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THE OPTIMAL STOPPING PROBLEM IN WHICH THE SUM OF THE ACCEPTED OFFER'S VALUE AND THE REMAINING SEARCH BUDGET IS AN OBJECTIVE FUNCTION

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Abstract The paper deals with an optimal stopping problem with a finite planning horizon where an available search budget, the total amount of money that can be invested in search activities throughout the planning horizon, is limited and where both the probability of an offer being obtained at each point in time and the probability distribution function of an obtained offer's value may depend on the search cost invested at that time. The objective is to maximize the expected present discounted value of the sum of the accepted offer's value and the remaining search budget at that point. The optimal decision strategy for the problem consists of the two decision rules: an *optimal investment rule*, prescribing how much of the search budget to invest in search activities of each point in time and an *optimal stopping rule*, prescribing how to stop the search with accepting an offer. The present paper examines the properties of the optimal decision strategy analytically and numerically. The most interesting and important properties revealed here are the two: (1) In a recall model, the optimal stopping rule has a *reservation value property*, and if the given search budget tends to infinity, the reservation value becomes time-independent, implying that the optimal stopping rule has a *myopic property*, and (2) in both of a recall model and a no recall model, the optimal investment does not always become monotone in the amount of search budget remaining then, possibly increasing or decreasing drastically with its very slight change.

1. Introduction

In almost all the optimal stopping problems that have been presented so far, it has been implicitly assumed that a search cost invested in each point in time is fixed and/or identical throughout a given planning horizon and that a search budget, the total amount of money available for search activities throughout the planning horizon, is either sufficiently large or infinite to allow the infinite number of searches. Not being so unrealistic for such small-scale search problems as a job search [11][12][24], the above assumptions might safely be said to be far removed from reality for such large-scale search problems as are seen in a research and development of new products in manufacturing companies [23]. Furthermore, with the exception of [11], it has not been taken into consideration that a more profitable offer can be obtained with a higher probability if a more search cost is invested.

In the present paper, adopting the three points below, we will set up a model of the optimal stopping problem that well reflects real situations to be applicable enough even to large-scale problems and examine the properties of its optimal decision strategy:

(1) *The amount of search budget available throughout the planning horizon is finite,*
(2) *A search cost invested at each point in time is a decision variable,*
(3) *A more profitable offer can be obtained with a higher probability if a more search cost is invested.*

2. Model

Consider the following discrete-time optimal stopping problem with a finite planning horizon. First, for convenience, let points in time be numbered backward from the final point in time of the horizon as time 0, time 1, and so on, equally spaced; an interval between time t and time $t-1$ is called a period t. Second, assume that the search process starts with a finite search budget i and that if c dollars out of the search budget is invested in search activities of each point in time, an offer can be obtained at the next point in time with a known probability $p(c)$ (offer probability) where $p(0) = 0$ and $p = \sup_{c\geq 0} p(c) \leq 1$. Let sequentially obtained offers $w, w', w'', \cdots$ be independent and identically distributed random variables having a known distribution function $F(w|c)$ (offer distribution), dependent on the search cost invested c, with a finite expectation $\mu(c)$ where $F(w|c) = 0$ on $w \leq 0$ for all $c \geq 0$ and $\mu = \sup_{c\geq 0} \mu(c) < \infty$. Then, assume that at most one of offers sequentially obtained within the planning horizon must be accepted. Finally, let a per-period discount factor be represented by $\beta = 1/(1+r)$ where r is a per-period rate of interest.

Now, in terms of availability in the future of an offer once inspected and passed up (Being available in the future means that the searcher can accept the offer at his own convenience at any time in the future), we consider the two cases [2][21] in the present paper: 1. it becomes instantly and forever unavailable (*no recall model*) and 2. it is forever available (*recall model*).

The objective of the search process is to maximize the *expected present discounted revenue*, the expected present discounted value of the sum of the offer w accepted and the search budget i remaining at that point. The decision strategy attaining the maximum consists of the two rules: an *optimal investment rule*, prescribing how much of the search budget to invest in search activities of each point in time and an *optimal stopping rule*, prescribing how to stop the search with accepting an offer (Refer to [3] for highly generalized discussions of an optimal stopping problem in which, in addition to stopping rule, another decision variable is involved.)

3. Preliminaries

Since no offer being obtained can be regarded as an offer 0 being obtained, the offer probability $p(c)$ and the offer distribution function $F(w|c)$ can be combined into the distribution function $G(w|c)$ whose probability density function is

$$g(w|c) = (1-p(c))I(w=0) + p(c)f(w|c)I(w>0) \tag{3.1}$$

where $I(S) = 1$ if a given statement S is true, or else $I(S) = 0$. Then, in general, for any given function $s(w)$,

$$\int_0^\infty s(w)dG(w|c) = (1-p(c))s(0) + p(c)\int_{0+}^\infty s(w)dF(w|c) \tag{3.2}$$

where the domain of integration in each of $\int_x^\infty$ and $\int_{x+}^\infty$ is, respectively, $x \leq w < \infty$ and $x < w < \infty$. For any real numbers $i \geq 0$ and x define

$$K(i,x) = \max_{0\leq c\leq i}\{\beta\int_0^\infty \max\{w,x\}dG(w|c) - c\} - x, \tag{3.3}$$

which can be rewritten as follows:

$$K(i,x) = \max_{0\leq c\leq i}\{\beta p(c)\int_{0+}^\infty \max\{w-x,0\}dF(w|c) - c\} + \beta x - x, \tag{3.4}$$

hence

$$K(\infty,0)=\max_{0\le c}\{\beta p(c)\mu(c)-c\}. \tag{3.5}$$

Define

$$h(i)=\sup\{x\mid K(i,x)>0\}, \tag{3.6}$$
$$h^*=\sup\{x\mid K(\infty,x)>0\}\ (=h(\infty)). \tag{3.7}$$

Let $\tilde{\beta}$ be the largest value of β for which $K(\infty,0)=0$; i.e.,

$$\tilde{\beta}=\max\{\beta\mid K(\infty,0)=0\} \tag{3.8}$$

where $\tilde{\beta}\ge 0$ because $K(\infty,0)=0$ if $\beta=0$.

Lemma 1.

(a) *$K(i,x)$ is nondecreasing in i and nonincreasing and continuous in x with $K(i,0)\ge 0$ for any i,*

(b) *If $\beta<1$, then $K(i,x)$ is strictly decreasing in x and diverging to $-\infty$ (∞) as $x\to\infty$ ($-\infty$) for all $i\le\infty$; hence, both $h(i)$ and h^* are given by the unique solutions of, respectively, $K(i,x)=0$ and $K(\infty,x)=0$,*

(c) *If $\beta<1$ and $K(i,0)>0$ for a certain i, then $h(j)>0$ for all $j\ge i$,*

(d) *$K(\infty,0)=0$ for $0\le\beta\le\tilde{\beta}$ and $K(\infty,0)>0$ for $\tilde{\beta}<\beta$.*

Proof: (a). The monotonicity in i is obvious from (3.3). The monotonicity in x and $K(i,0)\ge 0$ for any i is clear from (3.4). The continuity in x is evident.

(b). Clear from (3.4).

(c). Evident from (a) and (b).

(d). Immediate from the fact that $K(\infty,0)$ is nondecreasing in β. ∎

Throughout the subsequent sections, without stated otherwise, all the variables and parameters are assumed to be nonnegative.

4. Analysis: No Recall Model

A state of the search process at each point in time can be characterized by the vector (j,i,w) of the three elements:

$$j=\begin{cases}0: & \text{no offer has been accepted yet,}\\ 1: & \text{an offer has already been accepted,}\end{cases}$$
$$i=\text{the remaining search budget,}$$
$$w=\text{the current offer.}$$

Let us denote the set of all the possible states (j,i,w) by $\mathcal{I}$, called a state space.

A decision rule when in state (j,i,w) is given by the vector (a,c) consisting of the subsequent two actions, a and c:

$$a=\begin{cases}\begin{cases}0: & \text{don't accept the current offer,}\\ 1: & \text{accept the current offer,}\end{cases} & \text{if } j=0,\\ \quad 0: \text{ don't accept the current offer,} & \text{if } j=1,\end{cases}$$
$$c=\text{the search cost invested; } 0\le c\le i \text{ if } j=0, \text{ or else } 0$$

Let us denote the set of all the possible actions (a, c) when in state (j, i, w) by $\mathcal{A}(j, i, w)$.

Now states j and i change with time in the following fashion. First, clearly $j = 1$ is an absorbing state. If $j = 0$ at a certain point in time, then it changes at the next point in time into $j = 1$ if action $a = 1$ is taken, or else (i.e., if action $a = 0$ is taken) it remains unchanged. Next, the remaining search budget $i - c$ after having invested the search cost c will swell to $(i-c)(1+r) = (i-c)/\beta$ by the next point in time thanks to the rate of interest r; that is, the state i changes into $(i-c)/\beta$ if c ($\leq i$) is invested.

Let $g(j, i, w, a)$ represent an immediate reward gained when in state (j, i, w), provided that the action a is taken. Then $g(0, i, w, 0) = 0$, $g(0, i, w, 1) = i + w$, and $g(1, i, w, 0) = 0$. It should be noted here that the immediate reward does not depend directly on the search cost c invested; it is incorporated into the transition law of state i.

