Journal of the Operations Research Society of Japan
Vol. 35, No. 1, March 1992

© 1992 The Operations Research Society of Japan

GAME THEORETIC ANALYSIS FOR AN OPTIMAL STOPPING PROBLEM BY MEANS OF MOMENTS OF A DISTRIBUTION FUNCTION

Jun-ichi Nakagami
Chiba University

(Received June 4, 1990; Revised September 6, 1991)

Abstract Let $X_1, X_2, \cdots, X_n, \cdots$ be mutually independent random variables with a common cdf F, which is unknown but belongs to some class $\mathcal{F}$ of cdf's. The class $\mathcal{F} = \mathcal{F}(\mu, \sigma^2, M)$ is the set of all cdf's whose mean, variance and domain are $-\infty < \mu < \infty, 0 < \sigma^2 < \infty$, and $[\mu - M, \mu + M]$ respectively. It is assumed that they are known. Under an observation cost $c, 0 < c < \infty$, we consider a stopping problem $\phi(x, F)$ as a two-person zero-sum game in which the player 1 decides his stopping set $\{X > x\}, x \in R$, and the player 2 chooses her cdf F in $\mathcal{F}$. We analyze the upper bound problem $\phi^u = \sup_{x\in R}\sup_{F\in\mathcal{F}}\phi(x, F)$ and the game problem $\phi^s = \text{value}_{x\in R, F\in\mathcal{F}}\phi(x, F)$ to derive a simple and meaningful solution with the parameters c, μ, σ and M.

1. Introduction

1.1. Preliminaries

Let $X_1, X_2, \cdots, X_n, \cdots$ be mutually independent and identically distributed random variables with a common cdf $F(t) = P\{X \leq t\}$ such that $E[X^+] = \int_{R_+} tdF(t) < \infty$, where $R = (-\infty, \infty)$, $R_+ = [0, \infty)$. A positive observation cost $c (\in R_{++} = (0, \infty))$ is incurred to the observation of each $X_n, n \geq 1$. If the observation process is stopped after X_n is observed, a reward $X_n - nc$ is received.

When the cdf F is known precisely, an optimal stopping problem is to find a stopping time N which maximizes the expected reward $E[X_N - cN]$. An integer valued random variable N is called a stopping time if, for each n, the event $\{N = n\}$ is independent of $X_{n+1}, X_{n+2}, \cdots$. This is a typical example of the stationary stopping problem (cf. Chow, Robbins and Siegmund [1, p.56]), in which the optimal stopping time N is necessarily of the form: to stop at $N = \min\{n \mid X_n \in S\}$ for some stopping set $S \subset R$, and S is stationary and of a control-limit-type $\{X \geq x\}$ or $\{X > x\}$ for some $x \in R$, where x is called a stopping level. For this, we define that a stopping level x (or $x - 0$) means a stopping set $\{X > x\}$ (or $\{X \geq x\}$) respectively.

For any stopping level x and for any cdf F, we define an expected reward $\phi(x, F) = E[X_N - cN]$ of the stopping problem by

$$(1.1) \qquad \phi(x, F) = \frac{\int_{(x,\infty)} tdF(t) - c}{\overline{F}(x)} = x + \frac{\int_{(x,\infty)}(t - x)dF(t) - c}{\overline{F}(x)},$$

where $\overline{F}(x) = 1 - F(x)$. Note that $\overline{F}(x) \to 0$ and $\phi \to -\infty$ as $x \to \infty$ and that $\overline{F}(x) \to 1$ and $\phi \to \mu_F - c$ as $x \to -\infty$ where $\mu_F = E[X] = \int_R tdF(t)$. It means that to continue the observation process indefinitely gives us the reward $-\infty$ and to stop the process immediately gives us the reward $\mu_F - c$.

By the assumption $E[X^+] < \infty$, using integration by parts, we have

$$\int_{(x,\infty)}(t-x)dF(t) = -\int_{(x,\infty)}(t-x)d\overline{F}(t) = \int_x^\infty \overline{F}(t)dt = \int_{[x,\infty)}(t-x)dF(t) ,$$

which is continuous in x. To handle this expectation, it is convenient to define $T_F(x)$,

$$T_F(x) = \int_x^\infty (t-x)dF(t) = \int_x^\infty \overline{F}(t)dt . \tag{1.2}$$

It has some useful properties, which can be obtained by the elementary analytical technique. Then we give Lemma 1 without proof.

