdot \Pr\{T_{n-1} \leq t < T_{n-1} + V_n \text{ and } n \leq n^*\}, \\ n \geq 1, \text{Re}(s) > 0, |z| \leq 1, \text{Re}(\omega) > 0,$$

$$(4.47) \quad \Pi_n^{(1)}(s, z_1, z_2, \omega) = E \left[ e^{-st} z_1^{L_1(t)} z_2^{L_2(t)} e^{-\omega \hat{S}(t)} \mid T_{n-1} + V_n \leq t < T_n \text{ and } n \leq n^* \right] \cdot \Pr\{T_{n-1} + V_n \leq t < T_n \text{ and } n \leq n^*\}, \\ n \geq 1, \text{Re}(s) > 0, |z_1| \leq 1, |z_2| \leq 1, \text{Re}(\omega) > 0,$$

where  $T_n$  and  $n^*$  are defined in (3.22) and (3.23), respectively. We then have

$$(4.48) \quad \Omega_1^{(1)}(s, z, \omega) = \int_0^{\infty} dt \int_t^{\infty} dV(x) e^{-st} e^{-(\lambda - \lambda z)t} e^{-\omega(x-t)} \\ = \frac{V^*(s + \lambda - \lambda z) - V^*(\omega)}{\omega - s - \lambda + \lambda z},$$

where  $V(x)$  denotes the DF of a vacation period. By noting that

$$(4.49) \quad R_{n-1}(s, s + \lambda - \lambda z) - R_{n-1}^*(s) = E \left[ e^{-sT_{n-1}} z^{L(T_{n-1})} \mid n \leq n^* \right] \Pr\{n \leq n^*\},$$



We have the following equation for  $n \geq 2$ :

$$(4.50) \quad \Omega_n^{(1)}(s, z, \omega) = \{R_{n-1}(s, s + \lambda - \lambda z) - R_{n-1}^*(s)\} \Omega_1^{(1)}(s, z, \omega).$$

Next we consider the  $\Pi_n^{(1)}(s, z_1, z_2, \omega)$ .  $\Pi_1^{(1)}(s, z_1, z_2, \omega)$  can be derived by conditioning the number of customers at the end of the first vacation:

$$(4.51) \quad \Pi_1^{(1)}(s, z_1, z_2, \omega) = \sum_{i=1}^{\infty} \sum_{k=1}^i \int_0^{\infty} dt \int_0^t dV_i(x) dx \int_0^{t-x} dH^{(k-1)}(y) \int_{t-x-y}^{\infty} dH(u) \\ \cdot e^{-st} z_1^{i-(k-1)} e^{-(\lambda-\lambda z_2)(t-x)} e^{-\omega(x+y+u-t)},$$

where  $H^{(k)}(x)$  denotes the  $k$ -fold convolution of the service time DF  $H(x)$  with itself and  $V_i(x)$  denotes the DF of the length of a vacation period when  $i$  customers arrive to the system during that period. Note here that

$$(4.52) \quad \sum_{i=0}^{\infty} z^i \int_0^{\infty} e^{-sx} dV_i(x) = V^*(s + \lambda - \lambda z).$$

Thus, after some calculations, we get

$$(4.53) \quad \Pi_1^{(1)}(s, z_1, z_2, \omega) = \frac{\psi_1(s, s + \lambda - \lambda z_2) - V^*(s + \lambda - \lambda z_1)}{H^*(s + \lambda - \lambda z_2) - z_1} \cdot \frac{H^*(s + \lambda - \lambda z_2) - H^*(\omega)}{\omega - s - \lambda + \lambda z_2} \cdot z_1.$$

$\Pi_n^{(1)}(s, z_1, z_2, \omega)$  can be derived in a similar way. Some calculations yields

$$(4.54) \quad \Pi_n^{(1)}(s, z_1, z_2, \omega) = \sum_{i=1}^{\infty} \sum_{k=1}^i \int_0^{\infty} dt \int_0^t dRV_{n-1,i}(x) dx \int_0^{t-x} dH^{(k-1)}(y) \int_{t-x-y}^{\infty} dH(u) \\ \cdot e^{-st} z_1^{i-(k-1)} e^{-(\lambda-\lambda z_2)(t-x)} e^{-\omega(x+y+u-t)} \\ = \frac{R_n(s, s + \lambda - \lambda z_2) - \{R_{n-1}(s, s + \lambda - \lambda z_1) - R_{n-1}^*(s)\} V^*(s + \lambda - \lambda z_1)}{H^*(s + \lambda - \lambda z_2) - z_1} \\ \cdot \frac{H^*(s + \lambda - \lambda z_2) - H^*(\omega)}{\omega - s - \lambda + \lambda z_2} \cdot z_1, \quad n \geq 2,$$

where  $RV_{n,i}(x)$  denotes the joint DF which satisfies

$$(4.55) \quad \sum_{i=0}^{\infty} z^i \int_0^{\infty} e^{-sx} dRV_{n,i}(x) = \{R_n(s, s + \lambda - \lambda z) - R_n^*(s)\} V^*(s + \lambda - \lambda z).$$