Now let $\boldsymbol{d}_t(j, i, w)$ $(= (a_t, c_t) \in \mathcal{A}(j, i, w))$ denote an action that is taken at time t when in state (j, i, w), and define $\boldsymbol{d}_t = \{\boldsymbol{d}_t(j, i, w) \mid (j, i, w) \in \mathcal{I}\}$, prescribing a decision rule of time t. If it has come to time 0 without accepting any offer, then from the definition of the model an offer w obtained at that point must be accepted, so that $\boldsymbol{d}_0 = \{(1, 0), (0, 0)\}$ where (1,0) is for $j = 0$ and (0,0) is for $j = 1$. Let the time sequence of decision rules denoted by $\boldsymbol{f} = \{\boldsymbol{d}_0, \boldsymbol{d}_1, \cdots\}$, called a decision strategy, and let the set of all the possible decision strategies be denoted by $\mathcal{D}$.

Now let $u_t(j, i, w)$ denote the maximum expected present discounted *revenue* starting from time t when in state (j, i, w). Then we have

$$u_0(0, i, w) = i + w \ (= g(0, i, w, 1)), \tag{4.1}$$

$$u_t(1, i, w) = 0, \quad t \geq 0, \tag{4.2}$$

$$u_t(0, i, w) = \max_{\boldsymbol{f} \in \mathcal{D}} \boldsymbol{E}\Big[\sum_{\tau=0}^{t} \beta^{t-\tau} g(j_\tau, i_\tau, w_\tau, a_\tau)\Big|\boldsymbol{f}\Big], \quad t \geq 1, \ j_t = 0, \ i_t = i, \ w_t = w \tag{4.3}$$

where $\boldsymbol{E}[-]$ means taking an expectation with respect to all the related random variables and

$$\begin{aligned} j_{\tau-1} &= \begin{cases} 0 & \text{if } j_\tau = 0 \text{ and } a_\tau = 0, \\ 1 & \text{if } j_\tau = 0 \text{ and } a_\tau = 1 \text{ or if } j_\tau = 1, \end{cases} \\ i_{\tau-1} &= (i_\tau - c_\tau)/\beta, \\ w_{\tau-1} &= \text{a random sample from the distribution } F(w|c_\tau). \end{aligned}$$

Note here that the remaining search budget i_τ of time τ increases to $(i_\tau - c_\tau)/\beta$ at the next point in time if c_τ less than $100(1-\beta)$ % of the i_τ is invested, or else decreases to $(i_\tau - c_\tau)/\beta$.

Now let $\boldsymbol{f}$ attaining the maximum of the right hand side of (4.3) for all t, i, and w be denoted by $\boldsymbol{f}^*$, called the optimal decision strategy. Then, for any $\boldsymbol{f} \in \mathcal{D}$ always

$$\boldsymbol{E}\Big[\sum_{\tau=0}^{t-1} \beta^{t-\tau-1} g(j_\tau, i_\tau, w_\tau, a_\tau)\Big|\boldsymbol{f}^*\Big] \geq \boldsymbol{E}\Big[\sum_{\tau=0}^{t-1} \beta^{t-\tau-1} g(j_\tau, i_\tau, w_\tau, a_\tau)\Big|\boldsymbol{f}\Big]. \tag{4.4}$$

Theorem 1. *$u_t(0, i, w)$ satisfies the recurrence relation:*

$$u_t(0, i, w) = \max\{i + w, \ U_t(i)\}, \quad t \geq 1, \tag{4.5}$$

where

$$U_t(i) = \beta \max_{0\le c\le i} \int_0^\infty u_{t-1}(0,(i-c)/\beta,\xi)dG(\xi|c), \quad t\ge 1, \tag{4.6}$$

$$U_1(i) = K(i,0)+i. \tag{4.7}$$

Proof: First clearly we have

$$\begin{aligned} u_1(0,i,w) &= \max_{\boldsymbol{f}\in\mathcal{D}}\{g(0,i,w,a_1)+\beta \boldsymbol{E}[g(j_0,i_0,w_0,a_0)|\boldsymbol{f}]\} \\ &= \max\{g(0,i,w,1),\ \beta\max_{0\le c_1\le i}\boldsymbol{E}[g(0,(i-c_1)/\beta,w_0,1)]\} \\ &= \max\{i+w,\ \beta\max_{0\le c\le i}\int_0^\infty u_0(0,(i-c)/\beta,\xi)dG(\xi|c)\} \\ &= \max\{i+w, U_1(i)\} \end{aligned}$$

where

$$\begin{aligned} U_1(i) &= \beta\max_{0\le c\le i}\int_0^\infty u_0(0,(i-c)/\beta,\xi)dG(\xi|c) \\ &= \max_{0\le c\le i}\int_0^\infty (i-c+\beta\xi)dG(\xi|c) \\ &= K(i,0)+i. \end{aligned}$$

In general,

$$\begin{aligned} u_t(0,i,w) &= \max_{\boldsymbol{f}\in\mathcal{D}}\{g(0,i,w,a_t)+\beta\boldsymbol{E}\Big[\sum_{\tau=0}^{t-1}\beta^{t-\tau-1}g(j_\tau,i_\tau,w_\tau,a_\tau)\Big|\boldsymbol{f}\Big]\} \\ &= \max\{g(0,i,w,1),\beta\max_{0\le c_t\le i}\ \max_{\boldsymbol{f}\in\mathcal{D}}\int_0^\infty \boldsymbol{E}\Big[\sum_{\tau=0}^{t-1}\beta^{t-\tau-1}g(j_\tau,i_\tau,w_\tau,a_\tau)\Big|\boldsymbol{f}\Big]dG(w_{t-1}|c_t)\} \end{aligned}$$

where $j_{t-1}=0$ and $i_{t-1}=(i-c_t)/\beta$. Then, noting (4.4), we can interchange "$\max_{\boldsymbol{f}\in\mathcal{D}}$" and "$\int_0^\infty$" in the above expression,[†]so that

$$u_t(0,i,w) = \max\{g(0,i,w,1),\beta\max_{0\le c_t\le i}\int_0^\infty\max_{\boldsymbol{f}\in\mathcal{D}}\boldsymbol{E}\Big[\sum_{\tau=0}^{t-1}\beta^{t-\tau-1}g(j_\tau,i_\tau,w_\tau,a_\tau)\Big|\boldsymbol{f}\Big]dG(w_{t-1}|c_t)\},$$

which can be immediately expressed as (4.5). ∎

Now, for convenience of later discussions, we shall define

$$v_t(i,w) = u_t(0,i,w)-i, \tag{4.8}$$

$$V_t(i) = U_t(i)-i. \tag{4.9}$$

[†]In general, for a function $z(\xi,s)$ of ξ, $a\le\xi\le b$, with a parameter $s\in\mathcal{S}$, if $z(\xi,s^*)\ge z(\xi,s)$ for all ξ and s for a certain $s^*\in\mathcal{S}$, then

$$\max_{s\in\mathcal{S}}\int_a^b z(\xi,s)d\xi = \int_a^b\max_{s\in\mathcal{S}}z(\xi,s)d\xi.$$

The $v_t(i,w)$ implies the maximum expected present discounted *net profit* starting from time t in state $j=0$ with i and w. Then (4.5), (4.6), and (4.7) can be rewritten as follows:

$$v_t(i,w) = \max\{w, V_t(i)\}, \quad t \geq 1, \tag{4.10}$$

$$V_t(i) = \max_{0\leq c\leq i}\{\beta \int_0^\infty v_{t-1}((i-c)/\beta, \xi))dG(\xi|c) - c\}, \tag{4.11}$$

$$V_1(i) = K(i,0) \geq 0. \tag{4.12}$$

Note here that (4.11) can be expressed as follows.

$$V_t(i) = \max_{0\leq c\leq i}\{\beta \int_0^\infty \max\{\xi, V_{t-1}((i-c)/\beta)\}dG(\xi|c) - c\}, \quad t \geq 1. \tag{4.13}$$

Now (4.10) tells us that the *optimal stopping rule* of time t is prescribed as follows: if $w > V_t(i)$, stop with accepting the current offer w, or else continue; that is, $V_t(i)$ provides the reservation value of the model [5][6][11][12][23][24]. The *optimal investment* $c_t(i)$ of time t when the remaining search budget is i is given by $c = c^*$ attaining the maximum in the right hand side of (4.13); if there exist more than one c^*, then for convenience' sake let $c_t(i)$ be defined by the smallest of them.

Let $i_n(t,i)$, $V_n(t,i)$, and $c_n(t,i)$ represent the sequences of, respectively, states, reservation values, and optimal investments of times $n = 1, 2, \cdots, t$, starting from time t with a search budget i. Then by definition

$$i_t(t,i) = i, \quad V_t(t,i) = V_t(i), \quad c_t(t,i) = c_t(i), \tag{4.14}$$

and for $n = 1, 2, \cdots, t-1$

$$i_n(t,i) = i_{n+1}(t,i) - c_{n+1}(t,i), \tag{4.15}$$

$$V_n(t,i) = V_n(i_n(t,i)), \tag{4.16}$$

$$c_n(t,i) = c_n(i_n(t,i)). \tag{4.17}$$

Theorem 2.

(a) *If* $K(\infty,0) = 0$, *then* $V_t(i) = 0$ *for all* t *and* i,

(b) $V_t(i)$ *is nondecreasing in* t *and* i *with* $V_1(i) \geq 0$ *for all* i *and upper-bounded in* i *for all* t,

(c) $V_n(t,i)$ *is nondecreasing in* n *for all* i *and* t,

(d) $c_1(i)$ *is nondecreasing in* i.