Lemma 1. *$T_F(x)$ is continuous, non-negative, convex and non-increasing function of x. It satisfies that $T_F(x) \geq (\mu_F - x)^+$ for any $x \in R$ and that $T_F(x) \to +\infty$ as $x \to -\infty$ and $T_F(x) \to 0$ as $x \to +\infty$. T_F has a derivative a.e.. Moreover, if $T_F(x)$ is positive at any point x, it is strictly decreasing at x.*

Now, redefining the expected reward $\phi(x,F)$ by (1.1′) for any stopping level x and for any cdf F, we will have the optimal expected reward $\phi^o(F)$ for any cdf F:

$$\phi(x,F) = x + \frac{T_F(x)-c}{\overline{F}(x)} . \tag{1.1′}$$

$$\phi^o(F) \stackrel{\text{def}}{=} \sup_{x\in R} \phi(x,F) . \tag{1.3}$$

$$\frac{d\phi(x,F)}{dF(x)} = \frac{T_F(x)-c}{\overline{F}^2(x)} . \tag{1.4}$$

The right hand side of (1.4) changes the sign from + to − at most one time as x goes from $-\infty$ to $+\infty$. From Lemma 1, the equation $T_F(x) = c$ for any fixed $c(c > 0)$ has a unique solution $x^o(F) \stackrel{\text{def}}{=} (T_F)^{-1}(c)$, so that the set of optimal stopping levels $\mathbf{x}^o(F)$ (which must contain the point $x^o(F)$) of (1.3) is given by

$$\mathbf{x}^o(F) = \{x \mid F(x) = F(x^o(F))\} . \tag{1.5}$$

Since the cdf F is right-continuous, this set is an interval of the form $[a,b)$. If the cdf F has a positive density in the neighborhood of $x^o(F)$, the set $\mathbf{x}^o(F)$ contains only one point $x^o(F)$ i.e., $\mathbf{x}^o(F) = \{x^o(F)\}$.

We have the optimal expected reward $\phi^o(F)$,

$$\phi^o(F) = x^o(F) = \phi(\mathbf{x}^o(F), F) , \tag{1.3′}$$

where $\phi(A,F)$ means $\phi(x,F)$ for any x in a set A.

Let $a = \min\{x \mid x \in \mathbf{x}^o(F)\}$ and $b = \sup\{x \mid x \in \mathbf{x}^o(F)\}$, then the set $\mathbf{x}^o(F)$ of (1.5) is represented as the interval $[a,b)$. Since F has no probability mass in (a,b)

$$\overline{F}(a) = P\{X > a\} = P\{X \geq b\} = \overline{F}(b-0) \stackrel{\text{def}}{=} \lim_{x\uparrow b} \overline{F}(x), \quad \int_{(a,\infty)} tdF(t) = \int_{[b,\infty)} tdF(t),$$

Copyright © by ORSJ. Unauthorized reproduction of this article is prohibited.

and by (1.1) it holds that,

$$x^o(F) = \phi(a, F) = \lim_{x \uparrow b} \phi(x, F) \stackrel{\text{def}}{=} \phi(b-0, F) \ . \tag{1.3''}$$

Then, we have Lemma 2.

Lemma 2. *For any given cdf F, the following stopping sets or stopping levels $(i)(ii)(iii)$ are optimal, and the optimal expected reward is given by (1.3'');*
(i) the set $\{X > a\}$ or level a where $a = \min\{x \mid x \in \mathbf{x}^o(F)\}$,
(ii) the set $\{X \geq b\}$ or level $b-0$ where $b = \sup\{x \mid x \in \mathbf{x}^o(F)\}$,
(iii) the set $\{X > x\}$ ($\{X \geq x\}$) or level x $(x-0)$ where $\forall x \in (a, b)$.

1.2. Statement of the Problem

The underlying premise is that some information is available about certain moments of an unknown cdf F. This determines some class $\mathcal{F}$ of cdf's. This limited information is to be used in order to obtain an upper (or/and lower) bound for our stopping problem $\phi(x, F)$ of (1.1'). In this paper, we consider the classes $\mathcal{F}$ of all cdf's F which are unknown except for their first two moments and domain.

First, we shall consider the maximal bound ϕ^u for $\phi(x, F)$ on $R \times \mathcal{F}$

$$\phi^u = \sup_{x \in R} \sup_{F \in \mathcal{F}} \phi(x, F) = \sup_{F \in \mathcal{F}} \phi^o(F) = \phi(x^o(F^u), F^u) = \phi(x^u, F^u) \ , \tag{1.6}$$

where (x^u, F^u) is a joint maximizing point of $\phi(x, F)$.

Second, we shall consider $\phi(x, F)$ as a two-person zero-sum game in which the player 1 (gambler) decides his level x in R and the player 2 (nature) chooses her cdf F in $\mathcal{F}$, before the observation of $\{X_n; n \geq 1\}$. Then the minimax value ϕ^* and the maximin value ϕ_* on $R \times \mathcal{F}$,

$$\phi^* = \inf_{F \in \mathcal{F}} \sup_{x \in R} \phi(x, F) = \inf_{F \in \mathcal{F}} \phi^o(F) = \phi(x^o(F^*), F^*) = \phi(x^*, F^*) \ , \tag{1.7}$$

$$\phi_* = \sup_{x \in R} \inf_{F \in \mathcal{F}} \phi(x, F) = \phi(x_*, F_*) \ , \tag{1.8}$$

and the saddle value ϕ^s, the saddle point (x^s, F^s) in $R \times \mathcal{F}$,

$$\phi^s = \text{value}_{x \in R, F \in \mathcal{F}} \phi(x, F) = \phi(x^s, F^s) \ , \tag{1.9}$$

will be derived for the following two classes $\mathcal{F}(\mu, \sigma^2)$ and $\mathcal{F}(\mu, \sigma^2, M)$ of cdf's.

