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THE IMPORTANCE OF AN ITEM IN A MULTISTATE SYSTEM

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Abstract We define the utility of the specified up state of the system, $\sum X_i$, consisting of n independent multistate items. A measure of importance of an item and that of a given state of an item are discussed with respect to the expected utility function. It is shown that, for non-decreasing utility functions, the perfect state of an item yields maximum contribution to the expected utility function. However, such a choice of a state is not obvious when the utility function is non-monotonic. In this case, we propose the use of the linear programming technique to decide the availabilities of states of an item so that its contribution to the expected utility of the system is maximum. Further, we derive sufficient conditions to compare the overall and the state-wise impact of any two items on the expected utility of the system. A numerical example is given to illustrate the procedure.

1. Introduction

To begin the discussion on multistate monotone systems (MMS), let, the MMS have n items, X_i be the state of the i-th item taking values in {0, ..., M} and $\Phi(\underset{\sim}{X}) = \{(X_1,...,X_n)\}$.

Since Barlow and Wu[1] extended the theory of binary coherent structures (BCS) to MMS there have been several papers on the probabilistic aspects of MMS. For details see El-Neweihi and Proschan [2] and Natvig [6]. Also, the concept of the relevancy of an item to the BCS has been extended in various ways, for example, see El-Neweihi and Proschan [3] and Ohi and Nishida [8].

Common assumptions, in the literature on MMS, are

(i) $\Phi : \{0, ..., M\}^n \rightarrow \{0,...,M\}$,

where $\{0, ..., M\}^n$ is the Cartesian product of {0, ..., M} with itself n times.

(ii) $\min_{1 \le i \le n} X_i \le \Phi(\underset{\sim}{X}) \le \max_{1 \le i \le n} X_i$,

(iii) 0 : down state, M : perfect state, and

(iv) i : an up-state better than (i-1) and worse than (i+1) i = 1, ..., M-1.

Above restrictions do not permit the study of structure functions of the type $\Phi(\underset{\sim}{X}) = \sum_{i=1}^{n} X_i$. However, ΣX_i is a reasonable structure function while studying many real life situations. In fact, following Natvig [7], the $\Phi(\underset{\sim}{X})$ defined as above is the maximum flow through a parallel system with capacities X_i, i=1,...,n. For example, if there is a regionwide grid of n power generating stations and X_i MW is the supply capacity of the i-th station at a given time point then ΣX_i MW is the total capacity of the grid at that point. Note that ΣX_i is a surjective function from $\{0,\ldots,M\}^n$ to $\{0,\ldots, nM\}$.

A question, not relevant in the studies of BCS but of importance in the studies of MMS is the utility of a specified up-state to the system. Griffith [4], [5] has studied the MMS of binary items assuming that the utility function is monotonic non-decreasing. However, the utility of the i-th up-state would depend mainly upon the difference between the cost to run the system and the revenue from the system in that state. Hence, in general the utility function need not be a monotone or a linear function of the states of the system.

In the utility-studies of MMS composed of multi-state items it is essential to know the contribution of each item or of the specified state of a given item, to the expected utility of the system.

In this paper, we study the above questions pertaining to the structure function ΣX_i. Preliminaries are given in section 2 and results are summarized in section 3. Finally the paper is concluded with a few remarks.

2. Preliminaries

In addition to the symbols defined above we use the following notations :

$a_{ij} = P[X_i = j]$: the probability that the i-th component occupies state j, i=1,..., n, j = 0,...,M.

$A_i = (a_{io}, a_{i1}, \ldots, a_{iM})$: the availability vector of the i-th item

$U(\ell)$: utility of the system in state ℓ, $\ell = 0, \ldots, nM$, $U(0) = 0$.

$b(\ell)$: $U(\ell) - U(\ell-1)$,

Δ : $(\delta_o, \ldots, \delta_M)$, a vector in the M+1 dimensional Euclidean space, $\mathbb{R}^{M+1}$

$P\Delta$: $(\delta_o^*, \ldots, \delta_M^*)$ a permutation of a given vector Δ.

A_i^* : $(a_{io} + \delta_o^*, a_{i1} + \delta_1^*, \ldots, a_{iM} + \delta_M^*)$

$|S|$: Cardinality of set S.

Definition 2.1 : A vector $\Delta \in \mathbb{R}^{M+1}$ is a feasible vector for a change in A_i (hence further abbreviated FV-u) if there exists $P\Delta$ such that A_i^* is also an availability vector, i.e.

$$\sum_{j=0}^{M} (a_{ij} + \delta_j^*) = 1 \quad \text{and} \quad 0 \le a_{ij} + \delta_j^* \le 1$$

for $j = 0, 1, \ldots, M$.

