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ASYMPTOTIC EVALUATION OF FLOW RELIABILITY FOR COHERENT REPAIRABLE NETWORKS UNDER PERIODIC DEMAND VARIATION

Shun-ichi Abe
Aoyama-Gakuin University

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Abstract The purpose of the present paper is (i) to extend the stochastic models in the previous studies [1, 2 and 3] so as to analyze flow-reliability $R(t)$ of an arbitrary coherent repairable network under periodically changing demand $\phi(t)$, and (ii) to prove that the flow-reliability can also be evaluated asymptotically as an exponential function under mild assumptions. In the model, flow $\Phi(X(t))$ of the network is defined as a monotonic function of state-vector $X(t) = (X_1(t), X_2(t), \cdots, X_n(t))$ with $X_i(t) = 1$ in case of unit i being operative, and $X_i(t) = 0$ otherwise, at time t. Flow-reliability $R(t)$ is introduced as the probability that flow $\Phi(X(s))$ of the network is greater than or equal to demand $\phi(s)$ for all $s \in [0, t]$, i.e., $R(t) = P\{\Phi(X(s)) \geq \phi(s)$ for all $s \in [0, t]\}$; and the demand function $\phi(t)$ is given arbitrarily as a nonnegative periodic function with a certain period $T > 0$. It will finally be proved that the flow-reliability $R(t)$ of the network is asymptotically exponential, i.e., $R(t) = \exp(-\Lambda t) + \Theta(t)$, where the parameter Λ is evaluated by expected life-times, repair-time distributions of the units, the structure / logic to define the flow of the network, and the demand function $\phi(t)$. It will also be proved that the error $\Theta(t) = R(t) - \exp(-\Lambda t)$ of the approximation converges to 0 as the expected lifetimes of the units increase indefinitely under certain assumptions. In the course of the proof an extended renewal equation is first derived, from which a variational equation is yielded, and its unique solution, $R_0(t)$, is effective to characterize $\exp(-\Lambda t)$ and $\Theta(t)$ in the present model.

1. Introduction

The author has investigated an asymptotic evaluation of reliabilities of coherent systems constructed redundantly by repairable units [1, 2 and 3]. And, under certain mild assumptions he has proved that (i) reliabilities $R(t)$'s of the redundant repairable systems are asymptotically exponential, i.e., there holds a unified formula $R(t) = \exp(-\Lambda t) + \Theta(t)(t \geq 0)$, that (ii) the parameter Λ can be evaluated by using the structure of the systems, the life-time and repair-time distributions of the units and the number of repair facilities, and moreover that (iii) the error term $\Theta(t)$ of the exponential approximation $\exp(-\Lambda t)$ for system reliability $R(t)$ converges to 0 as expected lifetimes of the units increase indefinitely.

The purpose of the present paper[1)] is (i) to extend the stochastic models in the previous studies so as to analyze flow-reliability $R(t)$ of an arbitrary coherent repairable network under periodically changing demand $\phi(t)$, and (ii) to prove that the flow-reliability can also be evaluated asymptotically as an exponential function under mild assumptions. In the model, flow $\Phi(X(t))$ of the network is defined as a monotonic function of state-vector $X(t) = (X_1(t), X_2(t), \cdots, X_n(t))$ with $X_i(t) = 1$ in case of unit i being operative, and $X_i(t) = 0$ otherwise, at time t; flow-reliability $R(t)$ is introduced as the probability that flow $\Phi(X(s))$ of the network is greater than or equal to demand $\phi(s)$ for all $s \in [0, t]$, i.e., $R(t) = P\{\Phi(X(s)) \geq \phi(s)$ for all $s \in [0, t]\}$; and the demand function $\phi(t)$ is given arbitrarily

[1)] The summary of the paper was presented at the 1989 autumn meeting of the Operations Research Society of Japan. See [3].

as a nonnegative periodic function with a certain period $T > 0$.

The extended model of the flow-reliability will be described precisely in Assumptions in Section 2.

In Section 3, failure rate and local failure probabilities of the network will be evaluated under steady-state or stationariness condition. In Section 4 it will finally be proved that the flow-reliability $R(t)$ of the network is asymptotically exponential, i.e., $R(t) = \exp(-\Lambda t) + \Theta(t)$, where the parameter Λ is evaluated by expected life-times, repair-time distributions of the units, the structure / logic to define the flow of the network, and the demand function $\phi(t)$. It will also be proved that the error $\Theta(t) = R(t) - \exp(-\Lambda t)$ of the approximation converges to 0 as the expected lifetimes of the units increase indefinitely under the assumptions given in Section 2. It should be remarked here that in the course of the proof an extended renewal equation is first derived, from which a variational equation is yielded, and its unique solution, $R_0(t)$, is effective to characterize $\exp(-\Lambda t)$ and $\Theta(t)$ in the present model.

In Section 5, an example of the so-called "bridge structure" is shown, where Λ is evaluated analytically to obtain the asymptotic evaluation $\exp(-\Lambda t)$ for the network-flow-reliability $R(t)$.

Finally, in Section 6, the case of non-periodic demand function is briefly noted and a remark is given on the extension of the model to the case of stochastically varying demand function.

2. Assumptions and Definitions

The stochastic model in the present paper is described by Assumptions 1°~7°as follows.

Assumption 1°: Given an arbitrary system composed of n repairable units denoted here by $i = 1, 2, \cdots, n$, let flow $\Phi(X) \geq 0$ of the system be defined as a monotonic function of state vector $X = (X_1, X_2, \cdots, X_n)$ in the sense that $X^{(1)} \leq X^{(2)}$ implies $\Phi(X^{(1)}) \leq \Phi(X^{(2)})$, where $X^{(1)} = (X_1^{(1)}, X_2^{(1)}, \cdots, X_n^{(1)}) \leq (X_1^{(2)}, X_2^{(2)}, \cdots, X_n^{(2)}) = X^{(2)}$ if and only if $X_i^{(1)} \leq X_i^{(2)} (i \in N)$, and, moreover, $X_i = X_i(t)$ is 0 or 1 according to unit i being in failure (repair) or in operation respectively at time t for each $i \in N$, where $N \equiv \{1, 2, \cdots, n\}$. Conveniently, let $X_i(t) (i \in N)$ and $\Phi(X(t))$ be left-continuous functions of t. Obviously from the definition given above, there exist 2^n distinct state vectors. Let the range of $\Phi(X)$ be $\{\Phi_0, \Phi_1, \cdots, \Phi_f\}$ with conventional assumptions $0 = \Phi_0 \equiv \Phi(0, 0, \cdots, 0) \leq \Phi(X) \leq \Phi(1, 1, \cdots, 1) \equiv \Phi_f < \infty$, where $1 \leq f \leq 2^n$.

Assumption 2°: Each unit i has its life-time (operating time length) distribution function $F_i(t)$ and repair-time distribution function $G_i(t)$, both of which are continuous in $[0, \infty)$ with $F_i(0) = G_i(0) = 0$ and both have the respective density functions $f_i(t) = F_i'(t)$ and $g_i(t) = G_i'(t)$ respectively with the exception of at most finitely many points in $[0, \infty)$; there exist the expectations $\lambda_i^{-1} = \int_0^\infty t dF_i(t) > 0, \mu_i^{-1} = \int_0^\infty t dG_i(t) > 0$ and the second moments $\int_0^\infty t^2 dF_i(t) < \infty, \int_0^\infty t^2 dG_i(t) < \infty$; $F_i(t)$ includes the scale parameter θ_i such that, putting $F_i(t) = F_{i0}(t/\theta_i), F_{i0}(y)$ does not depend on λ_i but has the expectation $\ell_i^{-1} \equiv \int_0^\infty y dF_{i0}(y) = (\lambda_i \theta_i)^{-1} > 0$ for each $i \in N$[2]; denoting $\lambda_0 = \max\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$, there exists a constant m_i such that $\theta_i = m_i/\lambda_0$, and thus $\lambda_i = \ell_i/\theta_i = (\ell_i/m_i)\lambda_0$; moreover, for

[2] The assumption on the existence of the scale parameter θ_i was introduced in [2] in order to exclude the ambiguity on the asymptotic behaviour of $F_i(t)$ in the case of $\lambda_i \to 0$ in [1].

sufficiently small λ_0 the following inequalities

$$F_i(1|u_i) \leq a_0\lambda_0^\delta \quad (0 < a_0 < \infty, 0 < \delta < 1), \qquad \int_0^\infty \overline{F}_i(x|u_i)dx \leq a_1\lambda_0^{-\gamma},$$

$$\int_0^\infty \overline{G}_j(y|v_j)dy \leq b_0 \quad (0 < a_1 < \infty, \quad 1 \leq \gamma < 2, \quad 0 < b_0 < \infty)^{3)}$$

hold uniformly for every $u_i \geq 0, v_j \geq 0$ such that $\overline{F}_i(u_i) > 0$ and $\overline{G}_j(v_j) > 0$ for each $i, j \in N$, where $\overline{F}_i(x) \equiv 1 - F_i(x)$, $\overline{F}_i(x|y) \equiv \overline{F}_i(x+y)/\overline{F}_i(y)$ and $F_i(x|y) \equiv 1 - \overline{F}_i(x+y)/\overline{F}_i(y)$; $\overline{G}_j(x), \overline{G}_j(x|y)$ and $G_j(x|y)$ are also similarly defined.

Assumption 3°: The number, r, of the repair facilities is greater than r_0 to guarantee that the n operating and repairing processes are independent in the interval $[0, t]$ as far as any system failure (defined in Assumption 7°) does not occur in the interval. Each failed unit is sent instantly to a repair facility and repairing begins if it is vacant. Each unit can operate as a new one as soon as its repair is completed. Life-times and repair-times of the units of like or different kinds are mutually independent.

Assumption 4°: At time 0 each unit is newly installed in the system.