Proof: (a). Since $K(i,0)$ is nondecreasing in i with $K(0,0) = 0$, it follows from the assumption that $K(i,0) = 0$ for all i. Thus $V_1(i) = 0$ for all i. Suppose $V_{t-1}(i) = 0$ for all i. Then, from (4.13) we have $V_t(i) = K(i,0) = 0$ for all i.

(b). Since $\max\{\xi, V_1((i-c)/\beta)\} \geq \xi$, from (4.13) we have $V_2(i) \geq K(i,0) = V_1(i) \geq 0$ for all i. If $V_t(i) \geq V_{t-1}(i)$ for all i, then from (4.13) we have $V_{t+1}(i) \geq V_t(i)$ for all i. The monotonicity in i for all t can be also easily proved by induction starting with the fact that $V_1(i)$ is nondecreasing in i. The boundedness in i can be proved as follows. First, clearly $V_1(i) \leq K(\infty,0) < \infty$ for all i. Suppose $V_{t-1}(i) \leq (t-1)K(\infty,0) < \infty$. Then from (4.13) we have

$$V_t(i) \leq K(i,(t-1)K(\infty,0)) + (t-1)K(\infty,0)$$

$$\leq K(\infty,0)+(t-1)K(\infty,0)$$
$$= K(\infty,0) < \infty.$$

(c). Since $i_n(t,i) \leq i_{n+1}(t,i)$ from (4.15), we have $V_n(t,i) = V_n(i_n(t,i)) \leq V_n(i_{n+1}(t,i)) \leq V_{n+1}(i_{n+1}(t,i)) = V_{n+1}(t,i)$.

(d). Obvious from (4.12). ■

$V_t(i)$ converges as $i \rightarrow \infty$ for all t from Theorem 2(b). Let its limit be denoted by V_t. Now, since $V_{t-1}(i)$ is upper-bounded in i, the constraint " $0 \leq c \leq i$ " in the right hand side of (4.13) can be replaced by $0 \leq c$ for any sufficiently large i. Accordingly, as $i \rightarrow \infty$, (4.13) becomes,

$$V_t = \max_{0\leq c}\{\beta \int_0^\infty \max\{\xi, V_{t-1}\}dG(\xi|c) - c\}, \quad t \geq 1, \tag{4.18}$$

$$= K(\infty, V_{t-1}) + V_{t-1}, \tag{4.19}$$

where

$$V_1 = K(\infty,0) = \max_{0\leq c}\{\beta p(c)\mu(c) - c\} \geq 0. \tag{4.20}$$

Let us define the smallest c attaining the maximum of the right hand sides of (4.18) and (4.20) as a limiting optimal investment as $i \rightarrow \infty$, denoted by c_t.

Theorem 3. *Suppose $\beta < 1$. Then*
(a) *V_t is nondecreasing in β for all t and nondecreasing in t,*
(b) *If $\beta \leq \tilde{\beta}$, then $c_t = 0$ and $V_t = 0$ for all t, or else $c_t > 0$ and $V_t > 0$ for all t,*
(c) *If $\beta < 1$, then V_t converges to $h^* < \infty$ as $t \rightarrow \infty$,*
(d) *If $\beta < 1$, then $V_t(i)$ is upper-bounded in t and i.*

Proof: (a). V_1 is clearly nondecreasing in β from (4.20). If V_{t-1} is nondecreasing in β, so also is V_t from (4.18). The monotonicity in t is evident from Theorem 2(b).

(b). First, note that if $\beta \leq \tilde{\beta}$, then $V_t = 0$ for all t from Theorem 2(a) and Lemma 1(d). Next, define

$$\lambda_\beta(c,x) = \beta \int_0^\infty \max\{\xi, x\}dG(\xi|c) - c \geq \beta x - c$$

where $\lambda_\beta(0,x) = \beta x$. Then, from (4.20) we have

$$V_1 = K(\infty,0) = \max_{0\leq c} \lambda_\beta(c,0).$$

Hence, from the definition of $\tilde{\beta}$, if $\beta \leq \tilde{\beta}$, then since $K(\infty,0) = 0$, it must be that $\lambda_\beta(c,0) \leq 0$ for all $c \geq 0$, hence we have $c_1 = 0$, noting $\lambda_\beta(0,0) = 0$, or if $\tilde{\beta} < \beta$, then since $K(\infty,0) > 0$, there must exist at least one $c > 0$ for which $\lambda_\beta(c,0) > 0$, so that $c_1 > 0$ and $V_1 > 0$. Thus, the assertion in the theorem holds true for $t = 1$. Suppose it holds for $t-1$. Then, if $\beta \leq \tilde{\beta}$, we have

$$V_t = K(\infty,0) = \max_{0\leq c} \lambda_\beta(c,0) = 0$$

from the induction hypothesis and (4.18), hence $c_t = 0$, or if $\tilde{\beta} < \beta$, then from (4.18) and the induction hypothesis we have

$$V_t = \max_{0 \le c} \lambda_\beta(c, V_{t-1}) \ge \max_{0 \le c}\{\beta V_{t-1} - c\} = \beta V_{t-1} > 0,$$

thus it follows that $c_t > 0$.

(c). Suppose V_t diverges as $t \to \infty$. Then, for a t for which $V_t > h^*$ we have the contradiction of $V_{t+1} - V_t = K(\infty, V_t) < 0$. Thus V_t must converge, so its limit V satisfies $K(\infty, V) = 0$ from (4.19). Hence, from Lemma 1(b) we have

$$V = h^* < \infty. \tag{4.21}$$

(d). Clear from $V_t(i) \le V_t \le V = h^* < \infty$ for all t. ■

From Theorem 3(d) and Theorem 2(b) it follows that if $\beta < 1$, then $V_t(i)$ converges as $t \to \infty$. Let its limit be denoted by $V(i)$. Then from (4.13) we have

$$V(i) = \max_{0 \le c \le i}\{\beta \int_0^\infty \max\{\xi, V((i-c)/\beta)\} dG(\xi|c) - c\}. \tag{4.22}$$

Theorem 4. *When $\beta < 1$, the equation (4.22) has the unique solution that is upper-bounded.*

Proof: Suppose the equation (4.22) has another upper-bounded solution $U(i)$, i.e.,

$$U(i) = \max_{0 \le c \le i}\{\beta \int_0^\infty \max\{\xi, U((i-c)/\beta)\} dG(\xi|c) - c\}. \tag{4.23}$$

Let $\Delta = \sup_{0 \le i} |V(i) - U(i)|$ where $0 < \Delta < \infty$. Then by taking the difference of (4.22) and (4.23), we have immediately $|V(i) - U(i)| \le \beta\Delta$, hence $\Delta \le \beta\Delta$, yielding the contradiction of $1 \le \beta$. Therefore, the solution must be unique. ■

5. Analysis: Recall Model

A state of the search process at each point in time can be characterized by the three elements j, i, and y where the definitions of j and i are the same as in the no recall model and y is the best offer obtained so far. Two actions, a and c, a decision rule, a decision strategy, and the transition law of states j and i also can be defined in quite the same way as in the previous section.

Now let $u_t(j, i, y)$ denote the maximum expected present discounted *revenue* starting from time t when in state (j, i, y), and let $v_t(i, y) = u_t(0, i, y) - i$, the maximum expected present discounted *net profit* starting from time t in state $j = 0$ with i and y. Then, in almost the same fashion as in the no recall case, the $v_t(i, y)$ can be expressed as

$$v_t(i, y) = \max\{y, V_t(i, y)\}, \quad t \ge 1, \tag{5.1}$$

$$V_t(i, y) = \max_{0 \le c \le i}\{\beta \int_0^\infty v_{t-1}((i-c)/\beta, \max\{y, \xi\}) dG(\xi|c) - c\}, \tag{5.2}$$

$$V_1(i, y) = K(i, y) + y. \tag{5.3}$$

Now (5.1) tells us that the *optimal stopping rule* is prescribed as follows: if the best offer so far $y > V_t(i, y)$, stop with accepting it, or else continue. The *optimal investment* is given by the smallest $c = c^*$ attaining the maximum of the right hand side of (5.2); let it be denoted by $c_t(i, y)$.

Let $i_n(t,i,\boldsymbol{y})$ and $c_n(t,i,\boldsymbol{y})$, $n=1,2,\cdots,t$, denote, respectively, a state and an optimal investment of time n, starting from time t with a search budget i, provided that the vector of the best offers y_n of times $n=1,2,\cdots,t$ is $\boldsymbol{y}=(y_1, y_2,\cdots, y_t)$, $y_t \le y_{t-1} \le \cdots \le y_1$. By definition, clearly

$$i_t(t,i,\boldsymbol{y}) = i, \quad c_t(t,i,\boldsymbol{y}) = c_t(i,y_t), \tag{5.4}$$

and for $n=1,2,\cdots,t-1$,

$$i_n(t,i,\boldsymbol{y}) = i_{n+1}(t,i,\boldsymbol{y}) - c_{n+1}(t,i,\boldsymbol{y}), \tag{5.5}$$

$$c_n(t,i,\boldsymbol{y}) = c_n(i_n(t,i,\boldsymbol{y}), y_n). \tag{5.6}$$

Theorem 5.
(a) *$V_t(i,y)$ is nondecreasing in t, i, and y,*
(b) *If $K(\infty,0)=0$, then $V_t(i,y) \le y$ for all t, i, and y,*
(c) *If $K(\infty,0)>0$, then*
1. *if $\beta=1$, $V_t(i,y) \ge y$ for all t, i, and y,*
2. *if $\beta<1$, $V_t(i,y)$ is upper-bounded in t and i for all y,*

(d) *$c_1(i,y)$ is nondecreasing in i for all y.*

Proof: (a). Arranging (5.2) by substituting the inequality $v_{t-1}((i-c)/\beta, \max\{y,\xi\}) \ge \max\{y,\xi\}$ yields

$$V_t(i,y) \ge K(i,y) + y = V_1(i,y). \tag{5.7}$$

The monotonicity in t can be easily proved by induction starting with the above inequality. The monotonicity in i and y can be also easily verified by induction starting with the fact that $V_1(i,y)$ is nondecreasing in i and y.