Definition 2.2 : Θ_i, the set of all FV-i's is called the feasible set for the changes in A_i.

Remark 2.1 : Sufficient conditions for $\Delta \in \Theta_i$ are

(2.1) $$-\min_k (a_{ik}) \le \min_k \delta_k < \max_k \delta_k \le \min_k (1-a_{ik})$$

(2.2) $$\text{and} \quad \sum_0^M \delta_k = 0 .$$

Further, (2.1) implies that Θ_i is a non-empty set for a given A_i if $0 < a_{ij} < 1$, for $j = 0,\ldots,M$.

If $a_{ij} = 1$ (or 0) for some j then Δ obtained using condition (2.1) does not satisfy (2.2). However, an FV-i may be constructed by decreasing (increasing) a_{ij} and making suitable changes in rest of the a_{ik}s'.

3. Results

We, first, separate out the contribution of the i-th item to the expected utility of the system.

Proposition 3.1 : If ΣX_i is the MMS of n independent multistate items with availability vectors $A_1,\ldots,A_n$, then the expected utility of the system is

(3.1) $$E[U(\sum_{i=1}^{n} X_i)]$$

$$= \xi_i + \sum_{\ell=1}^{(n-1)M} U(\ell)\, P[(\sum_{j\neq i} X_j) = \ell]$$

where

$$(3.2)\qquad \xi_i = \sum_{k=1}^{M} a_{ik}\, \psi_{ik} ,$$

$$\psi_{io} = 0, \text{ and } \psi_{ik}=\psi_{ik+1} + \sum_{j=k}^{(n-1)M+k} b(j)P[\sum_{\ell\neq i} X_\ell = j-k],$$

$$k = 1, \ldots, M.$$

<u>Proof</u> : We proceed as in Griffith (1980),

$$E[U(\sum_{i=1}^{n} X_i)] = \sum_{j=1}^{nM} b(j)\ P[\sum_{\ell=1}^{n} X_\ell \geq j]$$

$$= \sum_{j=1}^{nM} b(j)\ \{ \sum_{k=0}^{M} P[(\sum_{\ell\neq i} X_\ell) \geq j-k,\ X_i = k]\}$$

$$= \sum_{j=1}^{(n-1)M} b(j)\ P[\sum_{\ell\neq i} X_\ell \geq j]$$

$$+ \sum_{j=1}^{nM} b(j)\ (\sum_{k=1}^{M} P[X_i=k]\{P[\sum_{\ell\neq i} X_\ell \geq j-k]-P[\sum_{\ell\neq i} X_\ell\geq j]\})$$

$$= \sum_{j=1}^{(n-1)M} b(j)P[\sum_{\ell\neq i} X_\ell\geq j]$$

$$+\sum_{k=1}^{M} P[X_i=k]\ (\sum_{j=1}^{(n-1)M+k} b(j)P[j-k \leq \sum_{\ell\neq i} X_\ell \leq j-1]\ )$$

$$= \sum_{j=1}^{(n-1)M} U(j)\ P[\sum_{\ell\neq i} X_\ell = j] + \sum_{k=1}^{M} a_{ik}\, \psi_{ik}$$

where

$$\psi_{ik} = \sum_{j=1}^{(n-1)M+k} b(j)\ P[j-k \leq \sum_{\ell\neq i} X_\ell \leq j-1]$$

Note that $\psi_{io} = 0$, and then,

$$\psi_{ik} = \psi_{ik-1} + \sum_{j=1}^{(n-1)M+k} b(j)\ P[\sum_{\ell \neq i} X_\ell = j-k]. \qquad \square$$

The second term on the R.H.S. of (3.1) is the joint contribution of the (n-1) items, except the i-th item, to the expected utility of the system. Hence ξ_i may be viewed as the contribution of the i-th item to the expected utility of the system. Further, we interprete ψ_{ik} as the utility of the k-th state of the i-th item.

If the utility of the system has to increase through the i-th item, efforts should be made to maintain the i-th item in the M-th (perfect) state which will yield maximum contribution of the i-th item to the expected utility of the system.

In the following discussion by an optimal FV-i we mean an FV-i which yields the maximum contribution of the i-th item to the expected utility of the system.