The class $\mathcal{F}(\mu, \sigma^2, M)$ is the set of cdf's whose mean μ, variance σ^2 and domain $[\mu - M, \mu + M]$ are assumed to be known.

$$\mathcal{F}(\mu, \sigma^2, M) = \{F \mid \int_A dF(t) = 1, \int_A t dF(t) = \mu, \tag{1.10}$$

$$\int_A t^2 dF(t) = \mu^2 + \sigma^2 \text{ where } A = [\mu - M, \mu + M] \ , \ M \geq \sigma\}$$

The class $\mathcal{F}(\mu, \sigma^2)$ is $\mathcal{F}(\mu, \sigma^2, M)$ where M is arbitrary in $[\sigma, \infty)$, and $\mathcal{F}(\mu)$ is $\mathcal{F}(\mu, \sigma^2)$ where σ^2 is arbitrary in R_{++}.

The organization of this paper is as follows. In Section 2, we study some basic results for our stopping problem $\phi(x, F)$ and $T_F(x)$ and properties of cdf's F in the class $\mathcal{F}$, which play fundamental roles in all that follows. Section 3 is a discussion about the existence and the

Copyright © by ORSJ. Unauthorized reproduction of this article is prohibited.

derivation of the maximizing point and the saddle point for $\phi(x, F)$ in the class $\mathcal{F} = \mathcal{F}(\mu, \sigma^2)$. In fact, the class $\mathcal{F}$ is so rich for the player 2 that the player 1 must stop immediately and may receive the reward of $\mu - c$ as the saddle value. In this case, the information of the value σ^2 is useless for the player 1.

To remove the non-interesting feature appearing in the game theoretic treatment in Section 3, we shall continue the discussion about the more interesting class $\mathcal{F} = \mathcal{F}(\mu, \sigma^2, M)$ in Section 4. In this case, the information of the value σ^2 and M is useful for the player 1. The player 1 can receive the reward of $\mu + (\sigma^2/M - c)^+ - c$ as the saddle value which is greater than $\mu - c$ in Section 3.

2. Some Fundamental Lemmas

The general theory of a duality theorem is presented in the textbook by Ekeland and Teman [2, Chapter 4]. It is hardly difficult to obtain the condition of the existence of a saddle point for our stopping problem $\phi(x, F)$. For an inventory problem where this condition is satisfied, we have derived a saddle point [4]. The following two propositions are used to check a candidate for a saddle point of $\phi(x, F)$ on $R \times \mathcal{F}$.

Proposition 1. [2, p.166] *If ϕ is a function defined on $R \times \mathcal{F}$ with real values, then*

$$\phi_* = \sup_{x\in R} \inf_{F\in\mathcal{F}} \phi(x, F) \leq \phi^* = \inf_{F\in\mathcal{F}} \sup_{x\in R} \phi(x, F) . \tag{2.1}$$

Proposition 2. [2, p.168] *If there exist $x^s \in R, F^s \in \mathcal{F}$ and a real value ϕ^s such that*

$$\begin{aligned} &\phi(x^s, F) \geq \phi^s \text{ for any } F \in \mathcal{F} , \\ &\phi(x, F^s) \leq \phi^s \text{ for any } x \in R , \end{aligned} \tag{2.2}$$

then (x^s, F^s) is a saddle point of ϕ and it holds that

$$\phi^s = \text{value}_{x\in R, F\in\mathcal{F}} \phi(x, F) = \sup_{x\in R} \inf_{F\in\mathcal{F}} \phi(x, F) = \inf_{F\in\mathcal{F}} \sup_{x\in R} \phi(x, F) . \tag{2.3}$$

Moreover, the set of saddle points of ϕ is of the form $(\mathbf{x}^s, \mathcal{F}^s)$, where $\mathbf{x}^s \subset R$ and $\mathcal{F}^s \subset \mathcal{F}$.

Before beginning our discussion, we shall establish some basic results for $\phi(x, F), T_F(x)$ and the class $\mathcal{F}$.

Let a random variable X has a mean μ with a cdf $F_\mu(t)$, then the new random variable $X - \mu$ has the mean 0 with the cdf $F_0(t) = F_\mu(t + \mu)$. The following Lemma 3 follows immediately from the definition (1.1) of $\phi(x, F)$.