(b). $V_1(i,y) = K(i,y) + y \le K(\infty,0) + y = y$ for all i and y, so the assertion is true for $t=1$. Suppose it is true for $t-1$. Then, since $v_{t-1}(i,y) = y$ for all i and y, we have $V_t(i,y) = K(i,y) + y = V_1(i,y) \le y$ for all i and y.

(c1). Immediate from (5.7) because if $\beta=1$, then $K(i,y) \ge 0$ for all i and y from (3.4) .

(c2). Let $s_t = (1+\beta+\beta^2+\cdots+\beta^{t-1})K(\infty,0)$. First, clearly $V_1(i,y) \le K(\infty,0)+y = s_1+y$ for all i and y. Next, suppose $V_{t-1}(i,y) \le s_{t-1}+y$ for all i and y. Then, since

$$\begin{aligned} v_{t-1}((i-c)/\beta, \max\{\xi,y\}) &\le \max\{\max\{\xi,y\}, s_{t-1}+\max\{\xi,y\}\} \\ &= s_{t-1} + \max\{\xi,y\}, \end{aligned}$$

arranging (5.2) by substituting the inequality yields

$$\begin{aligned} V_t(i,y) &\le \beta s_{t-1} + K(i,y) + y \\ &\le \beta s_{t-1} + K(\infty,0) + y \\ &= s_t + y. \end{aligned}$$

Accordingly, it follows by induction that $V_t(i,y) \le s_t + y \le K(\infty,0)/(1-\beta) + y$ for all t and i, i.e., upper-bounded in t and i.

(d). Immediate from (5.3). ■

Theorem 6. *If $\beta<1$, then $V_t(i,y)-y$ is strictly decreasing in y for all t and i and diverges to $-\infty$ $(+\infty)$ as $y \to +\infty$ $(-\infty)$.*

Proof: Let $Z_t(i,y) = V_t(i,y) - y$ and $z_t(i,y) = v_t(i,y) - y$ (= $\max\{0, Z_t(i,y)\}$). Since $Z_1(i,y) = K(i,y)$, the assertion obviously holds for $t = 1$ from Lemma 1(b). Suppose the assertion is true for $t-1$, so $z_{t-1}(i,y)$ is nonincreasing in y for all i. Now, from (5.2) we obtain

$$Z_t(i,y) = \max_{0\le c\le i}\{\beta \int_0^\infty z_{t-1}((i-c)/\beta, \max\{y,\xi\})dG(\xi|c) + \beta \int_0^\infty \max\{\xi - y, 0\}dG(\xi|c) - c\} - (1-\beta)y, \tag{5.8}$$

in which the inside of the braces is nonincreasing in y and the last term $-(1-\beta)y$ is strictly decreasing in y and diverges to $-\infty$ $(+\infty)$ as $y \to +\infty$ $(-\infty)$. Therefore, the assertion becomes true for t. ∎

Define

$$h_t(i) = \sup_y\{y \mid V_t(i,y) - y > 0\}, \tag{5.9}$$

so $h_1(i) = h(i)$ from (3.6).

Theorem 7. *If $\beta < 1$, then $h_t(i)$ is a unique, nonnegative solution to $V_t(i,y) - y = 0$ for all t and i, nondecreasing and upper-bounded in t and i.*

Proof: The existence of the unique solution is evident from Theorem 6. Being nonnegative follows from $Z_t(i,0) = V_t(i,0) \ge V_1(i,0) = K(i,0) \ge 0$ for all t and i. The monotonicity and the upper-boundedness in t and i is from the fact that $Z_t(i,y)$ is nondecreasing and upper-bounded in t and i from (5.8) and Theorem 5(a,c2). ∎

The above theorem tells us that the *optimal stopping rule* that was prescribed previously can be restated as follows: if the best offer so far $y > h_t(i)$, stop with accepting it, or else continue. That is, $h_t(i)$ provides a reservation value of the model.

Let $h_n(t,i,\boldsymbol{y})$ denote reservation values of times $n = 1, 2, \cdots, t$ starting from time t with i and $\boldsymbol{y} = (y_1, y_2, \cdots, y_t)$. By definition, $h_t(t,i,\boldsymbol{y}) = h_t(i)$, and for $n = 1, 2, \cdots, t-1$,

$$h_n(t,i,\boldsymbol{y}) = h_n(i_n(t,i,\boldsymbol{y})), \tag{5.10}$$

Now, Theorem 5(a,c2) guarantees that, if $K(\infty,0) > 0$ and $\beta < 1$, then $V_t(i,y)$ converges as $i \to \infty$ and as $i \to \infty$ and then $t \to \infty$. Let the limits be denoted by, respectively, $V_t(y)$ and $V(y)$, and furthermore, let $v_t(y) = \max\{y, V_t(y)\}$ and $v(y) = \max\{y, V(y)\}$. Then, from (5.2) we get

$$V_t(y) = \max_{0\le c}\{\beta \int_0^\infty v_{t-1}(\max\{\xi, y\})dG(\xi|c) - c\}, \quad t \ge 1, \tag{5.11}$$

$$V(y) = \max_{0\le c}\{\beta \int_0^\infty v(\max\{\xi, y\})dG(\xi|c) - c\} \ge 0 \tag{5.12}$$

where

$$V_1(y) = K(\infty, y) + y. \tag{5.13}$$

The limiting optimal investments, $c_t(y)$ and $c(y)$, are provided by the smallest c attaining the maximum of the right hand sides of (5.11) and (5.12).

Theorem 8. *Suppose $K(\infty,0)>0$ and $\beta<1$, so $h^*<\infty$. Then*
(a) *If $h^*\le y$, then $V_t(y)\le y$, hence $v_t(y)=y$,*
(b) *If $y\le h^*$, then $V_t(y)\ge y$, hence $v_t(y)=V_t(y)$,*
(c) *The above two assertions are also true for $V(y)$ and $v(y)$.*

Proof: (a,b). Clear for $t=1$ because $V_1(y)=K(\infty,y)+y$. Assume that the assertion is true for $t-1$. Suppose $h^*\le y$. Then, since $\max\{\xi,y\}\ge y\ge h^*$ for all ξ, we have $V_{t-1}(\max\{\xi,y\})\le\max\{\xi,y\}$ for all ξ from the induction hypothesis, yielding $V_t(y)\le K(\infty,y)+y\le y$. Thus (a) holds for t. Now since $V_t(y)\ge K(\infty,y)+y$ for all t and y from (5.7), we have $V_t(y)\ge y$ for $y\le h^*$; hence, (b) is true for t. Thus the induction is complete.
(c) Evident. ■

Theorem 9. *Suppose $K(\infty,0)>0$ and $\beta<1$. Then $V(y)=h^*<\infty$ for $y\le h^*$.*

Proof: It would suffice to prove the following two: (1) If the right hand side of (5.12) with $y\le h^*$ is arranged by substituting $V(y)=h^*$, $y\le h^*$, the resultant expression becomes equal to h^* and (2) the equation (5.12) has not another solution $U(y)$, $y\le h^*$. First, let us prove (1). Suppose $y\le h^*$. Then

$$\begin{aligned}\text{r.h.s. of (5.12)} &= \max_{0\le c}\{\beta\int_0^\infty (v(\max\{\xi,y\})I(\xi\le h^*)+v(\max\{\xi,y\})I(h^*<\xi))dG(\xi|c)-c\}\\ &= \max_{0\le c}\{\beta\int_0^\infty (h^*I(\xi\le h^*)+\xi I(h^*<\xi))dG(\xi|c)-c\}\\ &= \max_{0\le c}\{\beta\int_0^\infty \max\{\xi,h^*\}dG(\xi|c)-c\}\\ &= K(\infty,h^*)+h^*=h^*.\end{aligned}$$

Next, let us prove (2). Suppose the equation (5.12) has another solution $U(y)$, $y\le h^*$, that is monotone and as the same properties as $V(y)$. Then

$$U(y)=\max_{0\le c}\{\beta\int_0^\infty u(\max\{\xi,y\})dG(\xi|c)-c\}, \tag{5.14}$$

and let $\Delta=\sup_{0\le y\le h^*}|U(y)-V(y)|$ where $0<\Delta<\infty$. Then from (5.12) and (5.14), we have

$$|V(y)-U(y)|\le\beta\max_{0\le c}\int_0^\infty |v(\max\{\xi,y\})-u(\max\{\xi,y\})|dG(\xi|c)$$

Now, if $h^*\le\xi$, then since $\max\{\xi,y\}\ge h^*$, we have $v(\max\{\xi,y\})=u(\max\{\xi,y\})=\max\{\xi,y\}$ from Theorem 8. Accordingly, the above expression becomes

$$|V(y)-U(y)|\le\beta\max_{0\le c}\int_0^{h^*} |V(\max\{\xi,y\})-U(\max\{\xi,y\})|dG(\xi|c)\le\beta\Delta,$$

from which we have $\Delta\le\beta\Delta$, yielding the contradiction of $1\le\beta$. Thus the solution must be unique. ■

6. Numerical Analyses[††]

The right side of the expressions (4.13) and (5.2) are very intractable mathematically, so that it is quite difficult to analytically examine the relationship of $c_t(i)$, $V_t(i)$, $c_t(i,y)$, and $V_t(i,y)$ to t, i, and y. For the reason, in the section we will investigate it by numerical analyses. Although the conclusions obtained here are from only the limited number of numerical examples, some of them are quite counter-intuitive and beyond our comprehension.