However, in general U(.) is non-monotonic and the choice of an optimal FV-i is not obvious. In the following proposition we propose that a linear programming technique may be employed to obtain an optimal FV-i.

Proposition 3.2 : For the MMS discussed in Proposition 3.1, the optimal FV-i ∇, for the availability vector A_i is a solution to the linear programming problem.

$$\text{Max} : x_o = \sum_{k=o}^{M} \psi_{ik}\ \delta_k$$

$$\text{Subject to} \quad - a_{ik} \le \delta_k \le 1 - a_{ik},\ k = 1,\dots,M,$$

$$\text{and} \quad \sum_{k=0}^{M} \delta_k = 0\ . \qquad \square$$

Since ξ_i is proposed as a measure of contribution of the i-th item to the utility of the system, $\xi_i - \xi_\ell$ is proposed as a measure to study the importance of the i-th item over the ℓ-th item for the given utility function U. It can be shown that (see the Appendix)

$$(3.3) \qquad \xi_i - \xi_\ell = \sum_{k=1}^{M} \psi_k^{i\ell}\ \alpha_k^{i\ell}$$

where

$$\alpha_k^{i\ell} = a_{ik} - a_{\ell k} \quad \text{and}$$

$$\psi_k^{i\ell} = \sum_{j=1}^{(n-2)M+k} b(j)\, P[j-k \le \sum_{p\ne\ell,i} x_p \le j-1].$$

$(\xi_i - \xi_\ell)$ can be expressed in terms of the survival functions of the i-th and the ℓ-th items. In that case

$$(3.4)\quad \xi_i-\xi_\ell = \sum_{k=1}^{M} \{ \sum_{j=k}^{(n-2)M+k} b(j)P[\sum_{p\ne i,\ell} x_p = j-k]\} (\rho^i_{k-1} - \rho^\ell_{k-1})$$

where $\rho^j_{k-1} = \sum_{k=1}^{M} a_{jp}$.

Below we state a sufficient condition for $\xi_i > \xi_\ell$.

Let $\underset{\sim}{\alpha}^{i\ell}$ and $\underset{\sim}{\psi}^{i\ell}$ be the vectors in $\mathbb{R}^M$ having elements $\alpha_k^{i\ell}$ and $\psi_k^{i\ell}$, respectively. Let $\underset{\sim}{\alpha}^*$ be a rearrangement of $\underset{\sim}{\alpha}^{i\ell}$ such that $\alpha^*_1, \ldots, \alpha^*_p$ are negative and $\alpha^*_{p+1}, \ldots, \alpha^*_M$ are non-negative, and $\underset{\sim}{\psi}^* = (\psi^*_1, \ldots, \psi^*_M)$ be the rearrangement of $\underset{\sim}{\psi}^{i\ell}$ under the same permutation as in $\underset{\sim}{\alpha}^*$.

Further, let $K^+ = \{i \mid \psi^*_i > 0\}$, $K^0 = \{i \mid \psi^*_i = 0\}$, $K^- = \{\{1,\ldots,M\} - K^+ - K^0\}$ and $R = \{i \mid \alpha^*_i = 0\}$.

Define $Q = \{\{1,\ldots,p\} \cap K^+\} \cup \{\{p+1,\ldots,M\} \cap \{K^- - R\}\}$ and $P = \{1, \ldots, M\} - Q - K^0 - R$,

$$\text{i.e.}\quad \psi^*_k \alpha^*_k \begin{cases} > 0 & \text{if } k \in P \\ < 0 & \text{if } k \in Q \\ = 0 & \text{if } k \in K^0 \cup R \end{cases}$$

It may be noted that P, Q, K^0 are index sets. Further, if the $|\sum_Q \alpha^*_j \psi^*_j| < \sum_P \alpha^*_j \psi^*_j$ then $\xi_i > \xi_\ell$, i.e. the i-th item contributes more than the ℓ-th item to the expected utility of the system. Following two propositions give sufficient conditions for the comparison.

Proposition 3.3 : For the MMS discused in Proposition 3.1, the i-th (ℓ-th) item contributes more than the ℓ-th (i-th) item to the expected utility of the system if Q(P) is an empty set and $K^0 \cup R$ is proper subset of {1,...,M}.

Proof : Since Q is empty and $K^0 \cup R$ is a proper subset of {1,...,M}. $\psi_i^* \alpha_i^* \geq 0 \ \forall$ i with inequality for at least one value of i.

Hence $\sum_{i=1}^{M} \psi_i^* \alpha_i^* > 0$,

$\Rightarrow \xi_i > \xi_\ell$ from (3.3).