Assumption 5°: Demand function $\phi(t)$ is a left-continuous step function with period $T > 0$, i.e.,

$$\phi(t) = \phi(s + c_tT) = \phi(s) \text{ for every } s \in [0, T) \text{ and } t = s + c_tT,$$

where $c_t = [t/T]$ (Gauss notation) $= 0, 1, \cdots$. Let the discontinuity points of $\phi(s)(s \in [0, T))$ be $0 \equiv s_0 < s_1 < \cdots < s_{d_0}(< s_{d_0+1} \equiv T)$, where $s_0 = 0$ is or is not the discontinuity point according as $\phi(0+0) \neq \phi(0)$ or $\phi(0+0) = \phi(0)$, respectively.

Assumption 6°: Let the system reliability $R(t)$ be defined as the probability that $\Phi(X(s)) \geq \phi(s)$ holds for all $s \in [0, t]$ for every $t \geq 0$. Contrarily, the system is assumed in failure at time t if $\Phi(X(t)) < \phi(t)$. However, it is assumed that if the system is in failure at time s and the inequalities $\phi(u) \leq \Phi(X(u)) < \phi_{\max} \equiv \max_{i=0,1,\cdots,d_0} \phi(s_i)$ hold for every $u \in (s, t)(0 < s < t)$ and if $\Phi(X(t)) \geq \phi(t) = \phi(t+0) > \Phi(X(t+dt))$ or $\phi(t) \leq \Phi(X(t)) = \Phi(X(t+0)) < \phi(t+0)$ are satisfied, then, the deficiency of the flow in the interval $(t, t+dt)$ does imply not a *new* but a pretended occurrence of the system failure. In other words, it is assumed that the system-failure at time s has not *essentially* been repaired by time t, since the system at time t has never been free from the previous system-failure via $\Phi(X(u)) < \phi_{\max}$ for every $u \in (s, t]$. See Fig. 1(a) and 1(b).

Assumption 7°: If the system is in failure at time t, i.e., if $\Phi(X(t)) < \phi(t)$, then, the number, $d(t)$, of failed units at time t satisfies $d(t) \geq 2$ for every $t \geq 0$: The system is redundant in the sense that $\Phi(0_i, \mathbf{1}) \geq \phi_{\max}(i \in N)$, where the state vector $(0_i, \mathbf{1}) = (X_1, X_2, \cdots, X_n)$ is defined by $X_i = 0$ and $X_j = 1(j \neq i, j \in N)$ for each $i \in N$. For each unit $i \in N$, there exists at least a state vector $X = (X_1, X_2, \cdots, X_n)$ such that $\Phi(1_i, X) > \Phi(0_i, X)$, where $(x_i, X) \equiv (X_1, \cdots, X_{i-1}, x, X_{i+1}, \cdots, X_n)$.

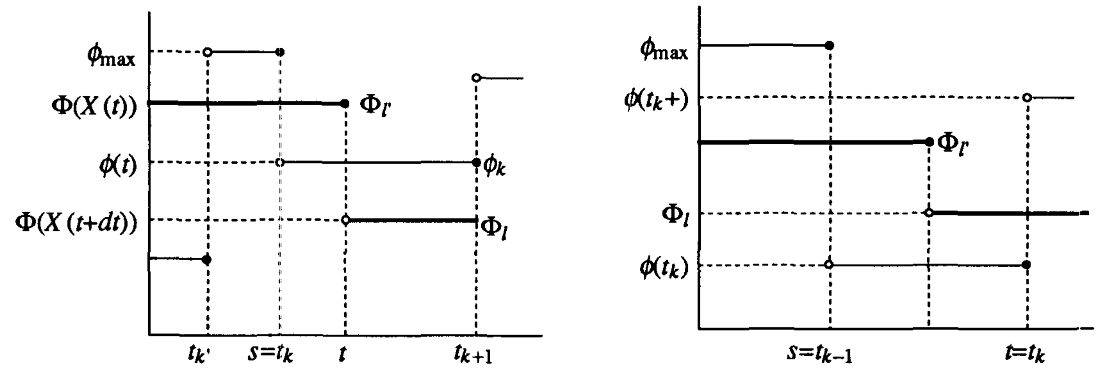
Definition 8°: If there exist times s and $t(0 \leq s \leq t)$ such that there hold $\Phi(X(s)) \geq \phi_{\max}, \Phi(X(u)) \geq \phi(u)$ for every $u \in (s, t)$ and $\Phi(X(t)) \geq \phi(t) = \phi(t+0) > \Phi(X(t+dt))$

3) Concerning sufficient conditions for these inequalities, see [2, pp. 333–335].

owing to a discontinuous decrease of the flow $\Phi(X(t)) > \Phi(X(t+dt))$ induced by a unit-failure in $(t, t+dt)$, then a system-failure occurs newly in the interval $(t, t+dt)$ as shown in Fig. 2(a). Let the system failures of this kind be called class- I -failures of the system. Contrarily, if there exist certain times s and $t (0 \le s \le t < t_k \equiv s_k + cT)$ such that a unit fails in the interval $(t, t+dt)$ and none of the units fails in the interval $(t+dt, t_k+dt)$ and that there hold $\Phi(X(s)) \ge \phi_{\max}, \Phi(X(u)) \ge \phi(u)$ for every $u \in (s, t_k)$, and $\phi(t_k) \le \Phi(X(t_k)) = \Phi(X(t_k+0)) < \phi(t_k+0)$, then a system-failure occurs newly at time $t_k + 0$ without accompanying any failure of the units, as exemplified in Fig. 2(b). Let the system-failures of this kind be termed class- II -failures of the system at $t_k + 0$. Hereafter, "$t+0$" is conventionally denoted by "$t+$" as in $\phi(t+0) = \phi(t+)$. Throughout the present paper, let ds, dt and du be infinitesimal and positive, and let us put $dh(t) = h(t+dt) - h(t)$ for every function $h(t)$. Define sets K, L and $L_k (k \in K)$ as

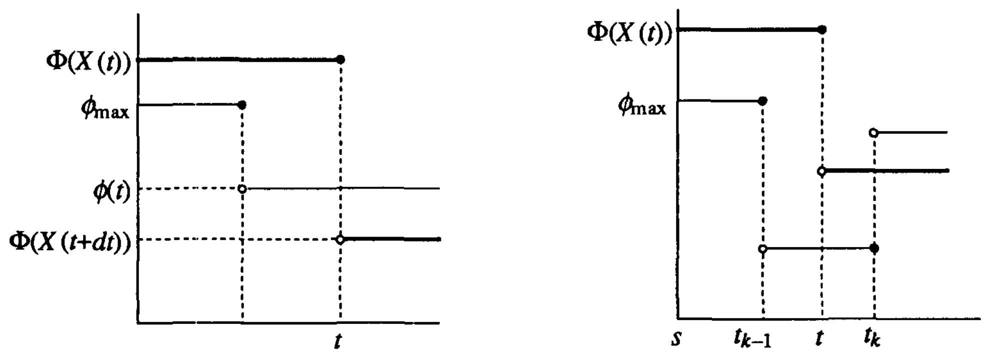
$$K \equiv \{k | \phi(s_k) < \phi(s_k+), k = 0, 1, \cdots, d_0\}, \ L \equiv \{0, 1, \cdots, f\},$$
$$L_k \equiv \{\ell | \phi(s_k) \le \Phi_\ell < \phi(s_k+0), \ell = 0, 1, \cdots, f\}.$$

If the set K is empty, the stochastic model is essentially reduced to the one in the previous studies [1, 2]. If K is not empty but $L_k (k \in K)$ is empty, the discontinuity point s_k is unessential, and it can be eliminated by revising the demand function $\phi(t)$ as $\phi(s_k+)$ for every $t \in (s_{k-1}, s_k]$ without changing the stochastic feature of the model. Hence, hereafter K and $L_k (k \in K)$ are assumed to be nonempty.



(a) Pretended class-I-failure in $(t,t+dt)$ **(b) Pretended class-II-failure at t_k+**

Fig. 1 Pretended occurrences of the system-failures



(a) Ordinary class-I-failure in $(t,t+dt)$ **(b) Ordinary class-II-failure at t_k+**

Fig. 2 Occurrences of system-failures

Finally, remark that if $r_0 = n$ in Assumption 3°, the n sequences of life- and repair-times of the respective units and of their direct descendants form the so-called independent alternating renewal processes, which will play an important role in the following section.

3. Asymptotic Evaluation of Class-I-Failure Rate and Class-II-Failure Probability of the System

For the moment assume that $r_0 = n$ in Assumption 3°, and that

(*) the n alternating renewal processes in the system are stationary [4] and mutually independent

in the present section, while the nonstationary case will be investigated in the following section.

From Assumption 5°, putting $t_k = s_k + cT, \phi(s_k+) = \phi(t_k+) \equiv \phi_k (k = 0, 1, \cdots, d_0; c = 0, 1, \cdots)$ and $t = s + c_t T (c_t = [t/T])$, there holds $\phi(t) = \phi_k = \phi(s)$ for every $s \in (s_k, s_{k+1}] \equiv S_k, t \in (t_k, t_{k+1}] \equiv T_k$ and for each $k = 0, 1, \cdots, d_0$. And, from Definition 8° a class-I-failure of the system newly induced in the interval $(t, t + dt) \subset T_k$ is characterized by a downward crossing of the flow $\Phi(X(t))$ of the system to the demand level $\phi_k : \Phi(X(t)) \geq \phi(t) = \phi_k = \phi(t + dt) > \Phi(X(t + dt))$ for $t \in T_k$ under the additional condition that there exists a certain $s \in [0, t]$ such that $\Phi(X(s)) \geq \phi_{\max}$ and $\Phi(X(u)) \geq \phi(u)$ for every $u \in [s, t]$.