6.1 Preliminaries

In the numerical analysis we consider only the following cases of $F(w|c)$ and $p(c)$:

$$F(w|c) = \begin{cases} \text{A continuous uniform distribution function on } [a,b],\ 0<a<b<\infty, \\ \text{for a } \textit{no recall model}, \\ \text{A discrete uniform distribution function with } M+1 \text{ mass points}, \\ \text{equally spaced, on } [a,b],\ 0<a<b<\infty, \text{ for a } \textit{recall model}. \end{cases}$$

$$p(c) = \begin{cases} \text{Case 1: } p(1-e^{-\lambda c}), & 1 \geq p > 0,\ \lambda > 0 \quad (6.1) \\ \text{Case 2: } g(c|p_1,\lambda_1,\rho_1), & 1 \geq p_1 \geq 0 \quad (6.2) \\ \text{Case 3: } g(c|p_1,\lambda_1,\rho_1) + I(s_2 < c)g(c-s_2|p_2,\lambda_2,\rho_2), & \\ & 1 \geq p_1+p_2 \geq 0,\ s_2 > 0 \quad (6.3) \end{cases}$$

where

$$g(c|p,\lambda,\rho) = p(\rho^{\lambda^c} - \rho)\ /\ (1-\rho), \quad c \geq 0, \quad p > 0, \quad 1 > \lambda > 0, \quad 1 > \rho > 0. \tag{6.4}$$

Note here that Case 1 is concave and that Case 2 and Case 3 have, respectively, one and two inflection points (*Figure 1*).

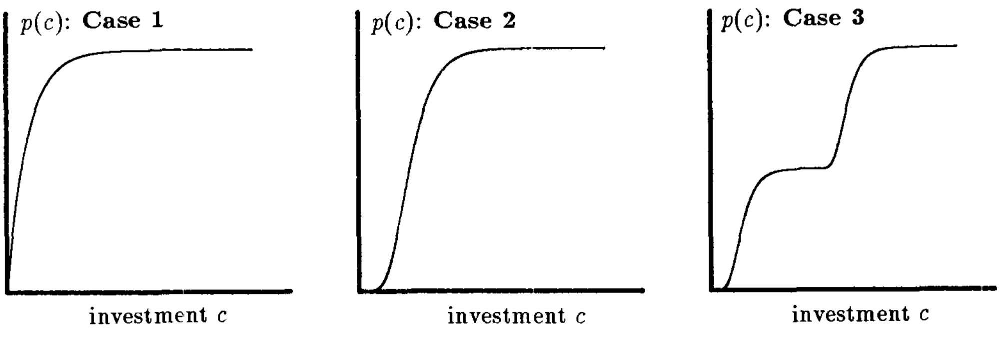

Figure 1

Offer probabilities $p(c)$ used in the numerical analyses: Case 1: $p = 1.0$, $\lambda = 0.5$ ($\tilde{c} = 24$), Case 2: $p_1 = 1.0$, $\lambda_1 = 0.5$, $\rho_1 = 10^{-5}$ ($\tilde{c} = 21$), Case 3: $p_1 = 0.5$, $\lambda_1 = 0.5$, $\rho_1 = 10^{-5}$, $s_2 = 15$, $p_2 = 0.5$, $\lambda_2 = 0.5$, $\rho_2 = 10^{-5}$ ($\tilde{c} = 35$)

[††]The computer used is a min-computer (Data General, MV10000), and the language and compiler used are, respectively, Fortran 77 and F77L (Lahey Computer Systems). In order to make computational error as small as possible, all the variables were defined on double precision. All computation results, enormous if sent to print, were stored into direct files in an external storage, and all the graphs were directly drawn using a 3-dimensional graph drawing software, CORE-PC (Mitubishi Research Institute), and X-Y plotter (GRAPHTEC, MP3200).

The existence of inflection points in the offer probability $p(c)$ reflects a real situation that, in order to make a critical breakthrough in technology development of an idea (for instance, as in the R&D of the Josephson device as a logic IC for computers), an investment at least up to a certain level, usually enormous, must be made. Now, for convenience of numerical computations, let us transform $p(c)$ as follows:

$$p(c) = \begin{cases} p(c) & \text{on } c \leq \tilde{c}, \\ p(\tilde{c}) & \text{on } \tilde{c} \leq c \end{cases} \tag{6.5}$$

where

$$\tilde{c} = \max\{c \mid p(c) \leq 0.99999 \times p(\infty)\}. \tag{6.6}$$

Then we can replace the constraints $0 \leq c \leq i$ in the right hand sides of (4.13) and (5.2) with $0 \leq c \leq \min\{i, \tilde{c}\}$. The reason is as follows. First, if $i < \tilde{c}$, it is obvious. Next suppose $\tilde{c} \leq i$. Let us now express (4.13) and (5.2) as, respectively, $V_t(i) = \max_{0\leq c\leq i} D_t(i,c)$ and $V_t(i,y) = \max_{0\leq c\leq i} D_t(i,y,c)$ where

$$D_t(i,c) = \beta\int_0^\infty \max\{\xi, V_{t-1}((i-c)/\beta)\}dG(\xi|c) - c, \qquad t \geq 1, \tag{6.7}$$

$$D_t(i,y,c) = \beta\int_0^\infty v_{t-1}((i-c)/\beta, \max\{\xi, y\})dG(\xi|c) - c, \quad t \geq 1. \tag{6.8}$$

Since $F(w|c)$ has been assumed to be independent of c and $p(c)$ has been transformed to be independent of $c \geq \tilde{c}$, $G(\xi|c)$ becomes also independent of $c \geq \tilde{c}$. Therefore both $D_t(i,c)$ and $D_t(i,y,c)$ become also nonincreasing in $c \geq \tilde{c}$ from Theorem 2(b) and Theorem 5(a). Thus it follows that

$$V_t(i) = \max\{\max_{0\leq c\leq\tilde{c}} D_t(i,c),\ \max_{\tilde{c}<c} D_t(i,c)\} = \max_{0\leq c\leq\tilde{c}} D_t(i,c)$$

and similarly $V_t(i,y) = \max_{0\leq c\leq\tilde{c}} D_t(i,y,c)$.

In order to calculate the reservation values $V_t(i)$ on $0 \leq i \leq I$ for certain given I and t using (4.13), $V_{t-1}(i)$ must be computed on $0 \leq i \leq I/\beta$ in advance of it because $0 \leq (i-c)/\beta \leq i/\beta \leq I/\beta$, and furthermore, in order to do this, similarly $V_{t-2}(i)$ must be obtained on $0 \leq i \leq I/\beta^2$ in advance of it, and in general $V_n(i)$ must be calculated on $0 \leq i \leq I/\beta^{t-n}$, $n = t-1, t-2, \cdots, 1$. The same argument holds for $V_t(i,y)$.

Finally, for convenience of numerical computations, we shall evaluate $V_t(i)$ only for

$$i \in \mathcal{A} = \{j\Delta \mid j = 0, 1, \cdots\}, \quad \Delta = \tilde{c}/N$$

for a given sufficiently large integer N. In the case, the argument of $V_{t-1}(\xi)$, $\xi = (i-c)/\beta$, is not always in $\mathcal{A}$ if $\beta < 1$. If $\xi \notin \mathcal{A}$, we approximate $V_{t-1}(\xi)$ by interpolation; i.e., if $j\Delta < \xi < (j+1)\Delta$, then

$$V_{t-1}(\xi) \approx \frac{(j+1)\Delta - \xi}{\Delta}V_{t-1}(j\Delta) + \frac{\xi - j\Delta}{\Delta}V_{t-1}((j+1)\Delta). \tag{6.9}$$

Similar for $V_{t-1}(\xi, y)$. It is of course that a sufficiently large N must be taken so as to attain a sufficiently reasonable accuracy of the approximation.

6.2 Results

Figure 2 to *Figure 9* below are all for the no recall model and *Figure 10* is for the recall model.

- *Figure 2*: The optimal reservation values $V_t(i)$.
- *Figure 3*: The optimal reservation values $V_n(t, i)$, $t = 10$.
- *Figure 4*: The optimal investments $c_t(i)$.
- *Figure 5*: The optimal investments $c_1(i)$, $c_2(i)$, and $c_3(i)$ in *Figure 4*.
- *Figure 6*: The optimal investments $c_n(t, i)$, $t = 10$.