Similarly, if P is empty and $K^0 \cup R$ is proper subset of {1,...,M},

$\xi_\ell > \xi_i$ □

Proposition 3.4 : For the MMS discussed in Proposition 3.1, if P and Q are non empty sets then a sufficient condition for $\xi_i \geq \xi_\ell$ is

$$(3.5) \qquad |Q| \max_Q |\alpha_i^*| \leq \frac{\underline{\psi}}{\bar{\psi}} \,||P| \min_P |\alpha_j^*| ,$$

where

$$\underline{\psi} = \min_P \{|\psi_j^*|\} ,$$

and

$$\bar{\psi} = \max_Q \{ |\psi_j^*|\}.$$

Proof : Let P and Q be nonempty. Let (3.5) be true, then

$$|Q| \max_Q |\alpha_i^*| \,.\, \bar{\psi}$$
$$\geq \sum_{j\in Q} |\alpha_j^*| \, |\psi_j^*| \geq | \sum_{j\in Q} \alpha_j^* \psi_j^* | \,.$$

Also

$$|P| \,.\, \underline{\psi} \min_P |\alpha_j^*| \leq \sum_{j\in P} |\psi_j^*| \, |\alpha_j^*|$$
$$= \sum_{j\in P} |\psi_j^* \alpha_j^*| = \sum_{j\in P} \psi_j^* \alpha_j^* \,.$$

Hence

$$\xi_i - \xi_\ell = \sum_{j=1}^{M} \alpha_j^* \psi_j^* = \sum_{j\in P} \alpha_j^* \psi_j^* - | \sum_{j\in Q} \alpha_j^* \psi_j^*| \geq 0. \qquad \square$$

In fact, for a linear utility function U(.) the following lemma shows that the importance of the i-th component depends only upon the 'average state' of the component.

Lemma 3.1 : For the MMS proposed in Proposition 3.1, if U(.) is a linear function with slope b then $\xi_i - \xi_\ell = b(EX_i - EX_\ell)$.

Proof : From equation (3.4)

$$\xi_i - \xi_\ell = b \sum_{k=1}^{M} (\rho^i_{k-1} - \rho^\ell_{k-1})$$

$$= b(EX_i - EX_\ell). \qquad \square$$

Further, note that

$$(3.6) \quad \psi_{ik}-\psi_{\ell k} = \sum_{m=1}^{M} \alpha^{\ell i}_m \Big\{ \sum_{j=(n-2)M+1}^{(n-2)M+m} (b(j)-b(j+k)) + \sum_{j=1}^{(n-2)M} (b(j)-b(j+m+1)-b(j+k+1)+b(j+m+k+1)) \cdot P[\sum_{p\neq i,\ell} X_p \leq j-1] \Big\}$$

Since, for the linear U(.) R.H.S. of (3.6) would be zero,

$$\psi_{ik} = \psi_{\ell k} \quad \text{for } k = 1, \ldots, m.$$

It may be noted that (3.6) can be writen alternatively, as :

$$(3.7) \quad \psi_{ik} - \psi_{\ell k} = [\psi_{ik-1}-\psi_{\ell k-1}] - \sum_{m=0}^{M} \alpha^{i\ell}_m \Theta^{i\ell}_m(k)$$

where

$$\Theta^{i\ell}_m(k) = \sum_{j=0}^{(n-2)M} b(j+m+k)\ P[\sum_{p\neq i,\ell} x_p = j].$$

Hence, it is possible to discuss the importance of the i-th and the ℓ-th items state-wise, for the given utility function. We consider few special cases of the utility function.

In view of (3.6) and (3.7) if either U(.) is increasing function or U(.) is a concave function with non-decreasing second differences and $\alpha^{i\ell}_m \geq 0$ for $m \geq 1$ then $\psi_{ik} \geq \psi_{\ell k}$ for each k. If the items are binary taking values 0 and M then this result agrees with that of Griffith ([4], p. 744).

However, if $\underset{\sim}{\alpha}^{i\ell}$ has positive as well as negative elements then we obtain a sufficient condition for $\psi_{ik} \geq \psi_{\ell k}$, as in Propositions 3.3 and 3.4. In the following discussion U(.) is an increasing convex function. As a consequence, $\Theta_k^{i\ell}(m)$ are non-negative and increasing in m.