The stationary component of the class -I- failure intensity of the system at $t \in T_k$ is asymptotically evaluated for sufficiently small λ_0 by

$$(1) \qquad \Lambda_k = \sum_{\alpha=1}^{\nu_k} \frac{\lambda_{i_{k\alpha}}}{\sigma_{i_{k\alpha}}} \prod_{i \in C_{k\alpha}} \frac{1}{\sigma_i} \prod_{j \in D_{k\alpha}} \frac{\rho_j}{\sigma_j} = \frac{1}{\sigma_0} \sum_{\alpha=1}^{\nu_k} \lambda_{i_{k\alpha}} \prod_{j \in D_{k\alpha}} \rho_j,$$

under the condition (*) just as in [2], where $I_{k\alpha} = \{i_{k\alpha}\}, C_{k\alpha}$ and $D_{k\alpha}$ are defined just as I_α, C_α and D_α respectively in [2, pp. 262-264], i.e.,

$$(2) \quad \begin{aligned} I_{k\alpha} &= \{i_{k\alpha}\} \\ &= \{i \mid X_i(t) = 1, X_i(t + dt) = 0, X_j(t) = X_j(t + dt) (j \neq i), \Phi(X(t)) \geq \phi(s_k+) \\ &\qquad > \Phi(X(t + dt)) \text{ for some } s \in S_k \text{ and } t = s + c_t T\}, \\ C_{k\alpha} &= \{i \mid X_i(t) = X_i(t + dt) = 1, \Phi(X(t)) \geq \phi(s_k+) > \Phi(X(t + dt)) \text{ for some} \\ &\qquad s \in S_k \text{ and } t = s + c_t T\}, \\ D_{k\alpha} &= N - I_{k\alpha} - C_{k\alpha} \quad (\alpha = 1, 2, \cdots, \nu_k), \end{aligned}$$

where ν_k is the whole number of the possible triplets $[I_{k\alpha}, C_{k\alpha}, D_{k\alpha}]$ for each $k = 0, 1, \cdots, d_0$, and

$$(3) \qquad \rho_j = \lambda_j / \mu_j, \ \sigma_j = 1 + \rho_j \ (j \in N) \text{ and } \sigma_0 = \sigma_1 \sigma_2 \cdots \sigma_n.$$

On the right side of (1), the quantities $\lambda_h dt / \sigma_h, 1/\sigma_i$ and ρ_j / σ_j are the probabilities $P\{X_h(t) = 1, X_h(t + dt) = 0\}, P\{X_i(t) = 1\}$ and $P\{X_j(t) = 0\}$, respectively for the stationary alternating renewal processes yielded by units h, i and j.

Note also that the right side of (1) may possibly include the terms that correspond to the pretended failures such as exemplified in Fig. 1(a). In fact, if $\phi_{\max} > \Phi(X(t)) = \Phi_{\ell'} \geq \phi_k > \Phi(X(t + dt))$ (for some $\ell' \in L_{k'}, k' \in K$) holds for $X(t)$ and $X(t + dt)$ with

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[4] Stationariness or steady-state of the alternating renewal process of unit, say i, is defined obviously by the intensity of the unit failure (or recovery) being $(\lambda_i^{-1} + \mu_i^{-1})^{-1} = \lambda_i / \sigma_i$ at any instant t.

respect to $[I_{k\alpha}, C_{k\alpha}, D_{k\alpha}]$ defined in (2), then, the pretended failure of the system occurs in $(t, t+dt)$, and the term corresponding to the triplet on the right side of (1) is of $O(\lambda_0^{d(t+dt)}) = O(\lambda_0^{1+|D_{k\alpha}|})$ [5]. In this case, let us define another triplet $[I_\beta^{\ell'}, C_\beta^{\ell'}, D_\beta^{\ell'}]$ and the state vectors $X(t_{k'})$ and $X(t_{k'}+)$ by

(4) $$X_i(t_{k'}) = X_i(t_{k'}+) = 1$$
$$(i \in C_\beta^{\ell'} \equiv I_{k\alpha} \cup C_{k\alpha}), X_j(t_{k'}) = X_j(t_{k'}+) = 0 \ (j \in I_\beta^{\ell'} \cup D_\beta^{\ell'} \equiv D_{k\alpha}).$$

Then, the flow deficiency of the system occurs at time $t_{k'}+$ by

$$\phi(t_{k'}) \leq \Phi(X(t_{k'})) = \Phi(X(t_{k'}+)) < \phi(t_{k'}+) \quad \text{(cf. Fig. 1(a))},$$

whose stationary probability, $\pi_{k'\beta}^{\ell'}$, is evaluated just as $\pi_{k\beta}^{\ell}$ given later in (7) to obtain $\pi_{k'\beta}^{\ell'} = O(\lambda_0^{d(t_{k'}+)}) = O(\lambda_0^{|D_{k\alpha}|})$, noticing (9) and (10). Thus, the term on the right side of (1) with respect to $[I_{k\alpha}, C_{k\alpha}, D_{k\alpha}]$ is of $O(\lambda_0^{1+|D_{k\alpha}|}) = O(\lambda_0 \pi_{k'\beta}^{\ell'})$, which can be negligible, compared with $\pi_{k'\beta}^{\ell'}$ or to the main term with the smallest order of λ_0 in the formula Λ defined later by (14).

The most important feature of the stochastic model of the present paper is the occurrence of the class-II-failures of the system characterized in Definition 8°, which could not appear in the models of the previous articles [1 and 2]. In the sequel let us evaluate asymptotically the probability of such failures at every increasing points $t_k = s_k + cT (k \in K)$ for sufficiently large positive integer c.

For the purpose, define constant(s) τ_k^ℓ $(k \in K, \ell \in L_k)$ (see Fig. 3) as

(5) $\tau_k^\ell \equiv$ the largest number τ such that the inequality $\Phi_\ell \geq \phi(t)$ holds for all $t \in (t_k - \tau, t_k]$.

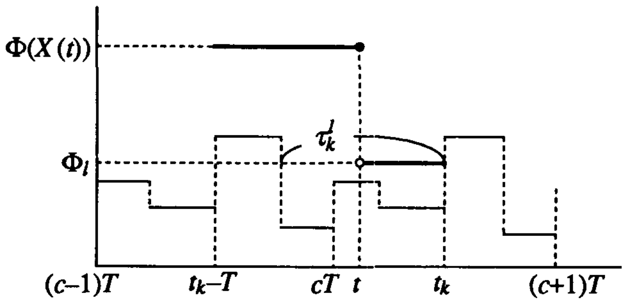


Fig. 3 Definition of the constant τ_k^ℓ

In order to characterize class-II-failures of the system more definitely, let us introduce the triplets of three sets $[I_\beta^\ell, C_\beta^\ell, D_\beta^\ell]$ with respect to Φ_ℓ $(\ell \in L_k, k \in K)$ so as to satisfy the following conditions:

[5] $|D|$ denotes the number of the elements in set D.

(i) $I^\ell_\beta \cup C^\ell_\beta \cup D^\ell_\beta = N$, $I^\ell_\beta \cap C^\ell_\beta = I^\ell_\beta \cap D^\ell_\beta = C^\ell_\beta \cap D^\ell_\beta = \phi$, $I^\ell_\beta = \{i^\ell_\beta\}$, $|I^\ell_\beta| = 1$;

(ii) defining the state vector $X^\ell_\beta = (X^\ell_{1\beta}, X^\ell_{2\beta}, \ldots, X^n_{n\beta})$ by $X^\ell_{i\beta} = 1$ $(i \in C^\ell_\beta)$ and $X^\ell_{i\beta} = 0$ $(i \in I^\ell_\beta \cup D^\ell_\beta)$, it satisfies $\Phi(X^\ell_\beta) = \Phi_\ell$;

(iii) for the state vector $(1_h, X^\ell_\beta)$ $(h = i^\ell_\beta)$ there holds $\Phi((1_{i^\ell_\beta}, X^\ell_\beta)) \geq \phi_{\max} > \Phi_\ell$.

Given $\ell \in L_k (k \in K)$, we can enumerate all of the triplets

(6) $[I^\ell_\beta, C^\ell_\beta, D^\ell_\beta]$ $(\beta = 1, 2, \ldots, \nu^\ell)$ satisfying Conditions (i) and (ii)

by a similar arguement as $[I_\alpha, C_\alpha, D_\alpha](\alpha = 1, 2, \ldots, \nu)$ in [2, pp. 262–264]. Now, if there exist certan times s and t $(0 \leq s \leq t < t_k \equiv s_k + cT)$ such that $t - s \leq T$,

$\Phi(X(s)) \geq \phi_{\max}$, $\Phi(X(u)) \geq \phi(u)$ for every $u \in (s, t_k]$, $t \in (t_{k-1}, t_k)$,

$X(t) = (1_h, X^{\ell'}_{\beta'})$, $X(t + dt) = (0_h, X^{\ell'}_{\beta'}) \leq X(t_k) = X(t_k+) = X^\ell_\beta$ $(\ell \in L_k, k \in K)$

for some $h \in N$ and state vector $X^{\ell'}_{\beta'}$ $(\ell' \in L)$,

and if none of the units fails newly in the time-interval $(t + dt, t_k + dt)$, then the class-II-failure of the system occurs newly at time t_k+ via $[I^\ell_\beta, C^\ell_\beta, D^\ell_\beta]$. See Fig. 4(b). The stationary probability of this event (denoted here as $E^\ell_{k\beta}$) is evaluated asymptotically by

(7)
$$\pi^\ell_{k\beta} = \frac{1}{\sigma_0}\lambda_h \int_0^{\tau^\ell_k} \overline{G}_h(\tau)\{\prod_{i \in C^\ell_\beta} \lambda_i \int_\tau^\infty \overline{F}_i(u_i)\, du_i \times \prod_{j \in D^\ell_\beta} \lambda_j \int_\tau^\infty \overline{G}_j(u_j)\, du_j\}\, d\tau\{1 + O(\lambda_0)\} \quad (h = i^\ell_\beta \in I^\ell_\beta)$$

under the condition $(*)$.