We now define  $\Omega^{(1)}(s, z, \omega)$  and  $\Pi^{(1)}(s, z_1, z_2, \omega)$  by

$$(4.56) \quad \Omega^{(1)}(s, z, \omega) = \sum_{n=1}^{\infty} \Omega_n^{(1)}(s, z, \omega), \quad Re(s) > 0, |z| \leq 1, Re(\omega) > 0,$$

$$(4.57) \quad \Pi^{(1)}(s, z_1, z_2, \omega) = \sum_{n=1}^{\infty} \Pi_n^{(1)}(s, z_1, z_2, \omega), \\ Re(s) > 0, |z_1| \leq 1, |z_2| \leq 1, Re(\omega) > 0.$$

It then follows from (4.48), (4.50), (4.53) and (4.54) that

$$(4.58) \quad \Omega^{(1)}(s, z, \omega) = \{1 - R^*(s) + R(s, s + \lambda - \lambda z)\} \frac{V^*(s + \lambda - \lambda z) - V^*(\omega)}{\omega - s - \lambda + \lambda z},$$

$$(4.59) \quad \Pi^{(1)}(s, z_1, z_2, \omega) = \left[ \frac{R(s, s + \lambda - \lambda z_2)}{H^*(s + \lambda - \lambda z_2) - z_1} - \frac{\{1 - R^*(s) + R(s, s + \lambda - \lambda z_1)\}V^*(s + \lambda - \lambda z_1)}{H^*(s + \lambda - \lambda z_2) - z_1} \right] \cdot \frac{H^*(s + \lambda - \lambda z_2) - H^*(\omega)}{\omega - s - \lambda + \lambda z_2} \cdot z_1,$$

where

$$(4.60) \quad R(\omega_1, \omega_2) = \sum_{n=1}^{\infty} R_n(\omega_1, \omega_2), \quad Re(\omega_j) > 0 \quad (j = 1, 2).$$

Similar to  $R^*(\omega)$ ,  $R(\omega_1, \omega_2)$  can be expressed to be

$$(4.61) \quad R(\omega_1, \omega_2) = \frac{\psi(\omega_1, \omega_2)}{1 + \phi(\omega_1)},$$

where  $\phi(\omega_1)$  and  $\psi(\omega_1, \omega_2)$  are given in (3.20) and (4.45), respectively. Thus, we obtain

$$(4.62) \quad \Omega^{(1)}(s, z, \omega) = \frac{1 + \psi(s, s + \lambda - \lambda z)}{1 + \phi(s)} \cdot \frac{V^*(s + \lambda - \lambda z) - V^*(\omega)}{\omega - s - \lambda + \lambda z},$$

$$(4.63) \quad \Pi^{(1)}(s, z_1, z_2, \omega) = \frac{\psi(s, s + \lambda - \lambda z_2) - \{1 + \psi(s, s + \lambda - \lambda z_1)\}V^*(s + \lambda - \lambda z_1)}{\{1 + \phi(s)\}\{H^*(s + \lambda - \lambda z_2) - z_1\}} \cdot \frac{H^*(s + \lambda - \lambda z_2) - H^*(\omega)}{\omega - s - \lambda + \lambda z_2} \cdot z_1.$$

With the LST  $\phi(s)$  of the renewal function for the regenerative cycle, we have

$$(4.64) \quad \Omega(s, z, \omega) = \{1 + \phi(s)\}\Omega^{(1)}(s, z, \omega),$$

$$(4.65) \quad \Pi(s, z_1, z_2, \omega) = \{1 + \phi(s)\}\Pi^{(1)}(s, z_1, z_2, \omega).$$

Theorem 4.1 immediately follows from the above equations. □

*Remark.* The function  $\psi(\omega_1, \omega_2)$  satisfies

$$(4.66) \quad \psi(\omega_1, \omega_2) = V^*(\omega_1 + \lambda - \lambda H^*(\omega_2))\{1 + \psi(\omega_1, \omega_1 + \lambda - \lambda H^*(\omega_2))\},$$

which can be verified with (3.16), (3.17) and (4.45).