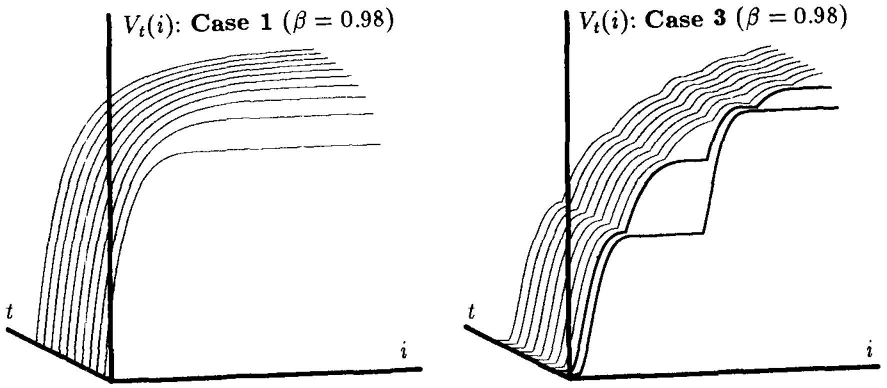

Figure 2
Optimal reservation values $V_t(i)$: $a = 50$, $b = 150$, $N = 2000$

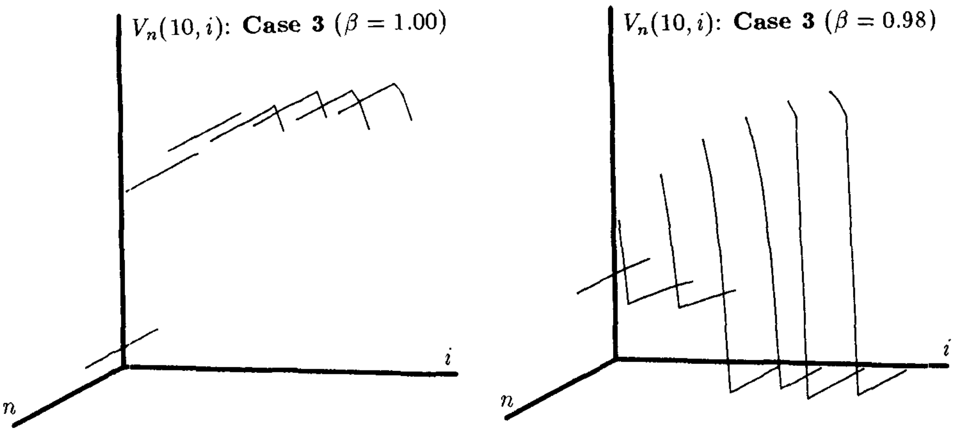

Figure 3
Optimal reservation values $V_n(10, i)$: $a = 50$, $b = 150$, $N = 2000$

• *Figure 7*: The limiting optimal reservation values V_t, $0 \le \beta \le 1$. The bold line curves in the graphs are for the first $t^* = t$ for which

$$\max_{0 \le \beta \le 1} |(V_t - V_{t-1})/V_t| < 0.000001;$$

they can be regarded as the approximations for the limits of V_t as $t \to \infty$.

• *Figure 8*: The limiting optimal investments c_t, $0 \le \beta \le 1$. The bold line curves in the graphs are for t^* in *Figure 7*.

• *Figure 9*: The graphs illustrates that how the optimal investment $c_n(t,i)$ and the optimal reservation values $V_n(t,i)$ are obtained from $c_t(i)$ and $V_t(i)$ (See (4.14) to (4.17)). In the graph, the search process is assumed to start from time $t = 10$ with a remaining search budget $i = |oa|$. First, the optimal investment of the starting point in time, $c_{10}(10,i)$, is given by $|bc|$. Therefore, the remaining search budget of time 9 reduces to $i_9(10,i) = |oa| - |bc|$ $= |oa'|$. Similarly the optimal investment of time 8 becomes $c_8(10,i) = |oa'| - |b'c'| = |oa''|$. On the other hand, the optimal reservation values, $V_{10}(10,i)$, $V_9(10,i)$, $\cdots$ are given by $|bd|$, $|b'd'|$, $|b''d''|$ and so on where the points $a, a', a'', \cdots$ and $b, b', b'', \cdots$ plotted on the (i,t) plane in the graph of $V_n(t,i)$ are the same as ones in the graph of $c_n(t,i)$.

• *Figure 10*: The optimal investments $c_t(i,y)$.

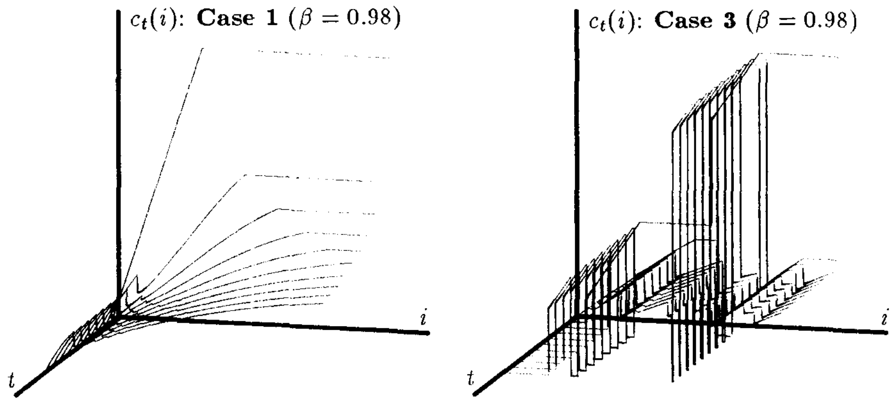

Figure 4

Optimal Investments $c_t(i)$: $a = 50$, $b = 150$, $N = 2000$

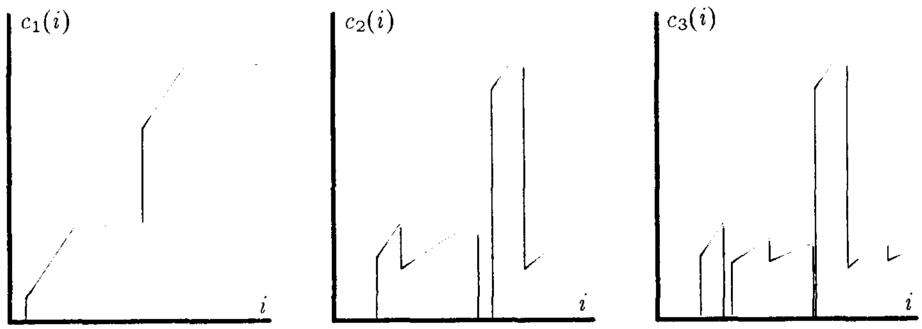

Figure 5

Optimal investments $c_1(i)$, $c_2(i)$, and $c_3(i)$ in Case 3 of Figure 4

7. Conclusions and Discussion

7.1 Case of $K(\infty,0)=0$

1. The equality $K(\infty,0)=0$ is equivalent to $\beta p(c)\mu(c)\leq c$ for all $c\geq 0$, implying that the expected present discounted value of an offer ξ obtained at the next point in time by paying *any* search cost c at a certain point in time, $\beta p(c)\mu(c)$, does not make up for the search cost. This suggests that if the equality holds, entering the search process will not yield any economical value. Theorem 2(a) and Theorem 5(b) claim that the interpretation is also theoretically true. In other words, when it holds, it is optimal to invest the whole available search budget in an investment opportunity with the rate of interest r (*financial investment*) with abandoning from the first an investment in search activities (*search investment*). Now it follows from Lemma 1(d) that with $\tilde{\beta}>0$, the above assertion is true for any $\beta<\tilde{\beta}$, so any $r>\tilde{r}=1/\tilde{\beta}-1$.

The conclusion reflects the economically rational interpretation that when the rate of interest r is sufficiently large, it is better to earn interest by allocating the entire search budget to a financial investment rather than to gain profit from an offer obtained by investing any part of it in search activities.

2. The lower bound $\tilde{r}$ can be obtained as follows. Let $\lambda(c)=\beta\delta(c)-c$ with $\delta(c)=p(c)\mu(c)$, so $K(\infty,0)=\max_{0\leq c}\lambda(c)$. The differential coefficient of $\lambda(c)$ at $c=0$ is $\lambda'(0)=\beta\delta'(0)-1$. Suppose $\lambda(c)$ is concave. In the case, if $\lambda'(0)\leq 0$, so $\beta<1/\delta'(0)$, then $\lambda(c)\leq 0$ for all $c\geq 0$, hence $K(\infty,0)=0$, or else $K(\infty,0)>0$. Thus, it follows that $\tilde{\beta}=1/\delta'(0)$, hence $\tilde{r}=\delta'(0)-1$. In case that $\lambda(c)$ is not concave, it is not so difficult, although rather complicated, to numerically obtain the lower bound $\tilde{r}$.

7.2 Case of $K(\infty,0)>0$

7.2.1 Optimal Stopping Rule

No Recall Model

3. The optimal reservation value $V_t(i)$ is nondecreasing in t and i with $V_1(i)\geq 0$ for all i and upper-bounded in i for all t (Theorem 2(b)). If $\beta<1$, its upper bound is $h^*<\infty$ (Theorem 3(c)). The monotonicity in t and i implies that with a larger planning horizon and/or with the more amount of search budget, it becomes more desirable to search for a higher-valued offer to accept. It should be noted here that the nondecreasing pattern of $V_t(i)$ in t and i is not always concave nor convex; for example, it can have such an uneven shape as *a fall of multiply stored cascades* (*Figure 2* (Case 3)).