Lemma 3.

$$\phi(x, F_\mu) = \mu + \phi(x - \mu, F_0) \text{ for any } x \in R . \tag{2.4}$$

Therefore, we may assume without loss of generality that all the cdf's in F have the mean 0. So that, we shall analyze the stopping problem in only two classes $\mathcal{F}(0, \sigma^2)$ and $\mathcal{F}(0, \sigma^2, M)$.

Lemma 4. *For cdf's F_i and non-negative numbers $\lambda_i, i = 1, 2, \cdots, n$, such that $\sum_{i=1}^n \lambda_i = 1$, let $F = \sum_{i=1}^n \lambda_i F_i$. Then*

$$\phi(x, F) = \sum_{j=1}^n \lambda_j(x) \phi(x, F_j) \text{ for any } x \in R, \text{ where } \lambda_j(x) = \frac{\lambda_j \overline{F}_j(x)}{\sum_{i=1}^n \lambda_i \overline{F}_i(x)}. \tag{2.5}$$

Copyright © by ORSJ. Unauthorized reproduction of this article is prohibited.

Proof. It follows at once from the definition (1.1) of ϕ. □

Note, from Lemma 4, that we find $\phi(x,F)$ is not necessarily convex in F for any fixed $x \in R$, and that the well-known sufficient condition for the existence of a saddle point is not satisfied.

Let G_n be a discrete cdf which has n probability masses p_i, $p_i > 0$, at n points t_i, $i = 1, 2, \cdots, n$, respectively ($\sum_{i=1}^{n} p_i = 1$), i.e., it is represented as

$$G_n(t) \;=\; (< t_1, \cdots, t_n >< p_1, \cdots, p_n >) \ , \tag{2.6}$$

and $\mathcal{G}_n(\mu,\sigma^2)$ be all discrete cdf's G_n in $\mathcal{F}(\mu,\sigma^2)$. Let

$$G_2(t;q) \;=\; (< -\frac{\sigma}{q}, \sigma q >< \frac{q^2}{1+q^2}, \frac{1}{1+q^2} >) \tag{2.7}$$

for any $q, 0 < q < \infty$. Then $G_2(t;q)$ is the only two-point cdf which has the mean 0 and the variance σ^2.

Lemma 5. *The class $\mathcal{G}_2(0,\sigma^2)$ of two-point cdf's is represented with a parameter q, $0 < q < \infty$, as follows,*

$$\mathcal{G}_2(0,\sigma^2) \;=\; \{G_2(\cdot;q) \;\mid\; 0 < q < \infty\} \ .$$

Let us define

$$T^u_{\mathcal{F}}(x) \;=\; \sup_{F\in\mathcal{F}} T_F(x) \ , \ T^\ell_{\mathcal{F}}(x) \;=\; \inf_{F\in\mathcal{F}} T_F(x) \ . \tag{2.8}$$

Then, from Lemma 1, we have the following lemma.

Lemma 6. *Suppose $\mathcal{F} = \mathcal{F}(0)$ so that $\mu_F = 0$ for all $F \in \mathcal{F}$, then $T^u_{\mathcal{F}}(x)$ and $T^\ell_{\mathcal{F}}(x)$ have the same property as $T_F(x)$ in Lemma 1 with μ_F replaced by 0 , except that $T^\ell_{\mathcal{F}}(x)$ is not always convex.*

From above Lemma 6, $T^u_{\mathcal{F}}(x)$ and $T^\ell_{\mathcal{F}}(x)$ have inverse functions $(T^u_{\mathcal{F}})^{-1}(c)$ and $(T^\ell_{\mathcal{F}})^{-1}(c)$ for all $c, c > 0$, respectively. Thus we have shown the existence of the values of ϕ^u and ϕ^*:

$$\phi^u \;=\; \sup_{F\in\mathcal{F}} \{x \mid T_F(x) = c\} \;=\; (T^u_{\mathcal{F}})^{-1}(c) \ , \tag{2.9}$$

$$\phi^* \;=\; \inf_{F\in\mathcal{F}} \{x \mid T_F(x) = c\} \;=\; (T^\ell_{\mathcal{F}})^{-1}(c) \ . \tag{2.10}$$

3. The Class $\mathcal{F}(\mu,\sigma^2)$

In this section, we shall derive the maximal bound ϕ^u and the saddle value ϕ^s for $\phi(x,F)$ in the class $\mathcal{F} = \mathcal{F}(0,\sigma^2)$.

$$\mathcal{F}(0,\sigma^2) = \{F \mid \int_R dF(t) = 1, \ \int_R t dF(t) = 0, \ \int_R t^2 dF(t) = \sigma^2\} \ . \tag{3.1}$$

3.1 The maximal bound ϕ^u

First, we prepare a useful proposition related to Schwartz inequalities.

Copyright © by ORSJ. Unauthorized reproduction of this article is prohibited.