#### 4.4 Various formulas

We can derive various formulas from Theorem 4.1.

*Theorem 4.5.* Let  $L(s, z)$  denote the transform of the time-dependent DF of the queue length  $L(t)$ , which is defined by

$$(4.67) \quad L(s, z) = E \left[ e^{-st} z^{L(t)} \right], \quad Re(s) > 0, |z| \leq 1.$$

Then

$$(4.68) \quad L(s, z) = \frac{H^*(s + \lambda - \lambda z)\{1 - (1 - z)V^*(s + \lambda - \lambda z)\} - z}{(s + \lambda - \lambda z)\{H^*(s + \lambda - \lambda z) - z\}} + \psi(s, s + \lambda - \lambda z) \frac{(1 - z)H^*(s + \lambda - \lambda z)\{1 - V^*(s + \lambda - \lambda z)\}}{(s + \lambda - \lambda z)\{H^*(s + \lambda - \lambda z) - z\}}.$$

*Proof.* By the definition,

$$(4.69) \quad L(s, z) = \lim_{\omega \rightarrow 0} \Omega(s, z, \omega) + \lim_{\omega \rightarrow 0} \Pi(s, z, z, \omega).$$

(4.68) is readily derived from Theorem 4.1 and (4.69). □

*Corollary 4.6.* The PGF  $L(z)$  of the limiting DF of the queue length is given by

$$(4.70) \quad L(z) = \frac{(1 - \rho)H^*(\lambda - \lambda z)\{1 - V^*(\lambda - \lambda z)\}}{\lambda E[V]\{H^*(\lambda - \lambda z) - z\}} \alpha(\lambda - \lambda z), \quad |z| \leq 1.$$

*Proof.* By the definition,

$$(4.71) \quad L(z) = \lim_{s \rightarrow 0} sL(s, z).$$

Note here that

$$(4.72) \quad \lim_{s \rightarrow 0} s\psi(s, s + \lambda - \lambda z) = \lim_{t \rightarrow \infty} \sum_{n=1}^{\infty} E[z^{L_n} | T_n = t] \Pr\{T_n = t\},$$

where  $T_n$  is defined in (3.7). To complete the proof, we need the following lemma.

*Lemma 4.7.*

$$(4.73) \quad \lim_{t \rightarrow \infty} \sum_{n=1}^{\infty} E[z^{L_n} | T_n = t] \Pr\{T_n = t\} = \frac{1 - \rho}{E[V]} \alpha(\lambda - \lambda z),$$

The proof of Lemma 4.4 is provided in Appendix. Thus, (4.70) immediately follows from (4.68) and (4.73). □

We note that (4.70) is identical to the known result in equilibrium (p.1124 in [9]).

*Remark.* Let  $\Pi^*(z, \omega)$  denote the joint transform of the limiting DF of the queue length  $L(\infty)$  and the forward recurrence time  $\hat{S}(\infty)$  of the service time. From (4.44) and Lemma 4.4, we have

$$(4.74) \quad \begin{aligned} \Pi^*(z, \omega) &= \lim_{s \rightarrow 0} s\Pi(s, z, z, \omega) \\ &= \Pi_{M/G/1}^*(z, \omega)\chi(z)\alpha(\lambda - \lambda z), \quad |z| \leq 1, \operatorname{Re}(\omega) > 0, \end{aligned}$$

where  $\chi(z)$  denotes the probability generating function of the number of customers at a random point in time given that the server is on vacation and  $\Pi_{M/G/1}^*(z, \omega)$  denotes the joint transform of the steady-state queue length and the forward recurrence service time in the corresponding M/G/1 queue without vacations, and these are given by (Takagi [16])

$$(4.75) \quad \chi(z) = \frac{1 - V^*(\lambda - \lambda z)}{\lambda E[V](1 - z)}, \quad |z| \leq 1,$$

$$(4.76) \quad \Pi_{M/G/1}^*(z, \omega) = \frac{\lambda(1 - \rho)z(1 - z)\{H^*(\lambda - \lambda z) - H^*(\omega)\}}{(\omega - \lambda + \lambda z)\{H^*(\lambda - \lambda z) - z\}}, \quad |z| \leq 1, \operatorname{Re}(\omega) > 0.$$

Thus the three-way decomposition property is valid for the joint distribution of the queue length and the forward recurrence service time, similar to the exhaustive service system (Takagi [16]). Recently, Takine and Hasegawa [18] show that this decomposition property including the forward recurrence service time holds in more diverse settings.