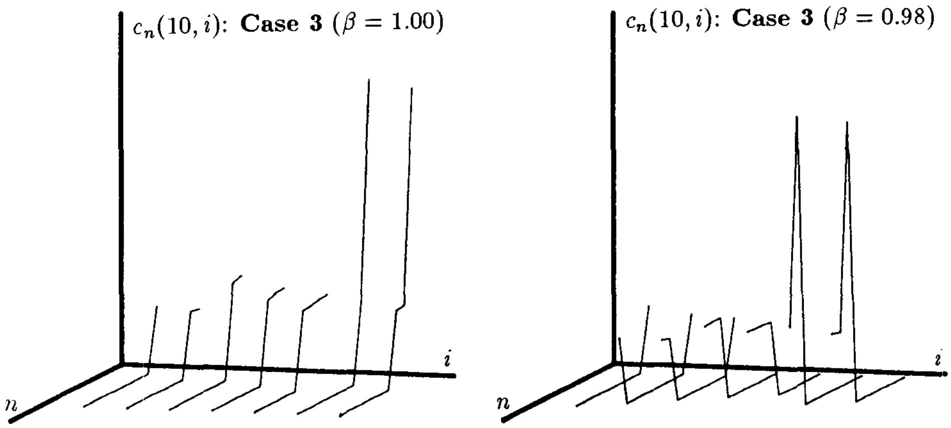

Figure 6

Optimal investments $c_n(10,i)$: $a=50$, $b=150$, $N=2000$

4. The optimal reservation value $V_t(i)$ converges to a limit V_t as $i \to \infty$ for all t (Theorem 2(b)), and when $\beta < 1$, the limit V_t, nondecreasing in t, converges as $t \to \infty$ to $h^* < \infty$ (Theorem 3(c)), the unique solution of $K(\infty, x) = 0$ (Lemma 1(b)). The limit V_t is nondecreasing in β with being equal to 0 on $0 \leq \beta < \tilde{\beta}$ (*Figure* 7, Theorem 3(a,b)). This implies that, if there exists an infinite search budget, the smaller the interest rate may be, we should search for a higher-valued offer to accept; on the contrary, the larger the interest rate may be, we should be contented more and more with accepting a lower-valued offer, and if it is furthermore large for V_t to become 0, we should accept a first appearing offer however low-valued it may be. If $\beta < 1$, then $V_t(i)$ converges to a limit $V(i) \leq h^* < \infty$ as $t \to \infty$ for all i (Theorem 3(a,c,d)).

5. The optimal reservation value $V_n(t,i)$, $n = 1, 2, \cdots, t$, starting from time t with a given search budget i, is nondecreasing in n (Theorem 2(c)), but the nondecreasing pattern is not always smooth (*Figure* 3); it can alter discontinuously even by a slight change in a starting search budget i and a discount factor β.

Recall Model

6. If $\beta = 1$, it is optimal to continue the search up to time 0 and accept the best offer obtained so far (Theorem 5(c1)).

7. If $\beta < 1$, there exists the nonnegative reservation value $h_t(i)$, the unique solution of the equation $V_t(i,y) - y = 0$, that is nondecreasing and upper-bounded in t and i (Theorem 7). Then the optimal stopping rule can be stated as follows; if $y > h_t(i)$ for the best offer y so far, stop with accepting it, or else continue. That is, in the case, the optimal stopping rule has a *reservation value property*.

8. Suppose $\beta < 1$ and $i \to \infty$. Then, at whatever point in time on the planning horizon, if the present best offer $y > h^*$, stop with accepting it, or else continue (Theorem 8). Here note that the optimal stopping rule is time-independent. This implies that whatever point in time the search process starts from, the optimal stopping rule at that point is the same as one at time 1 when the search process terminates at the next point in time. In other words, whatever planning horizon to go there remains, it is optimal to behave, in terms of stopping decision, *as if* there remains only a period of planning horizon to go. This, however, does not mean that when the best offer $y \leq h^*$, the search process must terminate at the next point in time with accepting the best offer at that point; it still proceeds. The property is usually called a *myopic property* [18][19].

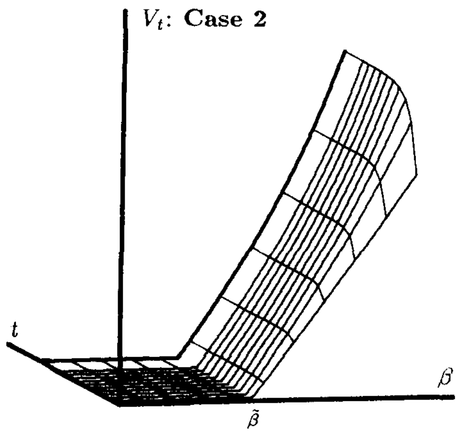

Figure 7
Limiting reservation values V_t: $a = 0$, $b = 100$, $N = 2000$

9. If $\beta < 1$, the limit of $V_t(y)$ $(= \lim_{i\to\infty} V_t(i,y))$ as $t \to \infty$, $V(y)$, becomes equal to h^* if $y \leq h^*$ (Theorem 9). This means that, in the limiting t and i, if the present best offer $y \leq h^*$, the expected present discounted net profit from continuing the search is always equal to h^*.

7.2.2 Optimal Investment

No Recall Model

10. One of the most interesting findings obtained from the numerical analysis is that the optimal investment $c_t(i)$, $t \geq 2$, does not always become monotone in the amount of search budget i that is currently available; $c_1(i)$ is always nondecreasing in i (Theorem 2(d)). Below we shall demonstrate the phenomenon by the simple example. Let $\beta = 1$, and let $p(c)$ be

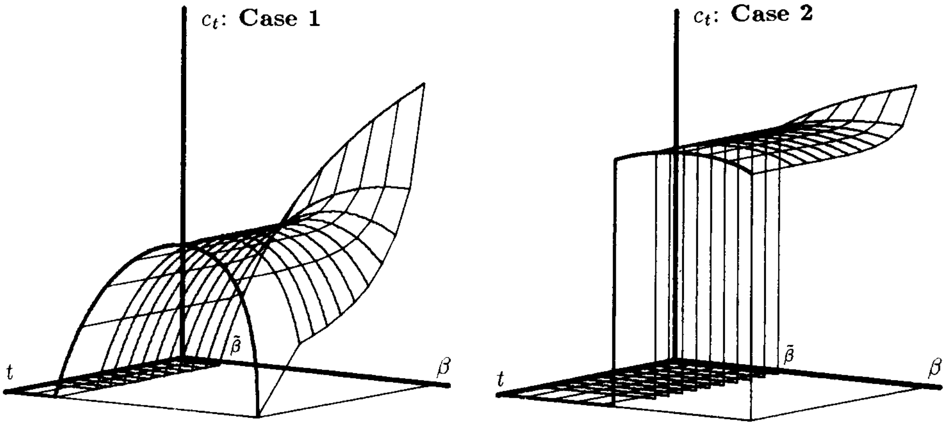

Figure 8

Limiting optimal investment c_t, $0 \leq \beta \leq 1$. Case 1: $a = 20$, $b = 20$, $N = 2000$, Case 2: $a = 0$, $b = 100$, $N = 2000$

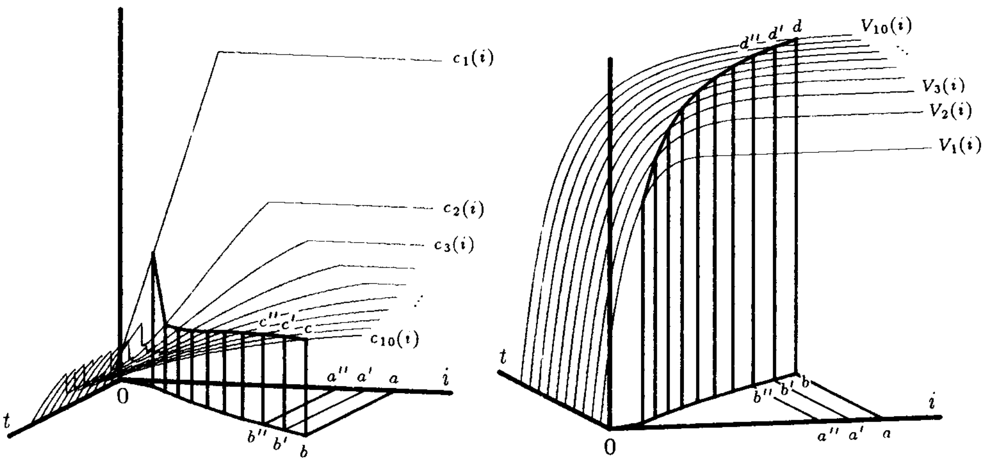

Figure 9

Relationships of $c_n(10,i)$ and $V_n(10,i)$ to $c_t(i)$ and $V_t(i)$

$$p(0.0) = 0.000,\ p(0.1) = 0.104,\ p(0.2) = 0.198,\ p(0.3) = 0.282,\ p(0.4) = 0.357,$$
$$p(0.5) = 0.424,\ p(0.6) = 0.485,\ p(0.7) = 0.539,\ p(0.8) = 0.587,\ p(0.9) = 0.631.$$

Suppose an offer obtained has the value of 50 or 150 million dollars, each with probability 0.5; hence its expectation is 100 million dollars. In the case, first we have

$$V_1(i) = \max_{0\le c\le i}\{100p(c) - c\}$$

where $i = 0.0,\ 0.1, \cdots$ million dollars and $c = 0.0,\ 0.1, \cdots, i$ million dollars. Then

$$V_1(0.0) = 0.0,\ V_1(0.1) = 10.3,\ V_1(0.2) = 19.6,\ V_1(0.3) = 27.9,\ V_1(0.4) = 35.3,$$
$$V_1(0.5) = 41.9,\ V_1(0.6) = 47.9,\ V_1(0.7) = 53.2,\ V_1(0.8) = 57.9,\ V_1(0.9) = 62.2.$$

Next let us compute $V_2(i)$ and $c_2(i)$ for $i = 0.7,\ 0.8$ using

$$V_2(i) = \max_{0\le c\le i}\{(1-p(c))V_1(i-c) + p(c)\{0.5\max\{50, V_1(i-c)\} + 0.5\max\{150, V_1(i-c)\}\} - c\}.$$

Then we have

$$V_2(0.7) = \max\{53.2000,\ 53.2184,\ 53.2038,\ \mathit{53.2454},\ 53.2397,\ 53.1896,\ 53.2045,\ 53.2000\},\ \text{so } c_2(0.7) = 0.3,$$
$$V_2(0.8) = \max\{57.9000,\ \mathit{58.1336},\ 58.0158,\ 57.9842,\ 57.9979,\ 57.9704,\ 57.9940,\ 57.9483,\ 57.9000\},\ \text{so } c_2(0.8) = 0.1.$$

That is, when $i = 0.7$ million dollars, the optimal investment is 0.3 million dollars, but when $i = 0.8$ million dollars, it is 0.1 million dollars. Like this, the optimal investment $c_t(i)$ does not always increases with a search budget, so even in case of $\beta = 1$.