On the right side of the formula (7), $\lambda_h \overline{G}_h(\tau)\, d\tau/\sigma_h$ is the stationary probability that the failure of unit h occurs in the time interval $[t_k - \tau, t_k - \tau + d\tau)$ and it cannot be repaired by time t_k (within time-length τ), and $\lambda_j \int_\tau^\infty \overline{G}_j(u)\, du/\sigma_j$ is the stationary probability that unit j is in failure throughout the time-interval $[t_k - \tau, t_k]$ $(0 < \tau \leq \tau^\ell_k)$ under the condition $(*)$ given above. Contrarily, $\lambda_i \int_\tau^\infty \overline{F}_i(u)\, du/\sigma_i$ is the stationary probability that unit i is in operation throughout the time-interval $[t_k - \tau, t_k]$ $(0 < \tau \leq \tau^\ell_k)$ under the same condition $(*)$. Thus, the main term on the right side of (7) is the stationary probability of the event $\subset E^\ell_{k\beta}$ such that $d(u) = d(t_k)$ for every $u \in (t + dt, t_k]$ as seen in Fig. 2(b). And the term $O(\lambda_0)$ on the right side of (7) represents the effect of the events $\subset E^\ell_{k\beta}$ with $d(t + dt) > d(t_k)$ as exemplified in Fig. 4(b), because in these cases the stationary probability of the events is of the order

(8)
$$O(\lambda_0^{d(t+dt)}) = O(\lambda_0 \pi^\ell_{k\beta})$$

under the condition $(*)$. Note here that

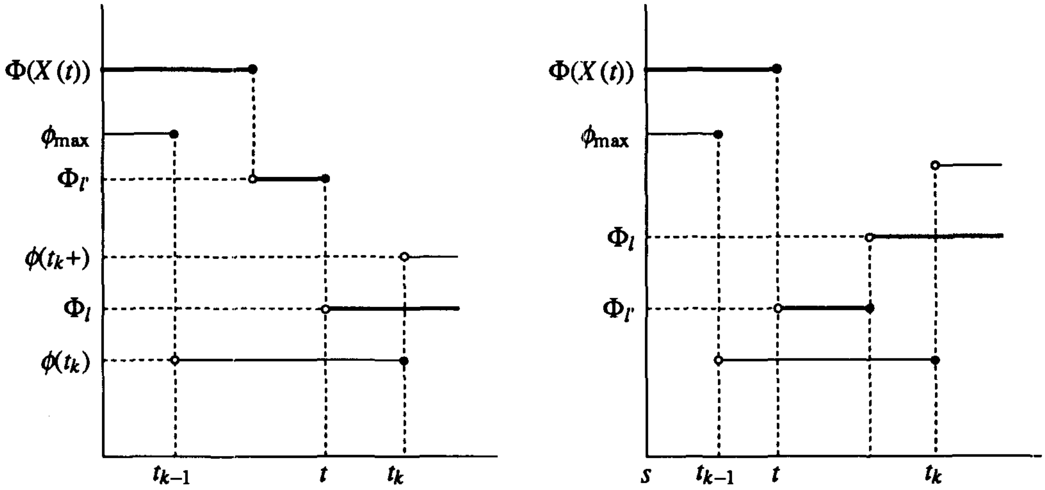
(9)
$$\lambda_i \int_\tau^\infty \overline{F}_i(u)\, du = \lambda_i[\int_0^\infty \overline{F}_i(u)\, du - \int_0^\tau \overline{F}_i(u)\, du] = 1 - \lambda_i \int_0^\tau \overline{F}_i(u)\, du,$$

where

(10) $$0 < \lambda_i \int_0^{\tau} \overline{F}_i(u)\,du \leq \lambda_i \tau \leq \lambda_i T \to 0 \quad (\lambda_0 \to 0).$$

Thus, in case of sufficiently small λ_0, applying (9) and (10) to (7), we get

(11) $$\pi_k^\ell = \frac{1}{\sigma_0} \sum_{\beta=1}^{\nu^\ell} \lambda_{i_\beta^\ell} \int_0^{\tau_k^\ell} \overline{G}_{i_\beta^\ell}(\tau) \{ \prod_{j \in D_\beta^\ell} \lambda_j \int_\tau^\infty \overline{G}_j(v_j)\,dv_j \}\,d\tau \{1 + O(\lambda_0)\}.$$



(a) (b)

Fig. 4 Occurrences of class-II-failures of the system at t_k+ (negligible cases)

Further, remark also that if there exists a set D_0^ℓ such that $I_\beta^\ell \cup D_\beta^\ell = D_0^\ell$ for $\beta = 1, 2, \cdots, \nu_0$ with $\nu_0 = |D_0^\ell|$, then there holds

(12) $$\begin{aligned} &\sum_{\beta=1}^{\nu_0} \lambda_{i_\beta^\ell} \int_0^{\tau_k^\ell} \overline{G}_{i_\beta^\ell}(\tau) \{ \prod_{j \in D_\beta^\ell} \lambda_j \int_\tau^\infty \overline{G}_j(v_j)\,dv_j \}\,d\tau \\ &= \int_0^{\tau_k^\ell} \frac{d}{d\tau} \{ - \prod_{i \in D_0^\ell} \lambda_i \int_\tau^\infty \overline{G}_i(v_i)\,dv_i \}\,d\tau \\ &= \prod_{j \in D_0^\ell} \rho_j - \prod_{j \in D_0^\ell} \lambda_j \int_{\tau_k^\ell}^\infty \overline{G}_j(v_j)\,dv_j \\ &= \prod_{j \in D_0^\ell} \rho_j \, [1 - \prod_{i \in D_0^\ell} \mu_i \int_{\tau_k^\ell}^\infty \overline{G}_i(v_i)\,dv_i]. \end{aligned}$$

In most cases of practical applications, the formula (12) is useful for us to simplify the right sides of the formula (11). See Example in Section 5.

Furthermore, it should be remarked that, on evaluating π_k^ℓ by using $[I_\beta^\ell, C_\beta^\ell, D_\beta^\ell]$ given in (6), the right side of (11) may include the probabilities of such events that are not the new but the pretended occurrences of class-II-failures of the system at time t_k+. See Fig. 1(b). On the other hand, application of Condition (iii) in addition to Conditions (i) and (ii) to enumerating the triplets in (6) may exclude the class-II-failures of the system at t_k+ with the flow level Φ_ℓ satisfying

$$\phi_{\max} > \Phi((1_h, X^\ell_\beta)) \equiv \Phi_{\ell'} \geq \Phi_\ell \quad (\ell \in L_k)\ (h \in I^\ell_\beta \cup D^\ell_\beta)$$

as exemplified in Fig. 4(a). In both cases such as mentioned above, putting $X^{\ell'}_{\beta'} = (1_h, X^\ell_\beta)$ (or applying the similar procedure iteratively if necessary) one can find some $k' \in K, \ell' \in L_{k'}$ and β' satisfying for instance

$$\Phi((1_j, X^{\ell'}_{\beta'})) \geq \phi_{\max} > \Phi_{\ell'} \quad (j \in I^{\ell'}_{\beta'} \cup D^{\ell'}_{\beta'}). \tag{13}$$

From these quantities one obtains $O(\pi^\ell_{k\beta}) = \lambda_0 O(\pi^{\ell'}_{k'\beta'})$, because $|D^{\ell'}_{\beta'}| \geq 1 + |D^\ell_\beta|$. Hence, the contributions of these cases to the right side of (11) can be neglected for sufficiently small λ_0, which implies that by using Condition (iii) one can exclude the ineffective triplets that yield negligible terms of orders of λ_0 on the right side of (14) to be given below.

Finally let us put

$$\Lambda = \Lambda_{\mathrm{I}} + \Lambda_{\mathrm{II}}, \tag{14}$$

$$\Lambda_{\mathrm{I}} = \sum_{k=0}^{d_0} (s_{k+1} - s_k)\Lambda_k / T, \tag{15}$$

$$\Lambda_{\mathrm{II}} = \sum_{k \in K} \pi_k / T, \quad \pi_k = \sum_{\ell \in L_k} \pi^\ell_k, \quad \Pi_{\mathrm{II}} = \sum_{k \in K} \pi_k, \tag{16}$$

where Λ may be interpreted as *the asymptotic averaged failure rate* of the system; Λ_{I} and Λ_{II} may be considered, respectively, as *the asymptotic averaged class-I- and class-II- failure rates* of the system, as shown in the following section.

Additionally, let us rewrite all of the triplets (6):

$$[I^\ell_\beta, C^\ell_\beta, D^\ell_\beta] \quad (\beta = 1, 2, \ldots, \nu^\ell; \ell \in L_k)$$

as

$$[I^*_{k\alpha}, C^*_{k\alpha}, D^*_{k\alpha}] \quad (\alpha = 1, 2, \ldots, \nu^*_k; \nu^*_k \equiv \sum_{\ell \in L_k} \nu^\ell) \quad (k \in K). \tag{17}$$

Then, the number, r_0, of the repair facilities in Assumption 3° is now given explicitly by

$$r_0 = \max\{ \max_{k=0,1,\ldots,d_0} \max_{\alpha=1,2,\ldots,\nu_k} |D_{k\alpha}|, \max_{k \in K} \max_{\alpha=1,2,\ldots,\nu^*_k} |I^*_{k\alpha} \cup D^*_{k\alpha}| \} \leq n. \tag{18}$$

The system with r_0 repair facilities behaves just as the one with n repair facilities until the first system failure occurs. Therefore, on analyzing reliability of the system with r_0 repair facilities, it suffices for us to assume $r_0 = n$, as was done at the top of the present section.