Next we consider the workload (also called unfinished work) in the system. The workload is defined as the sum of the remaining service times of all customers in the system. Let  $U(t)$  denote the workload at time  $t$  ( $\geq 0$ ).

*Theorem 4.8.* Let  $U(s, \omega)$  denote the LST of the time-dependent DF of the workload  $U(t)$ , which is formally defined by

$$(4.77) \quad U(s, \omega) = E \left[ e^{-st} e^{-\omega U(t)} \right], \quad \operatorname{Re}(s) > 0, \operatorname{Re}(\omega) > 0.$$

Then

$$(4.78) \quad U(s, \omega) = \frac{1 + \psi(s, s + \lambda - \lambda H^*(\omega)) - \psi(s, \omega)}{s + \lambda - \lambda H^*(\omega)} + \frac{\psi(s, s + \lambda - \lambda H^*(\omega)) - \psi(s, \omega)}{\omega - s - \lambda + \lambda H^*(\omega)}.$$

*Proof.* By the definition,

$$(4.79) \quad U(s, \omega) = \Pi(s, H^*(\omega), H^*(\omega), \omega) / H^*(\omega) + \lim_{\omega_1 \rightarrow 0} \Omega(s, H^*(\omega), \omega_1).$$

(4.78) is readily obtained from Theorem 4.1, (4.66) and (4.79).  $\square$

*Corollary 4.9.* The LST  $U^*(\omega)$  of the limiting DF of the workload is given by

$$(4.80) \quad U^*(\omega) = \frac{1 - V^*(\lambda - \lambda H^*(\omega))}{E[V](\lambda - \lambda H^*(\omega))} \cdot \frac{(1 - \rho)\omega}{\omega - \lambda + \lambda H^*(\omega)} \alpha(\lambda - \lambda H^*(\omega)), \quad \operatorname{Re}(\omega) > 0.$$

*Proof.* By the definition,

$$(4.81) \quad U^*(\omega) = \lim_{s \rightarrow 0} sU(s, \omega).$$

Note here that from Lemma 4.4

$$(4.82) \quad \lim_{s \rightarrow 0} s\psi(s, s + \lambda - \lambda H^*(\omega)) = \frac{1 - \rho}{E[V]} \alpha(\lambda - \lambda H^*(\omega)).$$

Further, with (2.15), (4.66) and (4.82)

$$(4.83) \quad \begin{aligned} \lim_{s \rightarrow 0} s\psi(s, \omega) &= \frac{1 - \rho}{E[V]} V^*(\lambda - \lambda H^*(\omega)) \alpha(\lambda - \lambda H^*(\omega)) \\ &= \frac{1 - \rho}{E[V]} \alpha(\omega). \end{aligned}$$

(4.80) is readily obtained from (4.81), (4.82) and (4.83).  $\square$

Note that (4.80) is known as a special case of the work decomposition in  $M/G/1$  vacation systems (Boxma [2]).

Next we consider the waiting time of a customer. Let  $W(t)$  denote the waiting time of a (virtual) customer that arrives to the system at time  $t$  ( $\geq 0$ ).

*Theorem 4.10.* Let  $W(s, \omega)$  denote the LST of the time-dependent DF of the waiting time  $W(t)$ , which is formally defined by

$$(4.84) \quad W(s, \omega) = E \left[ e^{-st} e^{-\omega W(t)} \right], \quad \operatorname{Re}(s) > 0, \operatorname{Re}(\omega) > 0.$$

Then

$$(4.85) \quad W(s, \omega) = \frac{\{1 - V^*(\omega)\} \psi(s, \omega) - V^*(\omega)}{\omega - s - \lambda + \lambda H^*(\omega)}.$$

*Proof.* According to the same reasoning as in Section 2,

$$(4.86) \quad W(s, \omega) = \Pi(s, H^*(\omega), H^*(\omega), \omega) \frac{V^*(\omega)}{H^*(\omega)} + \Omega(s, H^*(\omega), \omega).$$

Theorem 4.7 is readily obtained from Theorem 4.1, (4.66) and (4.86).  $\square$

With Theorem 4.7 and (4.83), we have again the LST  $W^*(\omega)$  of the limiting DF of the waiting time.

$$(4.87) \quad \begin{aligned} W^*(\omega) &= \lim_{s \rightarrow 0} sW(s, \omega) \\ &= \frac{(1 - \rho)\{1 - V^*(\omega)\}}{E[V]\{\omega - \lambda + \lambda H^*(\omega)\}} \alpha(\omega), \end{aligned}$$

which is identical to (2.13).