Proposition 3. (Feller [3, p.151]) *If F is an arbitrary cdf, then*

$$(\int_A u(t)v(t)dF(t))^2 \le (\int_A u^2(t)dF(t))\ (\int_A v^2(t)dF(t)) \tag{3.2}$$

for any set A and any functions u, v for which the integrals on the right exist. Furthermore, the equality sign holds if and only if

$$\int_A (au(t)+bv(t))^2 dF(t) = 0 \text{ for some } a, b \in R\ . \tag{3.3}$$

Remark on Proposition 3. If u and v are linearly dependent, i.e., for some $a, b \in R$, $au(t) + bv(t) = 0$, the condition (3.3) is satisfied for all $F \in \mathcal{F}$, and that if u and v are linearly independent, the condition (3.3) is satisfied only when the cdf F is degenerated at one point in a set A.

We shall calculate ϕ^u and the maximizing point (x^u, F^u) of the problem (2.9) by Proposition 3.

$$(\int_{(x,\infty)} (t-x)dF(t))^2 \le (\int_{(x,\infty)} dF(t))\ (\int_{(x,\infty)} (t-x)^2 dF(t))\ , \tag{3.4}$$

$$(\int_{(-\infty,x]} (t-x)dF(t))^2 \le (\int_{(-\infty,x]} dF(t))\ (\int_{(-\infty,x]} (t-x)^2 dF(t))\ . \tag{3.4'}$$

For all $F \in \mathcal{F}$, $T_F(x) = \int_{(x,\infty)}(t-x)dF(t) = c$ from (2.9) and $\int_R dF(t) = 1$, $\int_R t dF(t) = 0$, $\int_R t^2 dF(t) = \sigma^2$ from (3.1). Then, putting $p = \int_{(x,\infty)} dF(t)$, it holds that by adding (3.4) to (3.4′),

$$\frac{c^2}{p} + \frac{(x+c)^2}{1-p} \le \sigma^2 + x^2\ ,\ 0 < p < 1\ ,$$

$$-\frac{c}{p} - \sigma\sqrt{\frac{1-p}{p}} \le x \le -\frac{c}{p} + \sigma\sqrt{\frac{1-p}{p}}\ ,\ 0 < p < 1\ . \tag{3.5}$$

Putting $q = \sqrt{(1-p)/p}$, $0 < q < \infty$, the upper bound of x(the right-hand side of (3.5)) is represented by the parameter q.

$$\sup_{0<p<1} (\sigma\sqrt{\frac{1-p}{p}} - \frac{c}{p}) = \sup_{0<q<\infty} (\sigma q - c(1+q^2)) = \sup_{0<q<\infty} [-c(q - \frac{\sigma}{2c})^2 + \frac{\sigma^2}{4c} - c] = \frac{\sigma^2}{4c} - c\ , \tag{3.6}$$

where the supremum is attained at $q = \sigma/2c$ i.e., $p = 4c^2/(\sigma^2 + 4c^2)$.

Then, we obtain the maximal bound ϕ^u.

$$\phi^u = \sup_{F\in\mathcal{F}} \{x \mid T_F(x) = c\} = \frac{\sigma^2}{4c} - c\ . \tag{3.7}$$

Since the equality holds in two Schwartz inequalities (3.4) and (3.4′), from the remark of Proposition 3, the maximizing cdf F^u should be the two-point cdf. Then, we have

$$F^u(t) = G_2(t; \frac{\sigma}{2c}) = (< -2c, \frac{\sigma^2}{2c} >< \frac{\sigma^2}{\sigma^2+4c^2}, \frac{4c^2}{\sigma^2+4c^2} >), \tag{3.8}$$

Copyright © by ORSJ. Unauthorized reproduction of this article is prohibited.

(3.9) $$x^u \in \mathbf{x}^u = \mathbf{x}^o(F^u) = [-2c, \frac{\sigma^2}{2c}) .$$

Theorem 1. *For a class $\mathcal{F}(0,\sigma^2)$ of cdf's, the maximal bound ϕ^u is $\sigma^2/4c - c$ by (3.7) and all the maximizing points $(x^u, F^u) \in (\mathbf{x}^u, \{F^u\})$ are given by $F^u(t) = G_2(t;\sigma/2c)$ in (3.8) and $\mathbf{x}^u = [-2c, \sigma^2/2c)$ in (3.9).*

Remark on Theorem 1. From Lemma 2, the equation (3.9) means that the player 1 may decide a stopping level x^u for some $x^u \in [-2c, \sigma^2/2c)$ or $\sigma^2/2c - 0$. If the player 1 decides any of the above stopping levels, he stops the process whenever $X_n = \sigma^2/2c$ is observed because the player 2 chooses only one cdf given by (3.8).