4. Asymptotic Exponential Formula for System Reliability

In this section let us derive an asymptotic exponential formula for the flow reliability of the system under the periodic demand function from an extended renewal equation. For the purpose let us first define the age vector

$$\overline{u}_\alpha(s) = (u_{\alpha 1}(s), u_{\alpha 2}(s), \ldots, u_{\alpha n}(s)),$$

where, assuming that class-I- or class-II- failure of the system occurs in the interval $(s, s + ds)$, $u_{\alpha i}$ denotes the operating time-length (age) of unit i being in use at time s if $i \in I_{k\alpha} \cup C_{k\alpha}$ for class-I-failure with $s \in (t_k, t_{k+1})$, or if $i \in C^*_{k\alpha}$ for class-II-failure with $s = t_k$, and $u_{\alpha i}$ represents the repair duration time-length at time s if $i \in D_{k\alpha}$ for class-I-failure with $s \in (t_k, t_{k+1})$, or if $i \in I^*_{k\alpha} \cup D^*_{k\alpha}$ for class-II-failure with $s = t_k$. Under the same circumstances as the above, using the age vector $\overline{u}_\alpha(s)$, let us define $\overline{u}_\alpha(s+) = (u_{\alpha 1}(s+), u_{\alpha 2}(s+), \ldots, u_{\alpha n}(s+))$ as the age vector just after the system-failure by putting

$$u_{\alpha i}(s+) = \begin{cases} u_{\alpha i}(s) \text{ if } i \in C_{k\alpha} \cup D_{k\alpha} \text{ for class-I-failure at } s+ \\ \qquad \text{or if } i \in I^*_{k\alpha} \cup C^*_{k\alpha} \cup D^*_{k\alpha} \text{ for class-II-failure at } s+ = t_k+, \\ 0 \qquad \text{if } i \in I_{k\alpha} \text{ for class-I-failure at } s+. \end{cases}$$

Furthermore, using age vectors $\overline{u}_\alpha(s)$ and $\overline{u}_\alpha(s+)$, let us introduce two quantities as follows:

(19) $P_\alpha(ds, d\overline{u}_\alpha(s))$

= the probability that the first system-failure occurs in the interval $(s, s + ds)$ as a class-I-failure via $[I_{k\alpha}, C_{k\alpha}, D_{k\alpha}]$ for $s \in [s_k + c_s T, s_{k+1} + c_s T]$ $(k = 0, 1, \ldots, d_0)$ or as a class-II-failure via $[I^*_{k\alpha}, C^*_{k\alpha}, D^*_{k\alpha}]$ at $s+ = s_k + c_s T + 0$ $(k \in K)$ and that the age vector $\overline{u}_\alpha(s) \in (\overline{u}_\alpha, \overline{u}_\alpha + d\overline{u}_\alpha)$,

where $s > 0, c_s = [s/T]$ (Gauss notation), and $\overline{u}_\alpha(s) \in (\overline{u}_\alpha, \overline{u}_\alpha + d\overline{u}_\alpha)$ indicates that $u_{\alpha i}(s) \in (u_{\alpha i}, u_{\alpha i} + du_{\alpha i})$ for given $u_{\alpha i} \geq 0 (i \in N)$;

(20) $Q(dt \mid \overline{u}_\alpha(s+))$

= the conditional probability that a class-I- or class-II- failure of the system occurs in the interval $(t, t + dt)$, given that a class-I-failure of the system has occurred in the interval $(s, s + ds) \subset (s_k + c_s T, s_{k+1} + c_s T)$ via $[I_{k\alpha}, C_{k\alpha}, D_{k\alpha}]$ or a class-II-failure at time $s+ = s_k + c_s T + 0$ via $[I^*_{k\alpha}, C^*_{k\alpha}, D^*_{k\alpha}]$ $(k \in K)$ with age vector $\overline{u}_\alpha(s+)$,

where $0 \leq s < t$ and $\alpha = 1, 2, \ldots, \nu_k$ $(k = 0, 1, \ldots, d_0)$ for the class-I-failure, and $\alpha = 1, 2, \ldots, \nu^*_k$ $(k \in K)$ for the class-II-failure of the system. Similarly, define $P(dt)$ and $Q(dt)$ as follows:

(21) $P(dt) = \sum_{\alpha=1}^{\nu(t)} \int_{\overline{u}_\alpha(t)} P_\alpha(dt, d\overline{u}_\alpha(t))$

= the probability that the first system-failure occurs as a class-I- or a class-II-failure in the interval $(t, t + dt)$,

where $\int_{\overline{u}_\alpha(t)}$ indicates the integration with respect to $\overline{u}_\alpha(t) = (u_{\alpha 1}(t), u_{\alpha 2}(t), \ldots, u_{\alpha n}(t))$ on the whole of n-dimensional space $[0, \infty]^n$, and $\sum$ denotes the summation on $\alpha = 1, 2, \ldots, \nu_k$ $(\nu(t) = \nu_k)$ or $\alpha = 1, 2, \ldots, \nu^*_k$ $(\nu(t) = \nu^*_k)$ according as class-I-failure with $\nu(t) = \nu_k$ for every $t \in (t_k, t_{k+1})$ or class-II-failure with $\nu(t) = \nu^*_k$ for $t = t_k+$;

(22) $Q(dt) = Q(dt \mid \overline{u}_0)$

= the probability that a class-I- or class-II-failure of the system occurs in the interval $(t, t+dt)$,

where $\overline{u}_0 = (0, 0, \cdots, 0)$ is the age vector given by Assumption 4°.

Then, the following relation holds:

$$Q(dt) = P(dt) + \int_{s=0}^{t} \sum_{\alpha=1}^{\nu(s)} \int_{\overline{u}_\alpha(s)} P_\alpha(ds, d\overline{u}_\alpha(s)) Q(dt \mid \overline{u}_\alpha(s+)), \tag{23}$$

which can obviously be viewed as an *extended renewal equation* for $t > 0$. Note here that the conditional probability (20) can be put

$$Q(dt \mid \overline{u}_\alpha(s+)) = Q_0(dt) + \psi(dt \mid \overline{u}_\alpha(s+)), \tag{24}$$

where

$$Q_0(dt) = \begin{cases} \Lambda_k\, dt & \text{if } (t, t+dt) \subset (t_k, t_{k+1}), \\ \delta_k(K)\pi_k + \Lambda_k\, dt & \text{if } t = t_k \end{cases} \qquad (k = 0, 1, \ldots, d_0), \tag{25}$$

is the stationary (steady-state) factor of the quantity defined in (24), and $\psi(dt \mid \overline{u}_\alpha(s+))$ is its deviation from $Q_0(dt)$, and $\delta_k(K)$ is defined by

$$\delta_k(K) = \begin{cases} 0 & (k \notin K), \\ 1 & (k \in K). \end{cases} \tag{26}$$

Similarly, let us put

$$Q(dt \mid \overline{u}_0) = Q_0(dt) + \psi(dt \mid \overline{u}_0). \tag{27}$$

Substituting (24) and (27) to both sides of the equation (23), we can get

$$P(dt) = \{1 - P(t)\} Q_0(dt) + \omega(dt), \tag{28}$$

where

$$\begin{aligned} P(t) &= \int_0^t P(ds) \\ &= \text{the probability that the first system-failure occurs by time } t > 0 \\ &= 1 - R(t), \end{aligned} \tag{29}$$

and

$$\begin{aligned} \omega(dt) = &\psi(dt \mid \overline{u}_0) \\ &- \int_{s=0}^{t} \sum_{\alpha=1}^{\nu(s)} \int_{\overline{u}_\alpha(s)} P_\alpha(ds, d\overline{u}_\alpha(s)) \psi(dt \mid \overline{u}_\alpha(s+)). \end{aligned} \tag{30}$$

From (29) we obtain

$$P(dt) = P(t+dt) - P(t) = dP(t) = -dR(t). \tag{31}$$

Applying equations (29) and (31) to the both sides of (28), we have

$$-dR(t) = R(t)Q_0(dt) + \omega(dt). \tag{32}$$

In order to solve this equation, first it should be remarked that putting

$$\Psi(w) = \sup_{s\in[0,w]} \sup_{\alpha=0,1,\cdots,\nu(s)} \sup_{\overline{u}_\alpha(s)\geq\overline{0}} \int_s^w |\psi(dt \mid \overline{u}_\alpha(s+))|, \tag{33}$$

we can get

$$\begin{aligned} &\int_0^w |\psi(dt \mid \overline{u}_0)| \leq \Psi(w) = O(\lambda_0^{\delta_0}), \\ &\int_s^w |\psi(dt \mid \overline{u}_\alpha(s+))| \leq \Psi(w) = O(\lambda_0^{\delta_0}), \end{aligned} \tag{34}$$

which are proved in Appendix, where $\overline{0} = (0, 0, \cdots, 0)$, $\overline{u}_0 = \overline{0}$, $\delta_0 = \min\{\delta, 2-\gamma\}$. From the above we immediately have the evaluation

$$\begin{aligned} \int_0^t |\omega(ds)| \leq &\int_0^t |\psi(ds \mid \overline{u}_0)| \\ &+ \int_{x=0}^t \int_{s=0}^x \sum_{\alpha=1}^{\nu(s)} \int_{\overline{u}_\alpha(s)} P_\alpha(ds, d\overline{u}_\alpha(s))|\psi(dx \mid \overline{u}_\alpha(s+))| \\ \leq &2\Psi(t) = O(\lambda_0^{\delta_0}) \end{aligned} \tag{35}$$

after replacing $\int_{x=0}^t \int_{s=0}^x$ by $\int_{s=0}^t \int_{x=s}^t$.
Second, let us put for every $t \in [t_k, t_{k+1}]$,

$$R_0(cT) = [\exp(-\Lambda_I T) \prod_{k\in K} (1-\pi_k)]^c = [R_0(T)]^c, \tag{36}$$

$$R_0(t_k)/R_0(cT) = \exp\{-\sum_{i=1}^k \Lambda_i(t_i - t_{i-1})\} \prod_{j=1}^k (1-\pi_j)^{\delta_j(K,t_k)} \quad (k = 1, 2, \ldots, d_0), \tag{37}$$

$$R_0(t)/R_0(t_k) = \exp(-\Lambda_k(t-t_k))(1-\pi_k)^{\delta_k(K,t)}, \tag{38}$$

where $t_k = s_k + cT$ ($k \in K$, $c = 0, 1, 2, \cdots$) and

$$\delta_j(K, t) = \begin{cases} 0 & (j \notin K \text{ or } t_j \geq t), \\ 1 & (j \in K \text{ and } t_j < t). \end{cases} \tag{39}$$

Then, we can prove the following property:

Proposition 9. "The quantity $R_0(t)$ defined by (36)~(39) as $R_0(t) = R_0(cT) \cdot R_0(t_k)/R_0(cT) \cdot R_0(t)/R_0(t_k)$ is the unique solution of the variational equation:

$$-dR_0(t) = R_0(t)\, dQ_0(t) \quad (R_0(0) = 1), \tag{40}$$

where $dQ_0(t) = Q_0(dt)$ is given by (25) for every $t > 0$."