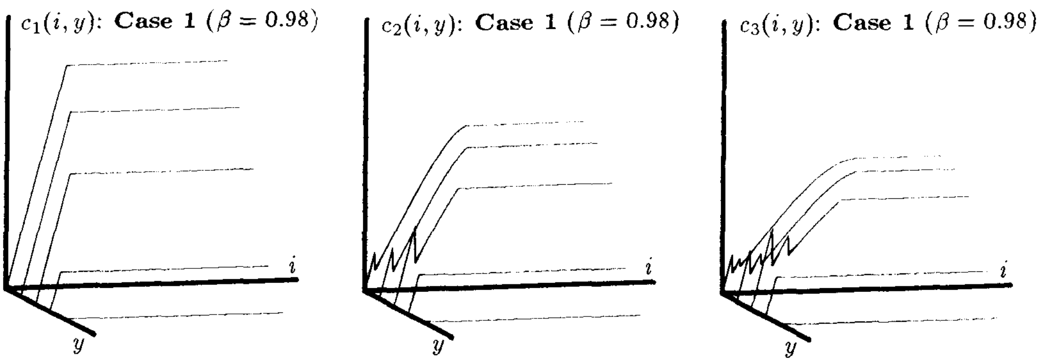

Figure 10

Optimal investments $c_t(i, y)$: $a = 50,\ b = 150,\ M = 20,\ N = 1000$

Now, *Figure 4* demonstrates that the nonmonotonicity of the optimal investment $c_t(i)$ can have almost unbelievably irregular patterns such as *rippling waves lapping a beach* (Case 1), *ditches in the Grand Canyon* (Case 3) and so on. The figure, especially in Case 3, indicates that the optimal investment is very sensitively fluctuate with a change in a remaining search budget i; its very slight increment may lift the optimal investment to a very high level (*the top of the mountain*) and reduce it drastically to zero (*the bottom of the ravine*). We might provide the following interpretation for the occurrence of the inversion phenomenon:

When a remaining search budget is small, it might be more reasonable to attempt to attain a total maximization through accepting a more profitable offer appearing with a higher probability by paying a larger search cost; by doing so, it might be possible to retrieve the disadvantageous situation of a small remaining search budget. Conversely, when a remaining search budget is large, it might be more reasonable to attempt to attain a total maximization through reserving the large remaining search budget instead of investing a larger search cost in search activities.

Figure 11 illustrates that how the inversion phenomenon occurs; the five curves and the dots on them are, respectively, the graphs of $D_2(i,c)$, $c \le i = 45, 100, 160, 340, 600$ and their maximum points. In the figure, the inversion occurs among $i = 100$ and $i = 160$ ($c_2(100) > c_2(160) > 0$) and among $i = 160$ and $i = 340$ ($c_2(160) > c_2(340) = 0$).

If the graphs of $c_1(i)$ and $c_2(i)$ are drawn on a three dimensional space by, in addition to c-axis and D-axis, taking i-axis as the third axis, the inversion phenomenon can be displayed more vividly as seen in *Figure 12* where the indented bold lines on the curved surface represents the loci of coordinates $(i, c_t(i), D_t(i, c_t(i)))$, $t = 1, 2$. Their vertical projections depicted on (i,c)-plane are the graphs of the optimal investments $c_t(i)$, $t = 1,2$, which are the same as graphs $c_1(i)$ and $c_2(i)$ of *Figure 5*, and their horizontal projections pictured as the increasing curved lines ***ab*** are the graphs of $V_t(i) = D_t(i, c_t(i))$, $t = 1, 2$, which are the same as the bold curved lines in Case 3 of *Figure 2*.

11. When a search budget is infinite, the optimal investment c_t has the following properties (*Figure 8*):

1. The c_t is equal to 0 on $0 \le \beta < \tilde{\beta}$ (Theorem 3(b)), implying that if a rate of interest is sufficiently large, it is optimal to allocate no part of the search budget to a search investment with allocating the whole to a financial investment.

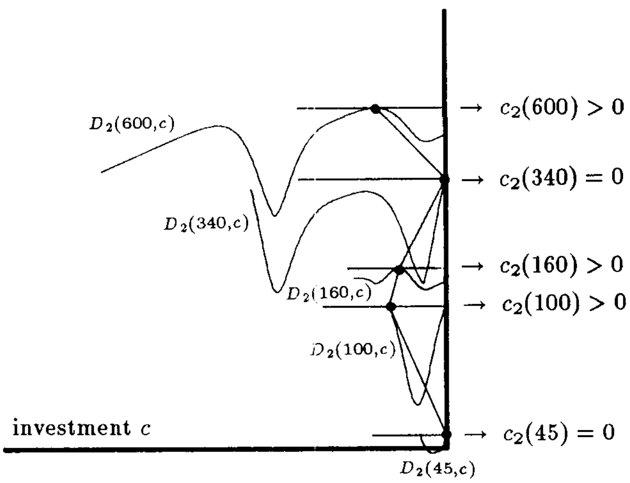

Figure 11

Inversion phenomenon of optimal investments $c_2(i)$: $a = 50$, $b = 150$, $N = 2000$

2. The c_t increases discontinuously in Case 1 and stepwise in Case 2 at $\beta = \tilde{\beta}$.

3. When there remains a sufficiently large planning horizon, there exists a discount factor at which c_t becomes maximal.

4. Note the pattern of the graphs of c_t for $\beta = 1$. As $t \to \infty$, the c_t converges to 0 in Case 1 while to a positive number in Case 2. The former case implies that, if starting with a sufficiently large planning horizon and a sufficiently large amount of search budget, it is optimal to continue to allocate the whole search budget to a *financial investment*. On the other hand, the latter case means that it is optimal to continue to allocate a positive amount of search cost to a *search investment* however large planning horizon remains.

Recall Model 12. The optimal investment $c_t(i, y)$ (*Figure 10*) depicts also an irregular pattern similarly to the optimal investment $c_t(i)$ in the no recall model (*Figure 4*).

8. Some limitations

We have abstracted certain other aspects of the optimal stopping problem, and many of the underlying assumptions of the current formulation are unrealistic. For a more realistic modification, the following provisions have to be taken into consideration: 1. Both $p(c)$ and $F(w|c)$ depend on the history of past search costs paid and offers obtained so far, 2. The availability in the future of an offer once inspected and passed up is uncertain (*uncertain recall*) [6][9][10], 3. The introduction of free search order [23]; the search order is fixed on the time axis in our model, 4. $p(c)$ and $F(w|c)$ have unknown parameters; they can and must be updated as the search process proceeds [8][15][18][19][20][21].

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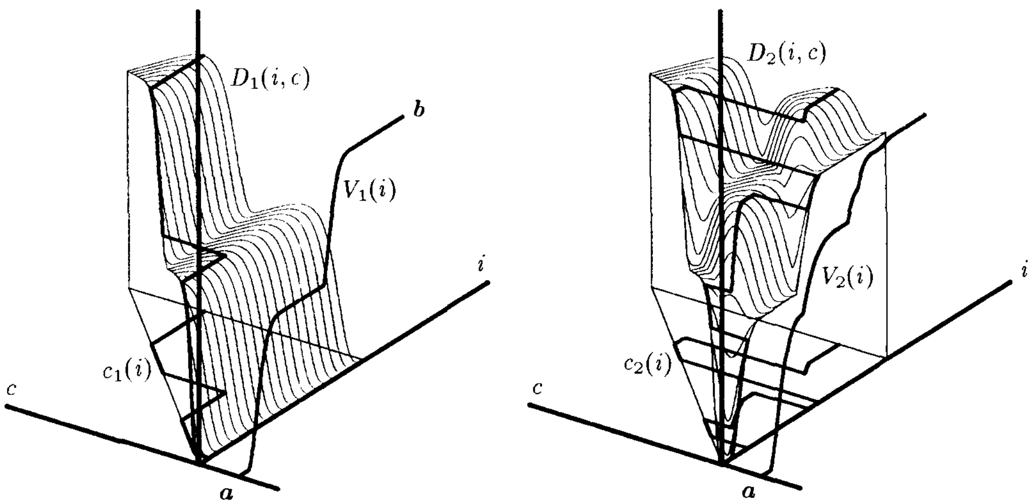

Figure 12

Optimal reservation values $V_t(i)$ and optimal investments $c_t(i)$ maximizing $D_t(i, c)$ on $0 \le c \le i$ and , $t = 1, 2$. Compare $V_1(i)$ and $V_2(i)$ with the two bold curved lines in Case 3 of Figure 2 and $c_1(i)$ and $c_2(i)$ with the graphs of $c_1(i)$ and $c_2(i)$ in Figure 5: $a = 50$, $b = 150$, $N = 2000$

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