Corollary 1. *For a class $\mathcal{F}(\mu,\sigma^2)$ of cdf's, the maximal bound ϕ^u is $\mu + \sigma^2/4c - c$ and all the maximizing points $(x^u, F^u) \in (\mathbf{x}^u, \{F^u\})$ are given by $F^u(t-\mu) = G_2(t;\sigma/2c)$ and $\mathbf{x}^u = [\mu - 2c, \mu + \sigma^2/2c)$.*

In the theory of extreme order statistics, by letting $Y = \max_{1 \le n \le \infty}[X_n - cn]$, it holds that $E[Y^+] < \infty$ if and only if $E[X^+] < \infty$ (See Pickands III [5, Cor. 2.2]). Corollary 1 also says that the maximal bound of $E[Y^+]$ is calculated by using μ and σ^2 if the cdf F is restricted in the class $\mathcal{F}(\mu,\sigma^2)$.

3.2 The minimax value ϕ^*

Second, we shall calculate the minimax value ϕ^* of (2.10) and the minimax-mizing point $(x^*, F^*) \in (\mathbf{x}^*, \mathcal{F}^*)$.

From Lemma 6, $T_{\mathcal{F}}^{\ell}(x) \ge (-x)^+$ for all $x \in R$. Then it holds that $T_{F^*}(x) = (-x)^+ \le T_{\mathcal{F}}^{\ell}(x)$ for $x \in (-\infty, -c]$ if a cdf F^*, which has all the mass on $[-c, \infty)$, is contained in $\mathcal{F}$. Since $T_{F^*}(x) = (-x)^+$ is strictly decreasing on $(-\infty, -c]$, we have

(3.10) $$\phi^* = \inf_{F \in \mathcal{F}}\{x \mid T_F(x) = c\} = \{x \mid T_{F^*}(x) = c\} = -c .$$

Such a class $\mathcal{F}^*$ of cdf's F^* always exists in $\mathcal{F}$ for all c, $c > 0$.

(3.11) $$\mathcal{F}^* = \{F \mid \int_{[-c,\infty)} dF(t) = 1, \ F \in \mathcal{F}\} .$$

In particular, we can find the class $\mathcal{G}_2^* = \mathcal{G}_2^*(0,\sigma^2)$ of two-point cdf's in $\mathcal{F}^*$ from Lemma 5.

(3.11′) $$\mathcal{G}_2^* = \{G_2(\cdot;q) \mid q \ge \frac{\sigma}{c}\} .$$

It is easily shown that for any $F^* \in \mathcal{F}^*$ it is optimal for the player 1 to stop the process immediately. That is,

(3.12) $$\mathbf{x}^* = \mathbf{x}^o(F^*) = (-\infty, -c) \text{ for all } F^* \in \mathcal{F} .$$

Theorem 2. *For a class $\mathcal{F}(0,\sigma^2)$ of cdf's, the minimax value ϕ^* is $-c$ by (3.10) and all the minimax-mizing points $(x^*, F^*) \in (\mathbf{x}^*, \mathcal{F}^*)$ are given by (3.11) and (3.12). In particular, there exists the class $\mathcal{G}_2^*$ of two-point cdf's in $\mathcal{F}^*$ by (3.11′).*

Corollary 2. *For a class $\mathcal{F}(\mu,\sigma^2)$ of cdf's, the minimax value ϕ^* is $\mu - c$, and all the minimax-mizing points $(x^*, F^*) \in (\mathbf{x}^*, \mathcal{F}^*)$ are given by*

$$\mathcal{F}^* = \{F \mid \int_{[\mu-c,\infty)} dF(t) = 1, \ F \in \mathcal{F}(\mu,\sigma^2)\}, \ \mathbf{x}^* = (-\infty, \mu - c) .$$

Copyright © by ORSJ. Unauthorized reproduction of this article is prohibited.

3.3 The saddle value ϕ^s

Now, we shall derive the saddle value ϕ^s for $\phi(x,F)$ in $\mathcal{F}=\mathcal{F}(\mu,\sigma^2)$. We have a candidate $(\mathbf{x}^*,\mathcal{F}^*)$ for a set of saddle points $(\mathbf{x}^s,\mathcal{F}^s)$ to be checked by Proposition 2. Since $(\mathbf{x}^*,\mathcal{F}^*)$ includes $(\mathbf{x}^s,\mathcal{F}^s)$ in general, we shall make an attempt to prove the inverse . From Theorem 2 it holds that $\phi(x,F)=-c$ for all $x\in\mathbf{x}^*$ and all $F\in\mathcal{F}^*$. Moreover, from (3.12), it holds that for any fixed $x\notin\mathbf{x}^*$, $\phi(x,F)\le -c$ for all $F\in\mathcal{F}^*$. It suffices to prove for any fixed $F\in\mathcal{F}$,

$$\phi(x,F)\ge -c \quad \text{for all } x\in\mathbf{x}^* \ . \tag{3.13}$$

Proof of (3.13). For any fixed F, it holds that $\phi(-\infty,F)=-c$ and $\phi(x,F)$ is a unimodal function of x and maximized at the point $x^o(F)$ from the remark of (1.1). When (i) $x^o(F)\ge -c$, $\phi(x,F)\ge -c$ for $x\in(-\infty,-c]$, i.e. (3.13) is true in the case (i). When (ii) $x^o(F)\le -c$, $\phi(x,F)\ge\phi(-c,F)$ for $x\in[x^o(F),-c]$, so that (3.13) is true in the case (ii) if we can prove $\phi(-c,F)\ge -c$.