Proof: (i) If $t_k < t < t + dt < t_{k+1}, \delta_k(K,t) = \delta_k(K, t + dt)$ holds. Hence, from (38) we have $-dR_0(t)/R_0(t_k) = \Lambda_k\{R_0(t)/R_0(t_k)\}\, dt$,

$$-dR_0(t) = \Lambda_k R_0(t)\, dt = R_0(t)\, dQ_0(t), \tag{41}$$

which assures the equation (40) in the case of $dQ_0(t) = \Lambda_k\, dt$ in the definition (25). Moreover, (ii) if $t = t_k$, then $\delta_k(K,t) = 0$ and $\delta_k(K, t + dt) = \delta_k(K)$, from which the equation (38) implies

$$\begin{aligned} -\{R_0(t_k + dt) &- R_0(t_k)\}/R_0(t_k) \\ &= -\{\exp(-\Lambda_k\, dt)(1 - \pi_k)^{\delta_k(K)} - 1\} \\ &= 1 - (1 - \Lambda_k\, dt)(1 - \pi_k)^{\delta_k(K)} \\ &= \Lambda_k\, dt + \delta_k(K)\pi_k - \delta_k(K)\Lambda_k\pi_k\, dt, \end{aligned}$$

neglecting higher order terms of dt. On the right side of the above, note that in case of $\delta_k(K) = 0$, $\delta_k(K)\Lambda_k\pi_k$ vanishes, and in case of $\delta_k(K) = 1$, $\Lambda_k\pi_k\, dt$ can be neglected compared to π_k; in either case the third term $\delta_k(K)\Lambda_k\pi_k\, dt$ is negligible. This implies that

$$-dR_0(t) = R_0(t)\{\Lambda_k\, dt + \delta_k(K)\pi_k\} \quad (t = t_k)$$

which asserts the equation (40) in the case of $t = t_k$ and $dQ_0(t) = \Lambda_k\, dt + \delta_k(K)\pi_k$ in (25). Thus, the function $R_0(t)$ defined by (36)–(39) satisfies (40) in either case. The uniqueness of the solution of (40) is easily proved as follows: If there exists another solution $R_*(t)$ of (40),

$$-dR_*(t) = R_*(t)\, dQ_0(t), \tag{42}$$

$$R_0(0) = R_*(0) = 1, \tag{43}$$

then we immediately obtain from (40) and (42) $R_0(t)\, dR_*(t) - R_*(t)\, dR_0(t) = 0$ for every $t \geq 0$ and hence (noting that $R_0(t) > 0$ for all $t \geq 0$)

$$d\left(\frac{R_*(t)}{R_0(t)}\right) = \frac{R_0(t)\, dR_*(t) - R_*(t)\, dR_0(t)}{R_0(t + dt)R_0(t)} = 0.$$

This asserts that for every $t \geq 0$

$$R_*(t)/R_0(t) = \text{const.}$$

Furthermore, from (43) the constant on the right side of the above must be unity, and finally we have $R_0(t) = R_*(t)$ for every $t > 0$. Q.E.D.

Proposition 10. "Under Assumptions $1° \sim 7°$, the flow reliability $R(t)$ of the system, with respect to the periodic variation of the demand levels, is represented by

$$R(t) = R_0(t) + \Theta(t), \tag{44}$$

where $R_0(t)$ is the left-continuous function given by (38)$\sim$(40) as

$$\begin{aligned} R_0(t) =& [\exp(-\Lambda_{\rm I} T) \prod_{i \in K} (1 - \pi_i)]^c \\ &\times \prod_{j=0}^{k-1} [\exp(-\Lambda_j(t_{j+1} - t_j))(1 - \pi_j)^{\delta_j(K,t_k)}] \\ &\times \exp(-\Lambda_k(t - t_k))(1 - \pi_k)^{\delta_k(K,t)} \end{aligned} \tag{45}$$

for every $t \in (t_k, t_{k+1}]$ with $t_k = s_k + cT$ $(k = 0,1,\cdots,d_0; c = 0,1,\cdots)$, and $\Theta(t)$ is characterized by $\Theta(0) = 0$,

$$\Theta(t) = -R_0(t)[I_k + J_k + \int_{a_k+}^{t} \{R_0(s)\}^{-1}\omega(ds)], \tag{46}$$

for every $t \in (a_k, a_{k+1}]$ $(k = 0,1,\cdots)$, denoting the set of all discontinuity points of the function $R_0(t)$ as $\{a_1, a_2, \ldots\}$ with $0 = a_0 < a_1 < a_2 < \cdots$ and putting $\{I_k\}$ and $\{J_k\}$ as $I_0 = J_0 = 0, I_k = \{\omega(a_k+) - \omega(a_k)\}/R_0(a_k+)$ and

$$J_k = \sum_{i=1}^{k}[I_{i-1} + \int_{a_{i-1}+}^{a_i} \{R_0(s)\}^{-1}\omega(ds)] \quad (k = 1,2,\cdots)$$

with $\omega(t) = \int_0^t \omega(ds)$ $(t \geq 0)$. Moreover,

$$|\Theta(t)| \leq 2\Psi(t) = O(\Lambda_0^{\delta_0}) \tag{47}$$

holds for every $t > 0$ such that $R_0(t) > 0$, where $\delta_0 = \min\{\delta, 2-\gamma\}$ in Assumption 2°."

Proof. Note that $R(t) = R(cT)\{R(t_k)/R(cT)\}\{R(t)/R(t_k)\}$, and the formula (45) is immediately obtained by the multiplication of right sides of the formulae (36)~(38). Note also that if $a_k < t < t + dt < a_{k+1}$, then from (46) and (40) we get the relation

$$\begin{aligned} -d\Theta(t) &= -dR_0(t) \cdot \Theta(t)/R_0(t) + \omega(dt) \\ &= \Theta(t)\, dQ_0(t) + \omega(dt), \end{aligned}$$

owing to the continuity of $R_0(t)$ in $t \in (a_k, a_{k+1})$ $(k = 0,1,\cdots)$. Furthermore, if $t = a_k < t+dt$, we have $\Theta(a_k) = -R_0(a_k)J_k$, $R_0(a_k+dt)I_k = \{1+O(R_0(a_k+dt) - R_0(a_k+))\}\{\omega(a_k+) -\omega(a_k)\}$, $R_0(a_k+dt)\int_{a_k+}^{a_k+dt}\{R_0(s)\}^{-1}\omega(ds) = \{1+O(R_0(a_k+dt) - R_0(a_k+))\}\{\omega(a_k+dt) -\omega(a_k+)\}$ and finally again

$$\begin{aligned} -d\Theta(t) &= -\Theta(a_k+dt) + \Theta(a_k) \\ &= R_0(a_k+dt)[I_k + J_k + \int_{a_k+}^{a_k+dt}\{R_0(s)\}^{-1}\omega(ds)] - R_0(a_k)J_k \\ &= J_k\, dR_0(t) + \{\omega(a_k+dt) - \omega(a_k)\}\{1 + O(R_0(a_k+dt) - R_0(a_k+))\} \\ &= -J_kR_0(a_k)dQ_0(t) + \omega(dt)\{1 + O(R_0(a_k+dt) - R_0(a_k+))\} \\ &= \Theta(t)dQ_0(t) + \omega(dt) \quad (t = a_k), \end{aligned}$$

neglecting higher order terms of dt. This relation with (40) implies that $R(t)$ given by (44) satisfies the variational equation (32). From (46) and (35) we obtain

$$|\Theta(t)| \leq \int_0^t |\omega(ds)| \leq 2\Psi(t) = O(\lambda_0^{\delta_0}),$$

which proves (47). Q.E.D.

Proposition 11. "Under the same assumptions and notations as in Proposition 10, $R_0(t)$ is approximated well by $\exp(-\Lambda t)$:

$$R_0(t) = \exp(-\Lambda t) + \theta(t), \tag{48}$$

and the error $\theta(t)$ is evaluated as

(49) $$\theta(t) = O(\lambda_0^2)$$

for sufficiently small value of $\lambda_0 > 0$ in case $\Lambda t = O(1)$, where Λ is given by the formula (14)."