Let $\underset{\sim}{\alpha}^*$ be a permutation of $\underset{\sim}{\alpha}^{i\ell}$ such that $\alpha_j^* < 0$ for $j = \{1,...,p\}$ and $\alpha_j^* \geq 0$ for $j \in \{P+1,...,m\}$. Let Θ^* be the corresponding rearrangement of $\underset{\sim}{\Theta}^{i\ell}(k)$ $(=(\Theta_1^{i\ell}(k), \ ..., \ \Theta_m^{i\ell}(k))$. Now $\Theta_m^{i\ell}(k) \geq 0$ for m in $\{1,...,M\}$. Hence $Q = \{1,...,p\}$ and $P = \{p+1,...,M\}$, where Q and P are as defined earlier. Then

Proposition 3.5 : For the MMS discussed in Proposition (3.1) a sufficient condition for $\psi_{ik} \geq \psi_{\ell k}$ is

$$p \min_{1\leq i \leq p} |\alpha_i^*| \geq \frac{\bar{\psi}}{\underline{\psi}} (M-P) \max_{p+1 \leq i \leq M} |\alpha_i^*| \text{, where}$$

$$\bar{\psi} = \max_{p+1 \leq j \leq M} \Theta_j^*(k) \quad \text{and} \quad \underline{\psi} = \min_{1 \leq j \leq p} \Theta_j^*(k) \ .$$

Proof : Similar to Proposition 3.4. □

To summarize, if the i-th item is viewed as a binary item with states k and $\bar{0}$ (= {0,...,k-1,k+1,...,M}), then ψ_{ik} can be used as a measure of importance of the k-th state of the i-th item.

Concluding Remarks

As mentioned in the introduction, this paper deals with the maximum flow in a parallel network and a given utility function. Proposition 3.2 and 3.4 hold for general utility functions. However, in proposition 3.5, we need U(.) to be non-decreasing convex function. For maintenance, priorities to the items should be given according to the orders of $\sum_{k=1}^{m} \psi_{ik} a_{ik}^*$ where a_{ik}^*'s are the availabilities for the i-th item obtained from Proposition 3.2, if it is possible to change A_1 to A_1^*. Such a change may be possible by using redundent items.

However, when changes in A_1 are not possible then the ordering of ξ_is' should help to decide the maintenance schedule of the items.

Below we give an illustrative simple example.

Example : Let n = 3 and M = 2, i.e. we have an MMS with 3 items with {0,1,2} being the states of an item. Then for the structure function $\sum_{i=1}^{3} X_i$, the states of the system are {0,1,2,3,4,5,6}.

Let the utility function U(.) be defined as follows

$$U(j) = \begin{cases} 2j, & j = 0,1,2, \\ 13-2j, & j = 3,4,5,6, \end{cases}$$

and the availability vectors be :

$A_1 = (0.2,\ 0.2,\ 0.6)$, $A_2 = (0.6,\ 0.2,\ 0.2)$, $A_3=(0.2,0.4,0.4)$.

Table 1 aids us in the computation of ξ_1 and the quantities in Proposition 3.3 and 3.4.

Note that in this case

$$\xi_1 = \sum_{k=1}^{2} a_{1k}\ \psi_{1k}\ .$$

Table 1

j	U(j)	b(j)	$P[X_2+X_3=j-1]$	$P[X_2+X_3=j-2]$	b(j). $P[X_2+X_3=j-1]$	b(j). $P[X_2+X_3=j-2]$
0	0	0	0	0	0	0
1	2	2	0.12	0	0.24	0
2	4	2	0.28	0.12	0.56	0.24
3	7	3	0.36	0.28	1.08	0.84
4	5	-2	0.16	0.36	-0.32	-0.72
5	3	-2	0.08	0.16	-0.16	-0.32
6	1	-2		0.08		-0.16
Total					1.40	-0.12

Now $\psi_{11} = \psi_{10} + \sum_{j=1}^{5} b(j)\ P[X_2+X_3 = j-1]$

$= 0 + 1.40 = 1.40$

$\psi_{12} = \psi_{11} + \sum_{j=2}^{6} b(j)\ P[X_2+X_3 = j-2]$

$= 1.40 + (-0.12) = 1.28$

and $\xi_1 = 1.40 \times 0.2 + 1.28 \times 0.6 = 0.280 + 0.768 = 1.048.$

To compare items 1 and 3 we need the following :

$\alpha_1^{13} = -0.2 = (a_{11} - a_{31})$, $a_2^{13} = 0.2$

$\psi_1^{13} = \sum_{j=1}^{3} b(j)\ P[X_2 = j-1] = 2 \times 0.6 + 2 \times 0.2 + 3 \times 0.2 = 2.2$