For $F\in\mathcal{F}$, F has the mean 0, so that $\overline{F}(-c)>0$ and

$$\int_{(-c,\infty)} tdF(t) = -\int_{(-\infty,-c]} tdF(t)\ge c(1-\overline{F}(-c)) \ .$$

Thus, we have from (1.1),

$$\phi(-c,F)=\frac{\int_{(-c,\infty)} tdF(t)-c}{\overline{F}(-c)}\ge\frac{c(1-\overline{F}(-c))-c}{\overline{F}(-c)}=-c \ . \tag{3.14}$$

This completes the proof. □

Hence, we have the main theorem in this chapter.

Theorem 3. *For a class $\mathcal{F}(0,\sigma^2)$ of cdf's, the saddle value ϕ^s is $-c$ and all the saddle points $(x^s,F^s)\in(\mathbf{x}^s,\mathcal{F}^s)$ are given by $\mathbf{x}^s=\mathbf{x}^*,\mathcal{F}^s=\mathcal{F}^*$ and $\mathcal{G}_2^s=\mathcal{G}_2^*\subset\mathcal{F}^s$ defined in Theorem 2.*

Corollary 3. *For a class $\mathcal{F}(\mu,\sigma^2)$ of cdf's, the saddle value ϕ^s is $\mu-c$ and all the saddle points $(x^s,F^s)\in(\mathbf{x}^s,\mathcal{F}^s)$ are given by $\mathbf{x}^s=\mathbf{x}^*$, $\mathcal{F}^s=\mathcal{F}^*$ defined in Corollary 2.*

Theorem 3 says the class $\mathcal{F}(\mu,\sigma^2)$ is so rich for the player 2 that the player 1 must stop immediately. In this case, the information of the value σ^2 is useless for the player 1.

If we consider the problem in the class

$$\mathcal{F}(\mu,\cdot,M) \ = \ \{F \mid \int_A dF(t)=1,\ \int_A tdF(t)=\mu,\ A=[\mu-M,\mu+M]\} \ , \tag{3.15}$$

the player 2 may choose the cdf which is degenerated at the point μ and the player 1 receives the reward of $\mu-c$, in which the information of the value M is useless for the player 1 .

To remove the non-interesting strategies for the player 1 in the above two cases $\mathcal{F}(\mu,\sigma^2)$ and $\mathcal{F}(\mu,\cdot,M)$, we shall analyze the problem in the more restrictive class $\mathcal{F}(\mu,\sigma^2,M)$ of (1.10) in the next section.

4. The Class $\mathcal{F}(\mu,\sigma^2,M)$

In this section, we shall derive the maximal bound ϕ^u and the saddle value ϕ^s in the more restrictive and interesting class $\mathcal{F}=\mathcal{F}(0,\sigma^2,M)$ (see (1.10)).

$$\mathcal{F}(0,\sigma^2,M) \ = \ \{F \mid \int_A dF(t)=1,\ \int_A tdF(t)=0,\ \int_A t^2dF(t)=\sigma^2,\ A=\ [-M,M],\ \sigma\le M\}.$$

Copyright © by ORSJ. Unauthorized reproduction of this article is prohibited.

If $\sigma = M > 0$, the class $\mathcal{F}$ contains only one cdf $G_2(t;1) = (< -M, M >< 1/2, 1/2 >)$. So, it is assumed that $\sigma < M$.

We calculate ϕ^u and the maximizing point (x^u, F^u) in the same fashion as (3.6) with the additional constraint $\sigma/M \le q \le M/\sigma$,

$$\phi^u = \max_{\sigma/M \le q \le M/\sigma} [\sigma q - c(1+q^2)] . \tag{4.1}$$

Theorem 4. *For a class $\mathcal{F}(0,\sigma^2,M)$ of cdf's, $\sigma < M$, the maximal bound ϕ^u and all the maximizing points $(x^u, F^u) \in (\mathbf{x}^u, \mathcal{F}^u)$ are as follows:*
(i) When $0 \le c \le \sigma^2/2M$,

$$\phi^u = M - c(1+\frac{M^2}{\sigma^2}) \quad \mathbf{x}^u = [-\frac{\sigma^2}{M}, M) ,$$

$$F^u(t) = G_2(t; \frac{M}{\sigma}) = (< -\frac{\sigma^2}{M}, M >< \frac{M^2}{\sigma^2+M^2}, \frac{\sigma^2}{\sigma^2+M^2} >) .$$