Proof. First, notice that from the definition (16) we have

(50) $$\begin{aligned}&\prod_{k\in K} [(1-\pi_k)/\exp(-\Lambda_{\mathrm{II}}T)]^c \\ &\quad = \exp[c\sum_{k\in K}\{\log(1-\pi_k)+\pi_k\}] \\ &\quad = \exp(-c\sum_{k\in K}\sum_{j=2}^{\infty}\pi_k^j/j) \\ &\quad = 1+\pi_0 O(c\Lambda_{\mathrm{II}}T) \quad (\pi_0 = \max_{k\in K}\{\pi_k\}),\end{aligned}$$

(51) $$\begin{aligned}&\prod_{k\in K}(1-\pi_k)^{\delta_k(K,t)}/\exp(-\Lambda_{\mathrm{II}}(t-cT)) \\ &\quad = \exp[\sum_{k\in K}\{\delta_k(K,t)\log(1-\pi_k)+(s/T)\pi_k\}] \quad (s = t-cT) \\ &\quad = 1+O(\sum_{k\in K}\pi_k).\end{aligned}$$

Second, remark that from $t \in [t_k, t_{k+1})$ and $t_j = s_j + cT$ $(j = 0, 1, \cdots, k+1)$ we get

(52) $$\begin{aligned}&c\Lambda_{\mathrm{I}}T + \sum_{i=0}^{k-1}\Lambda_i(t_{i+1}-t_i) + \Lambda_k(t-t_k) - \Lambda_{\mathrm{I}}t \\ &\quad = O(\Lambda_0 T) \quad (\Lambda_0 = \max_{i=1,2,\ldots,d_0}\{\Lambda_i\}).\end{aligned}$$

From the above evaluations (50)~(52) and $\Lambda = \Lambda_{\mathrm{I}} + \Lambda_{\mathrm{II}}$ we easily obtain

$$\begin{aligned}R_0(t)/\exp(-\Lambda t) = &\{1+O(\Lambda_0 T)\}\{1+\pi_0 O(\Lambda_{\mathrm{II}}t)\} \\ &\times\{1+|K|O(\pi_0)\},\end{aligned}$$

where the assumption $\Lambda t = O(1)$ implies $\Lambda_{\mathrm{II}}t = O(1)$, and the evaluations (1),(11) and (16) induce $\Lambda_0 = O(\lambda_0^2)$, $\pi_0 = O(\lambda_0^2)$. Therefore, we finally conclude that

$$R_0(t)/\exp(-\Lambda t) = 1 + O(\lambda_0^2),$$

which proves the error evaluation formula (49). Q.E.D.

From Propositions 9~11, we easily get

Proposition 12. "Under the same assumptions as in the previous proposition, we have

(53) $$R(t) = \exp(-\Lambda t) + O(\lambda_0^{\delta_0}) = \exp(-\Lambda^0 t) + O(\lambda_0^{\delta_0}) \quad (t \geq 0)$$

for sufficiently small λ_0 and $\Lambda t = O(1)$, where $\Lambda = \Lambda^0\{1+O(\lambda_0)\}$, and Λ^0 is the main term of Λ obtained by neglecting its terms of higher orders of λ_0."

5. An Example

Let us consider the so-called bridge structure shown in Fig. 5, where $c_1 = 55, c_2 = 60, c_3 = 60, c_4 = 55, c_5 = 40$. The minimal cuts of the system as the usual 0-1 system are $B_1 = \{1,2\}, B_2 = \{3,4\}, B_3 = \{1,4,5\}, B_4 = \{2,3,5\}$. The demand function is given as

$$\phi(s) = \begin{cases} 30 & (0 < s \le 8), \\ 55 & (8 < s \le 18), \\ 40 & (18 < s \le 24) \end{cases}$$

and its period is $T = 24$. Therefore, $\phi(t) = \phi(s + c_t T) = \phi(s)$, where $c_t = [t/T]$. Let the flow $\Phi(X(t))$ of the system be defined by the usual max-flow min-cut principle as follows:

$$\Phi(X(t)) = \min_{j=1,2,3,4} \sum_{i \in B_j} c_i X_i.$$

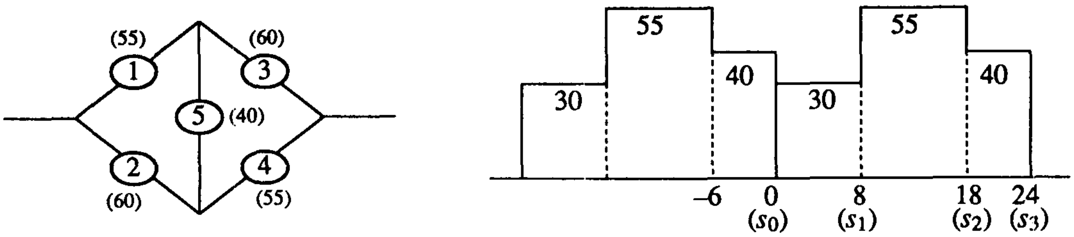


Fig. 5 The bridge structure and the demand function $\phi(t)$

Let us agree that the model described above satisfies Assumptions 1° $\sim$ 7°. Therefore, $\Phi_0 = \Phi(0,0,*,*,*) = \Phi(*,*,0,0,*) = \Phi(0,*,*,0,0) = \Phi(*,0,0,*,0) = 0$,

$$\Phi_1 = \Phi(0,1,1,0,1) = \sum_{i \in B_3} c_i X_i = \Phi(1,0,0,1,1) = \sum_{i \in B_4} c_i X_i = 40,$$

$$\Phi_2 = \Phi(1,0,1,*,*) = \sum_{i \in B_1} c_i X_i = \Phi(*,1,0,1,*) = \sum_{i \in B_2} c_i X_i = 55, \cdots,$$

where the units denoted by $*$ may be 0 or 1. Note here that we can neglect the system failures induced by failures of three or more units, since they have probabilities of $O(\lambda_0^3)$. Thus, the class-I-failures of the system in the time-interval $(s, s+ds) \subset (0,8]$ or $(s, s+ds) \subset (18,24]$ are induced by the four cases:

$$D(s) = \{1\} \subset D(s+ds) = \{1,2\},$$
$$D(s) = \{2\} \subset D(s+ds) = \{1,2\},$$
$$D(s) = \{3\} \subset D(s+ds) = \{3,4\},$$
$$D(s) = \{4\} \subset D(s+ds) = \{3,4\},$$

neglecting the ones with $|D(s+ds)| \ge 3$, where $D(s) = \{i \mid X_i(s) = 0, i \in N\}$. The class-I-failures of the system in the interval $(s, s+ds) \subset (8,18]$ are induced not only by the four cases given above but also by

$$D(s) = \{1\} \subset D(s+ds) = \{1,4\},$$
$$D(s) = \{4\} \subset D(s+ds) = \{1,4\},$$
$$D(s) = \{2\} \subset D(s+ds) = \{2,3\},$$
$$D(s) = \{3\} \subset D(s+ds) = \{2,3\},$$

neglecting the cases with $|D(s+ds)| \geq 3$. Therefore, we have

$$\Lambda_0 := \frac{1}{\sigma_0}(\lambda_1\rho_2 + \lambda_2\rho_1 + \lambda_3\rho_4 + \lambda_4\rho_3),$$
$$\Lambda_1 := \Lambda_0 + \frac{1}{\sigma_0}(\lambda_1\rho_4 + \lambda_4\rho_1 + \lambda_2\rho_3 + \lambda_3\rho_2),$$
$$\Lambda_2 := \Lambda_0.$$

The class-II-failures of the system is induced only when the flow level $\Phi_1 = 40$ is preserved during the time interval $(s,8]$ for $s \in (-6,8)$, where $\Phi(X(s)) > 40 = \Phi(X(u))$ for all $u \in (s+ds, 8] : K = \{1\}$ $(\phi(s_1) < \phi(s_1+0)$ with $s_1 = 8$, $\phi(s_i) > \phi(s_i+0)(i=0,2))$. $L_1 = \{1\}$ from (4)$(\Phi(X(8)) = \Phi(X(8)) = \Phi_1 = 40 > \phi(8) = 30, \Phi(X(8+)) = \Phi_1 = 40 < \phi(8+) = 55)$, and $\tau_1^1 = (24-18)+8 = 14$ from (5) and Fig. 3. In this case, $D(8) = D(8+) = \{1,4\},\{2,3\}$: That is, $D_0^1 = \{1,4\}$ and $\{2,3\}$. From Formula (16), using (12), we obviously get the asymptotic class-II-failure probability

$$\Pi_{\rm II} = \sum_{k\in K}\sum_{\ell\in L_k} \pi_k^\ell = \pi_1^1$$
$$= H_{14}(14) + H_{23}(14),$$

where

$$H_{14}(14) = \frac{\rho_1\rho_4}{\sigma_0}[1 - \mu_1\int_{14}^{\infty}\overline{G}_1(u)\,du\mu_4\int_{14}^{\infty}\overline{G}_4(v)\,dv],$$
$$H_{23}(14) = \frac{\rho_2\rho_3}{\sigma_0}[1 - \mu_2\int_{14}^{\infty}\overline{G}_2(u)\,du\mu_3\int_{14}^{\infty}\overline{G}_3(v)\,dv].$$

Finally, we obtain from Proposition 12 the asymptotic averaged system-failure rate

$$\Lambda = \frac{1}{24}[8\Lambda_0 + 10\Lambda_1 + 6\Lambda_0 + \pi_1]$$
$$= \frac{1}{24}[24(\lambda_1\rho_2 + \lambda_2\rho_1 + \lambda_3\rho_4 + \lambda_4\rho_3)/\sigma_0$$
$$+ 10(\lambda_1\rho_4 + \lambda_4\rho_1 + \lambda_2\rho_3 + \lambda_3\rho_2)/\sigma_0 + H_{14}(14) + H_{23}(14)]$$

and the exponential approximation for the flow-reliability of the system

$$R(t) = \exp(-\Lambda t) + O(\lambda_0^{\delta_0}).$$

6. Conclusive Remarks

Remark 13. The asymptotic micro-scopic (or local) characterization of the system failures, i.e., Λ_i, π_k and $R_0(t)$ given by (1), (16) and (45) respectively, may be effective for some cases of practical applications. However, in most cases the macro-scopic (or overall) evaluation of the system failures, i.e., Λ and $\exp(-\Lambda t)$ given by (14) and (53) respectively may certainly be useful, because they are very simple and clear-cut.

Remark 14. Note that in Propositions 11 and 12 the periodicity of the demand function introduced in Assumption 5° makes an essential role to derive the exponential reliability functions in (48) and (53). If Assumption 5° is deleted and if the asymptotic class-I-failure rate Λ_k is evaluated for $t \in (t_k, t_{k+1})$ just as in Formula (1) and if the asymptotic class-II-failure probability π_k is calculated just as in Formula (16), then, the system reliability under the given non-periodic stepwise demand function $\phi(t)$ may surely be given as

$$R(t) = \prod_{i=1}^{k}(1-\pi_i)^{\delta_i(K,t)}\exp[-\sum_{j=0}^{k-1}\Lambda_j(t_{j+1}-t_j) - \Lambda_k(t-t_k)] + O(\lambda_0^{\delta_0}),$$

where $0 = t_0 < t_1 < t_2 < \cdots$ and $K = \{k \mid \phi(t_k) < \phi(t_k + 0), k = 0, 1, 2, \cdots\}$.