$\psi_2^{13} = \sum_{j=1}^{4} b(j)\ \{P[X_2 = j-2] + P[X_2 = j-1]\}$

$$= 2 \times (0.6) + 2 \times (0.8) + 3 \times (0.4) + (-2) \times (0.2) = 3.6.$$

Then, $\underset{\sim}{\alpha}^* = (-0.2, 0.2)$, i.e. $p = 1$ and $\alpha_1^* = \alpha_1^{13}$ and

$$\underset{\sim}{\psi}^* = (2.2, 3.6).$$

Hence $Q = \{1\}$, $K^+ = \{1,2\}$, K^- = empty set = K^0 and $P = \{2\}$. The left hand side of expression (3.5) is

$$1 \times (0.2) = 0.2$$

whereas the right hand side is

$$((3.6)/(2.2)) \times 1 \times 0.2 ,$$

that is, the inequality in (3.5) holds. Thus the contribution of the first item is more than that of the third item.

The above can also be verified by actually computing ξ_3 and noting that

$$\xi_1 - \xi_3 \ (= 0.28) \ > 0 .$$

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APPENDIX

(Proof of (3.3))

For $k \in \{1, \ldots, M\}$,

$$\psi_{ik} = \sum_{j=1}^{(n-1)M+k} b(j)\, P[j - k \le \sum_{t\ne i} X_t \le j-1]$$

$$= \sum_{j=1}^{(n-1)M+k} b(j)\{P[\sum_{t\ne i} X_t \ge j-k] - P[\sum_{t\ne i} X_t \ge j]\}$$

$$= \sum_{j=k}^{(n-1)M+k} b(j)\{\sum_{s=0}^{M} P[\sum_{t\ne i} X_t \ge j-k,\ X_\ell = s] - \sum_{s=0}^{M} P[\sum_{t\ne i} X_t \ge j,\ X_\ell = s]\}.$$

Using the independence of X_t's and the fact that $a_{\ell 0} = 1 - \sum_{s=1}^{M} a_{\ell s}$,

$$\psi_{ik} = \sum_{j=1}^{(n-1)M+k} b(j)\{(1-\sum_{s=1}^{M} a_{\ell s})(P[\sum_{t\ne i,\ell} X_t \ge j-k] - P[\sum_{t\ne i,\ell} X_t \ge j])$$

$$+ \sum_{s=1}^{M} a_{\ell s}(P[\sum_{t\ne i,\ell} X_t \ge j-k-s] - P[\sum_{t\ne i,\ell} X_t \ge j-s])\}.$$

Now, $\sum_{t\ne i,\ell} X_t$ takes values in $\{0, \ldots, (n-2)M\}$.

Hence

$$\psi_{ik} = \sum_{j=1}^{(n-2)M+k} b(j)\, P[j-k \le \sum_{t\ne i,\ell} X_t \le j-1]$$

$$- \sum_{j=1}^{(n-2)M+k} b(j) \sum_{s=1}^{M} a_{\ell s} \sum_{p=1}^{k} \{P[\sum_{t\ne i,\ell} X_t = j-p]$$

$$- P[\sum_{t\ne i,\ell} X_t = j-s-p]\} \qquad (1)$$

Similarly, $\psi_{\ell k} =$

$$\sum_{j=1}^{(n-2)M+k} b(j)\, P[j-k \le \sum_{t\ne i,\ell} X_t \le j-1]$$

$$- \sum_{j=1}^{(n-2)M+k} b(j) \sum_{s=1}^{M} a_{is} \sum_{p=1}^{k} \{P[\sum_{t\ne i,\ell} X_t = j-p]$$

$$- P[\sum_{t\neq i,\ell} X_t = j-s-p]\}, \; k = 1,\ldots, M \qquad (2)$$

Now,

$$\xi_i - \xi_\ell = \sum_{k=1}^{M} a_{ik}\,\psi_{ik} - \sum_{k=1}^{M} a_{\ell k}\,\psi_{\ell k}\,.$$

Substitution of ψ_{ik}, $\psi_{\ell k}$ from (1) and (2) and cancelation of common terms provides

$$\xi_i - \xi_\ell = \sum_{k=1}^{M} (a_{ik} - a_{\ell k}) \sum_{j=1}^{(n-2)M+k} b(j)P[j-k \le \sum_{t\neq i,\ell} X_t \le j-1].$$

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