Lastly we consider the LCFS (last-come first-served)  $M/G/1$  vacation system with gated service discipline. In the LCFS system, customers are served in reverse order of arrivals during each service period. Recall that, in the gated vacation system, only customers that are waiting at the end of a vacation are served in the following service period, and customers that arrive to the system during a service period have to wait for their services until the next service period even in the LCFS system. Thus, the LCFS discipline applies to only those who are waiting at the end of a vacation. Note that the queue length and the workload processes are identical to those in the FCFS system. Let  $W_{LCFS}(t)$  denote the waiting time of a (virtual) customer that arrives to the LCFS system at time  $t$ .

*Theorem 4.11.* Let  $W_{LCFS}(s, \omega)$  denote the LST of the time-dependent DF of the waiting time  $W_{LCFS}(t)$ , which is formally defined by

$$(4.88) \quad W_{LCFS}(s, \omega) = E \left[ e^{-st} e^{-\omega W_{LCFS}(t)} \right], \quad \operatorname{Re}(s) > 0, \operatorname{Re}(\omega) > 0.$$

Then

$$(4.89) \quad W_{LCFS}(s, \omega) = \frac{\{1 + \psi(s, s)\} V^*(s)}{\omega + \lambda - \lambda H^*(\omega) - s} - \frac{\{1 + \psi(s, \omega + \lambda - \lambda H^*(\omega))\} V^*(\omega + \lambda - \lambda H^*(\omega))}{\omega + \lambda - \lambda H^*(\omega) - s}.$$

*Proof.* When the server is on vacation at time  $t$ ,  $W_{LCFS}(t)$  is given by the sum of the forward recurrence time of the current vacation and the service times of customers that

arrive to the system during the forward recurrence time of the current vacation. On the other hand, when a customer is served at time  $t$ ,  $W_{LCFS}(t)$  is given by the sum of the forward recurrence time of the current service, the service times of  $L_1(t) - 1$  customers, a vacation period, and the service times of customers that arrive to the system during the interval from time  $t$  to the end of the vacation period. Thus we have

$$(4.90) \quad W_{LCFS}(s, \omega) = \Pi(s, H^*(\omega + \lambda - \lambda H^*(\omega)), 1, \omega + \lambda - \lambda H^*(\omega)) \cdot \frac{V^*(\omega + \lambda - \lambda H^*(\omega))}{H^*(\omega + \lambda - \lambda H^*(\omega))} + \Omega(s, 1, \omega + \lambda - \lambda H^*(\omega)).$$

Theorem 4.8 immediately follows from Theorem 4.1 and (4.90). □

*Corollary 4.12.* Let  $W_{LCFS}^*(\omega)$  denote the LST of the limiting DF of the waiting time in the LCFS system. Then for  $Re(\omega) > 0$

$$(4.91) \quad W_{LCFS}^*(\omega) = \frac{(1 - \rho)\{1 - \alpha(\omega + \lambda - \lambda H^*(\omega))V^*(\omega + \lambda - \lambda H^*(\omega))\}}{E[V](\omega + \lambda - \lambda H^*(\omega))}.$$

*Proof.* By the definition,

$$(4.92) \quad W_{LCFS}^*(\omega) = \lim_{s \rightarrow 0} sW_{LCFS}(s, \omega).$$

Substituting one for  $z$  in Lemma 4.4, we have

$$(4.93) \quad \lim_{s \rightarrow 0} s\psi(s, s) = \frac{1 - \rho}{E[V]}.$$

(4.91) follows from (4.83), (4.89), (4.92) and (4.93). □

Note that (4.91) is equivalent to the known result in equilibrium ((5.61) in [15]).