Remark also that we can analyze the case where there exist two or more patterns of demand levels and where one of them chosen stochastically at time cT is applied to the system during the time-interval $(cT, (c+1)T]$ $(c = 0, 1, \cdots)$. See [4].

It should be noted here that cases with continuous demand curves can easily be reduced to ones with discrete demand levels, since the number of possible flow-levels of the system is finite. Using this fact, Nahman and Graovac have shown a method for evaluating a kind of averaged failure intensity of power systems [5], although their model and method are quite different from the ones in the present paper.

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Appendix : Proof of (34)

First let us define

(A.1) $Q_{k\beta}(dt \mid \overline{u}_\alpha(s+))$

= the conditional probability that a class-I-failure of the system occurs in the interval $(t, t+dt) \subset (t_k, t_{k+1})$ via $[I_{k\beta}, C_{k\beta}, D_{k\beta}]$, given that the age vector at time $s+$ is $\overline{u}_\alpha(s+)$

$= \Lambda_{k\beta} dt + \psi_{k\beta}(dt \mid \overline{u}_\alpha(s+))$,

where $0 < s < t$; $\beta = 1, 2, \cdots, \nu^k$; $k = 0, 1, \cdots, d_0$, $\Lambda_{k\beta} = (1/\sigma_0)\lambda_{i_{k\beta}} \prod_{j \in D_{k\beta}} \rho_j$, and $\overline{u}_\alpha(s+)$ is precisely defined in the first paragraph of Section 4. Then, we obviously have

$$\text{(A.2)} \qquad \begin{aligned} &Q_k(dt \mid \overline{u}_\alpha(s+)) \\ &\equiv \sum_{\beta=1}^{\nu^k} Q_{k\beta}(dt \mid \overline{u}_\alpha(s+)) \\ &= \Lambda_k dt + \psi_k(dt \mid \overline{u}_\alpha(s+)) \end{aligned}$$

for every $t \in (t_k, t_{k+1})$ with $t_k = s_k + cT$, where Λ_k is the stationary class-I-failure rate of the system given in (1), and $\psi_k(dt \mid \overline{u}_\alpha(s+)) = \sum_{\beta=1}^{\nu^k} \psi_{k\beta}(dt \mid \overline{u}_\alpha(s+))$ is the deviation of the conditional probability (A.2) from the stationary value $\Lambda_k dt$. In case of $\alpha = s = 0$ Assumption 4° implies $\overline{u}_0(0+) = \overline{0}$, and one gets

$$\text{(A.3)} \qquad \begin{aligned} &\sum_{c=0}^{m}\sum_{k=0}^{d_0}\int_{s_k+cT}^{s_{k+1}+cT} |\psi_k(dt \mid \overline{u}_0(0+))| \\ &\leq \sum_{c=0}^{m}\sum_{k=0}^{d_0}\int_{s_k+cT}^{s_{k+1}+cT} \sum_{\beta=1}^{\nu_k} |\psi_{k\beta}(dt \mid \overline{u}_0(0+))| \\ &\leq \sum_{k=0}^{d_0}\sum_{\beta=1}^{\nu^k}\int_0^\infty |\psi_{k\beta}(dt \mid \overline{u}_0(0+))| \\ &= O(\lambda_0^{\delta_0}) \qquad (0 \leq m < \infty) \end{aligned}$$

which can be proved just as (6.7.84) in [2, p. 306]. Similarly, putting $\psi_k(dt \mid \overline{u}_\alpha(s+)) = 0$ for $s > t$,

$$\text{(A.4)} \qquad \sum_{c=0}^{\infty}\sum_{k=0}^{d_0}\int_{s_k+cT}^{s_{k+1}+cT} |\psi_k(dt \mid \overline{u}_\alpha(s+))| = O(\lambda_0^{\delta_0})$$

is obtained just as (A.3) given above.

Second, let us define

(A.5) $Q^\ell_{k\beta}(dx, d\overline{v} \mid \overline{u}_\alpha(s+))$

= the conditional probability that $\Phi(X) = \Phi_\ell(\ell \in L_k)$ is attained in the interval $(x, x+dx)$ via $[I^\ell_\beta, C^\ell_\beta, D^\ell_\beta]$ and the age vector $\overline{u}(x) \in (\overline{v}, \overline{v}+d\overline{v})$ at time x, given that age vector at time $s+$ is $\overline{u}_\alpha(s+)$

$$= \frac{dx}{\sigma_0}\lambda_h f_h(v_h)dv_h \prod_{i\in C^\ell_\beta} \lambda_i \overline{F}_i(v_i)dv_i \prod_{j\in D^\ell_\beta} \lambda_j \overline{G}_j(v_j)dv_j + \psi^\ell_{k\beta}(dx, d\overline{v} \mid \overline{u}_\alpha(s+))$$

$(h = i^\ell_\beta \in I^\ell_\beta)$,

where $0 \leq s < x < t$, and $[I^\ell_\beta, C^\ell_\beta, D^\ell_\beta]$ is defined in (6). Furthermore, define

(A.6) $Q^\ell_{k\beta}(\,\cdot \mid \overline{u}_\alpha(s+))$

= the conditional probability that the flow level Φ_ℓ $(\ell \in L_k)$ is attained in the interval (s^ℓ_k, t_k) and it is preserved up to time t_k without changing the state vector and that a class-II-failure of the system via $[I^\ell_\beta, C^\ell_\beta, D^\ell_\beta]$ occurs at t_k+, given that the age vector at time $s+$ is $\overline{u}_\alpha(s+)$,

where $0 \leq s < t_k, t_k = s_k + cT, s^\ell_k = \max\{s, t_k - \tau^\ell_k\}$ and $[I^\ell_\beta, C^\ell_\beta, D^\ell_\beta]$ is defined in (6). Now, noticing

$$\begin{aligned} Q^\ell_{k\beta}(\,\cdot \mid \overline{u}_\alpha(s+)) = \int_{x=s^\ell_k}^{t_k} \{ \int_{\overline{v}} Q^\ell_{k\beta}(dx, d\overline{v} \mid \overline{u}_\alpha(s+)) \overline{G}_h(t_k - x) \\ \times \prod_{i\in C^\ell_\beta} \overline{F}_i(t_k - x \mid v_i) \prod_{j\in D^\ell_\beta} \overline{G}_j(t_k - x \mid v_j)\} \quad (h \equiv i^\ell_\beta \in I^\ell_\beta), \end{aligned}$$

and inserting (A.5) to the right side of the above, we get

$$Q^{\ell}_{k\beta}(\,\cdot\,\mid \overline{u}_\alpha(s+)) = \pi^{\ell 0}_{k\beta} + \psi^{\ell}_{k\beta}(\,\cdot\,\mid \overline{u}_\alpha(s+)),$$

where, neglecting higher order terms of λ_0 and the case of $s > t_k - \tau^{\ell}_k$ from Assumption 6,

$$\pi^{\ell 0}_{k\beta} = \frac{1}{\sigma_0}\lambda_h \int_0^{\tau^{\ell}_k} \overline{G}_h(\tau)\{\prod_{i\in C^{\ell}_\beta} \lambda_i \int_\tau^\infty \overline{F}_i(v_i)dv_i \prod_{j\in D^{\ell}_\beta} \lambda_j \int_\tau^\infty \overline{G}_j(v_j)dv_j\}d\tau,$$

$$\text{(A.7)} \qquad |\psi^{\ell}_{k\beta}(\,\cdot\,\mid \overline{u}_\alpha(s+))| \le \int_{x=s^{\ell}_k}^{t_k} \int_{\overline{v}} |\psi^{\ell}_{k\beta}(dx, d\overline{v} \mid \overline{u}_\alpha(s+))|.$$

Obviously, for sufficiently small λ_0

$$\text{(A.8)} \qquad \pi^{\ell}_k = \sum_{\beta=1}^{\nu^{\ell}} \pi^{\ell 0}_{k\beta}$$

is given by (11), and owing to (A.7)

$$\begin{aligned}\text{(A.9)} \qquad & \sum_{c=0}^{m}\sum_{k=0}^{d_0}\sum_{\ell\in L_k}\sum_{\beta=1}^{\nu_\ell} \delta_k(K,s)|\psi^{\ell}_{k\beta}(\,\cdot\,\mid \overline{u}_\alpha(s+))| \\ & \le \sum_{c=0}^{\infty}\sum_{k=0}^{d_0}\sum_{\ell\in L_k}\sum_{\beta=1}^{\nu^{\ell}} \int_{x=s^{\ell}_k}^{t_k}\int_{\overline{v}} |\psi^{\ell}_{k\beta}(dx, d\overline{v}|\overline{u}_\alpha(s+))|\overline{\delta}_k(K,s) \\ & \le \sum_{k\in K}\sum_{\ell\in L_k}\sum_{\beta=1}^{\nu^{\ell}} \int_{x=s+}^{\infty}\int_{\overline{v}} |\psi^{\ell}_{k\beta}(dx, d\overline{v} \mid \overline{u}_\alpha(s+))| \\ & = O(\lambda_0^{\delta_0}) \qquad (\overline{\delta}_k(K,s) = 1 - \delta_k(K,s))\end{aligned}$$

is assured just as (6.7.84) in [2, p. 306].

Finally, (A.3), (A.4) and (A.9) establish the validity of (34), since the left sides of (34) are obviously dominated by the sum of the left sides of (A.4) and (A.9), or, of (A.3) and (A.4) with $\alpha = s = 0$.

Shun-ichi Abe
Department of Industrial
& Systems Engineering
Aoyama-Gakuin University
6-16-1, Chitosedai, Setagaya-ku,
Tokyo 157
Japan