### 5. Depletion Time Distribution

In this section, we analyze the depletion time of the gated vacation system based on the results in the previous sections. The depletion time  $D(t)$  at time  $t$  is defined as an interval from time  $t$  to the first instant that there are no customers after time  $t$ . We define  $D(t) = 0$  when there are no customers at time  $t$ . Thus, for  $Re(s) > 0$  and  $Re(\omega) > 0$

$$(5.94) \quad \begin{aligned} E \left[ e^{-st} \mid D(t) = 0 \right] \Pr\{D(t) = 0\} &= \lim_{\omega \rightarrow 0} \Omega(s, 0, \omega) \\ &= \{1 + \phi(s)\} \frac{1 - V^*(s + \lambda)}{s + \lambda}. \end{aligned}$$

Next we consider the case that there are some customers at time  $t$ . Let  $I(t)$  denote the initial delay of the depletion time at time  $t$ , which is defined as the interval from time  $t$  to the first end of a service period after time  $t$ .

*Proposition 5.13.* Let  $I_1(s, z, \omega)$  denote the joint transform of the time-dependent DF's of the initial delay  $I(t)$  and the queue length  $L(t + I(t))$ , which is formally defined by

$$(5.95) \quad I_1(s, z, \omega) = E \left[ e^{-st} z^{L(t+I(t))} e^{-\omega I(t)} \right], \quad Re(s) > 0, |z| \leq 1, Re(\omega) > 0.$$

Then

$$(5.96) \quad I_1(s, z, \omega) = \frac{\psi(s, s + \lambda - \lambda z)}{\omega - s} - \frac{\psi_1(\omega, \omega + \lambda - \lambda z)\psi(s, \omega + \lambda - \lambda z)}{(\omega - s)\psi_1(s, \omega + \lambda - \lambda z)} - \{1 + \phi(s)\} \frac{V^*(s + \lambda) - \psi_1(\omega, \omega + \lambda - \lambda z)}{\omega - s - \lambda H^*(\omega + \lambda - \lambda z)}.$$

*Proof.* By the definition, for  $Re(s) > 0$ ,  $|z| \leq 1$  and  $Re(\omega) > 0$

$$(5.97) \quad E \left[ e^{-st} z^{L(t+I(t))} e^{-\omega I(t)} \mid \text{on vacation at time } t \right] \Pr\{ \text{on vacation at time } t \} \\ = \Omega(s, H^*(\omega + \lambda - \lambda z), \omega + \lambda - \lambda H^*(\omega + \lambda - \lambda z)) \\ - \Omega(s, 0, \omega + \lambda - \lambda H^*(\omega + \lambda - \lambda z)) \\ = \frac{\psi(s, \omega + \lambda - \lambda z) \{ \psi_1(s, \omega + \lambda - \lambda z) - \psi_1(\omega, \omega + \lambda - \lambda z) \}}{(\omega - s)\psi_1(s, \omega + \lambda - \lambda z)} \\ - \{1 + \phi(s)\} \frac{V^*(s + \lambda) - \psi_1(\omega, \omega + \lambda - \lambda z)}{\omega - s - \lambda H^*(\omega + \lambda - \lambda z)},$$

$$(5.98) \quad E \left[ e^{-st} z^{L(t+I(t))} e^{-\omega I(t)} \mid \text{on service at time } t \right] \Pr\{ \text{on service at time } t \} \\ = \Pi(s, H^*(\omega + \lambda - \lambda z), z, \omega + \lambda - \lambda z) \frac{1}{H(\omega + \lambda - \lambda z)} \\ = \frac{\psi(s, s + \lambda - \lambda z) - \psi(s, \omega + \lambda - \lambda z)}{\omega - s}.$$

$I_1(s, z, \omega)$  is given by the sum of the right-hand sides of the last equalities in (5.97) and (5.98). □

We now prove the following theorem.

*Theorem 5.14.* Let  $D(s, \omega)$  denote the LST of the time-dependent DF of the depletion time  $D(t)$ , which is formally defined by

$$(5.99) \quad D^*(s, \omega) = E \left[ e^{-st} e^{-\omega D(t)} \right], \quad Re(s) > 0, \quad Re(\omega) > 0.$$

Then

$$(5.100) \quad D^*(s, \omega) = \{1 + \phi(s)\} \frac{1 - V^*(s + \lambda)}{s + \lambda} + \frac{\sum_{n=1}^{\infty} \prod_{k=1}^{n-1} \phi_k(\omega) I_n(s, 0, \omega)}{1 + \phi(\omega)},$$

where for  $Re(s) > 0$ ,  $|z| \leq 1$  and  $Re(\omega) > 0$

$$(5.101) \quad I_n(s, z, \omega) = I_{n-1}(s, H^*(\omega + \lambda - \lambda z), \omega), \quad n \geq 2.$$

*Proof.* Let  $V'_n$  and  $S'_n$  ( $n = 1, 2, \dots$ ) denote the random variables for the lengths of the  $n$ th vacation period and the  $n$ th service period, respectively, after the end of the initial delay  $I(t)$ . We define  $D_n(s, z, \omega)$  by for  $Re(s) > 0$ ,  $|z| \leq 1$  and  $Re(\omega) > 0$

$$(5.102) \quad D_n(s, z, \omega) = E \left[ e^{-st} z^{L(t+T_{n-1}(t))} e^{-\omega T_{n-1}(t)} \mid n \leq \tilde{n} \right] \Pr\{n \leq \tilde{n}\}, \quad n \geq 1,$$

where

$$(5.103) \quad T_n(t) = I(t) + \sum_{j=1}^n (V'_j + S'_j),$$

$$(5.104) \quad \tilde{n} = \inf \{ n \mid \text{no customers at time } t + T_n(t) \}.$$

By the definition, we have

$$(5.105) \quad D_1(s, z, \omega) = I_1(s, z, \omega),$$

and for  $n \geq 2$

$$(5.106) \quad \mathcal{D}_n(s, z, \omega) = \{D_{n-1}(s, H^*(\omega + \lambda - \lambda z), \omega) - D_{n-1}(s, 0, \omega)\} \psi_1(\omega, \omega + \lambda - \lambda z).$$

Note that (5.106) takes the similar form to (3.26). Therefore we have for  $Re(s) > 0$  and  $Re(\omega) > 0$

$$(5.107) \quad \begin{aligned} D_n^*(s, \omega) &= E \left[ e^{-st} e^{-\omega D(t)} \mid \tilde{n} = n \right] \Pr\{\tilde{n} = n\} \\ &= D_n(s, 0, \omega) \\ &= \sum_{j=1}^{n-1} b_{n,j}(s, \omega) \prod_{k=1}^j \phi_k(\omega) + \prod_{k=1}^{n-1} \phi_k(\omega) I_n(s, 0, \omega), \quad n \geq 1, \end{aligned}$$

where

$$(5.108) \quad b_{1,1}(s, \omega) = 1,$$

$$(5.109) \quad b_{n,j}(s, \omega) = b_{n-1,j-1}(s, \omega), \quad 2 \leq j \leq n,$$

$$(5.110) \quad b_{n,1}(s, \omega) = - \sum_{j=1}^{n-2} b_{n-1,j}(s, \omega) \prod_{k=1}^j \phi_k(\omega) + \prod_{k=1}^{n-2} \phi_k(\omega) I_{n-1}(s, 0, \omega), \quad n \geq 2.$$

The expression of the right-hand side of the last equality in (5.107) can be verified by the direct substitution in (5.105) and (5.106). Thus, with (5.94), we get

$$(5.111) \quad D^*(s, \omega) = \{1 + \phi(s)\} \frac{1 - V^*(s + \lambda)}{s + \lambda} + \sum_{n=1}^{\infty} D_n^*(s, \omega).$$

Similar to the derivation of  $R^*(\omega)$ , we can obtain the following expression by noting that  $b_{n,1}(s, \omega) = -D_{n-1}^*(s, \omega)$  for  $n \geq 2$ .

$$(5.112) \quad \sum_{n=1}^{\infty} D_n^*(s, \omega) = \frac{\sum_{n=1}^{\infty} \prod_{k=1}^{n-1} \phi_k(\omega) I_n(s, 0, \omega)}{1 + \phi(\omega)}.$$

(5.100) immediately follows from (5.111) and (5.112). □

Finally the following result is stated without proof.

*Corollary 5.15.* The LST  $D^*(\omega)$  of the limiting DF of the depletion time is given by

$$(5.113) \quad D^*(\omega) = \frac{1 - \rho}{E[V]} \left[ \frac{1 - V^*(\lambda)}{\lambda} + \frac{\sum_{n=1}^{\infty} \prod_{k=1}^{n-1} \phi_k(\omega) \tilde{I}_n(0, \omega)}{1 + \phi(\omega)} \right], \quad Re(\omega) > 0,$$

where for  $|z| \leq 1$  and  $Re(\omega) > 0$ ,

$$(5.114) \quad \tilde{I}_1(z, \omega) = \frac{1 - \rho}{E[V]} \left[ \frac{\alpha(\lambda - \lambda z)}{\omega} - \frac{\psi_1(\omega, \omega + \lambda - \lambda z) \alpha(\omega + \lambda - \lambda z)}{\omega \psi_1(0, \omega + \lambda - \lambda z)} - \alpha(\lambda) \cdot \frac{V^*(\lambda) - \psi_1(\omega, \omega + \lambda - \lambda z)}{\omega - \lambda H^*(\omega + \lambda - \lambda z)} \right],$$

$$(5.115) \quad \tilde{I}_n(z, \omega) = \tilde{I}_{n-1}(H^*(\omega + \lambda - \lambda z), \omega), \quad n \geq 2.$$



**Appendix**

*Proofs of Lemma 2.2 and Lemma 4.4.*

In order to prove Lemma 2.2 and Lemma 4.4, we formulate the queue length process  $L_n$  ( $n = 1, 2, \dots$ ) just after the  $n$ th service period by an imbedded Markov renewal process in accordance with Çinlar [3]. In the followings, it is assumed that there are no customers at the beginning of the first vacation. We define the semi-Markov kernel  $Q(i, j, t)$  ( $i, j = 0, 1, \dots, t \geq 0$ ) and the transition probabilities  $P(i, j)$  ( $i, j = 0, 1, \dots$ ) by

$$(A.1) \quad Q(i, j, t) = \Pr\{L_{n+1} = j, (V_{n+1} + S_{n+1}) \leq t \mid L_n = i\},$$

$$(A.2) \quad P(i, j) = \lim_{t \rightarrow \infty} Q(i, j, t).$$

Let  $\nu$  denote the invariant probability vector associated with  $P(i, j)$ , which satisfies

$$(A.3) \quad \sum_{i=0}^{\infty} \nu(i)P(i, j) = \nu(j), \quad j = 0, 1, \dots,$$

$$(A.4) \quad \sum_{i=0}^{\infty} \nu(i) = 1.$$

We define the probability generating function  $F(z)$  for  $\nu$  by

$$(A.5) \quad F(z) = \sum_{i=0}^{\infty} \nu(i)z^i, \quad |z| \leq 1.$$

Note that

$$(A.6) \quad F(z) = \alpha(\lambda - \lambda z),$$

$$(A.7) \quad \nu(0) = \alpha(\lambda),$$

where  $\alpha(\omega)$  is defined in (2.5). Let  $m(i)$  denote the average sojourn time of state  $i$  ( $i = 0, 1, \dots$ ). It follows from (2.15) and (A.6) that

$$(A.8) \quad \begin{aligned} \sum_{i=0}^{\infty} \nu(i)m(i) &= E[V] + E[S] \\ &= \frac{E[V]}{1 - \rho}, \end{aligned}$$

because  $m(i) = (1 + \rho)E[V] + iE[H]$ . Thus we have

$$(A.9) \quad E[S] = \frac{\rho E[V]}{1 - \rho}.$$

Further, Markov renewal functions  $R(j, t)$ 's ( $j = 0, 1, \dots$ ) are defined by

$$(A.10) \quad R(j, t) = \sum_{n=1}^{\infty} \Pr\{L_n = j, \sum_{i=1}^n (V_i + S_i) \leq t\}, \quad j = 0, 1, \dots, t \geq 0.$$

Thus, for any function  $g(j, t)$  which is directly integrable with respect to  $\nu(i)$ 's, we have ((4.17) in Çinlar [3])

$$(A.11) \quad \lim_{t \rightarrow \infty} \sum_{j=0}^{\infty} \int_0^t R(j, du)g(j, t - u) = \frac{1 - \rho}{E[V]} \int_0^{\infty} \sum_{j=0}^{\infty} \nu(j)g(j, u)du.$$

When  $g(j, t) = 1 - V(t)$  in (A.11), we have Lemma 2.2:

$$(A.12) \quad \lim_{t \rightarrow \infty} \Pr\{\text{the server is on vacation at } t\} = 1 - \rho,$$

where  $V(t)$  denotes the DF of a vacation period. Also, when  $g(j, t) = z^j \delta(t)$  for  $|z| \leq 1$  in (A.11), we have Lemma 4.4:

$$(A.13) \quad \lim_{t \rightarrow \infty} \sum_{n=1}^{\infty} E[z^{Ln} | T_n = t] \Pr\{T_n = t\} = \frac{1 - \rho}{E[V]} F(z),$$

where  $\delta(t)$  denotes Dirac's delta function.

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Tetsuya Takine and Toshiharu Hasegawa:  
Department of Applied Mathematics and  
Physics, Faculty of Engineering, Kyoto  
University, Kyoto 606-01, Japan