(ii) When $\sigma^2/2M \le c \le M/2$, the same result as Theorem 1 holds, i.e.,

$$\phi^u = \frac{\sigma^2}{4c} - c , \quad \mathbf{x}^u = [-2c, \frac{\sigma^2}{2c}) ,$$

$$F^u(t) = G_2(t; \frac{\sigma}{2c}) = (< -2c, \frac{\sigma^2}{2c} >< \frac{\sigma^2}{\sigma^2+4c^2}, \frac{4c^2}{\sigma^2+4c^2} >) .$$

(iii) When $M/2 \le c \le M$,

$$\phi^u = \frac{\sigma^2}{M} - c(1+\frac{\sigma^2}{M^2}) , \quad \mathbf{x}^u = [-M, \frac{\sigma^2}{M}) ,$$

$$F^u(t) = G_2(t; \frac{\sigma}{M}) = (< -M, \frac{\sigma^2}{M} >< \frac{\sigma^2}{\sigma^2+M^2}, \frac{M^2}{\sigma^2+M^2} >) .$$

Remark on Theorem 4. When $c > M$, i.e. the observation cost is larger than the maximal value of the observation, it is natural for the player 1 to stop immediately, i.e., $\phi^u = -c$, $\mathcal{F}^u = \mathcal{F}$ and $\mathbf{x}^u = (-\infty, -M)$. The shape of ϕ^u as the function of c is illustrated in Figure 1 at the end of this paper. Furthermore, if the results for the class $\mathcal{F}(\mu, \sigma^2, M)$ are required, Lemma 2 is useful for the theorems in this section.

Now, we shall derive the saddle value ϕ^s. By calculating $T_F(x)$ of (1.2) for the cdf's $G_2(\cdot; M/\sigma)$ and $G_2(\cdot; \sigma/M)$, we have, from (2.8) and Lemma 6,

$$T^\ell_{\mathcal{F}}(x) \ge (-x)^+ = -x = T_{G_2(\cdot;M/\sigma)}(x) \text{ for } x \in [-M, -\sigma^2/M] , \tag{4.2}$$

$$T^\ell_{\mathcal{F}}(x) \ge (-x)^+ = 0 = T_{G_2(\cdot;\sigma/M)}(x) \text{ for } x \in [\sigma^2/M, M] . \tag{4.3}$$

If the observation cost c is a constant such that $M \ge c \ge \sigma^2/M$, the equality $\phi^* = (T^\ell_{\mathcal{F}})^{-1}(c) = -c$ is satisfied from (4.2). In this case, the saddle value ϕ^s and the saddle point (x^s, F^s) are the same ones given in Theorem 3, because the player 2 can choose a cdf $G_2(\cdot, M/\sigma)$ in this restrictive class $\mathcal{F} = \mathcal{F}(0,\sigma^2,M)$, i.e., $G_2(\cdot; M/\sigma) \in \mathcal{F}(0,\sigma^2,M) \subset \mathcal{F}(0,\sigma^2)$. So that we confine our consideration to the case:

$$0 < c < \sigma^2/M . \tag{4.4}$$

Copyright © by ORSJ. Unauthorized reproduction of this article is prohibited.

On the other hand, from (4.2) and (4.3), it holds that

$$\inf_{F\in\mathcal{F}} \phi(x,F) \le \phi(x, G_2(\cdot; M/\sigma)) = -c \text{ for } x \in [-M, -\sigma^2/M)\ , \tag{4.2'}$$

$$\inf_{F\in\mathcal{F}} \phi(x,F) \le \phi(x, G_2(\cdot; \sigma/M)) = -\infty \text{ for } x \in [\sigma^2/M, M]\ , \tag{4.3'}$$

because the player 1 stops immediately in the case of (4.2′) or he cannot stop in the case of (4.3′). Then, the player 1 must decide his stopping level x in the interval

$$\mathbf{x}^M \stackrel{\text{def}}{=} [-\frac{\sigma^2}{M}, \frac{\sigma^2}{M})\ , \tag{4.5}$$

in order not to make his reward $\inf_{F\in\mathcal{F}} \phi(x,F) \le -c$, where $-c$ is the reward of immediately stopping or the saddle value $\phi^s = -c$ in Section 3.

Hereafter, we consider the stopping problem $\phi(x,F)$ under the restriction of (4.4) and (4.5). To begin with, we prepare the following three lemmas.

Lemma 7. *If F is an arbitrary cdf defined on a finite interval $[a,b]$ with the mean $\mu(a \le \mu \le b,\ a < b)$, then*

$$\int_{[a,b]} (t-\mu)^2 dF(t) \le (\mu - a)(b-a).$$

Furthermore, the equality holds if and only if

$$F = (< a, b >< \frac{b-\mu}{b-a}, \frac{\mu-a}{b-a} >). \tag{4.6}$$

Proof. Since $f(t) = (t-\mu)^2$ is strictly convex, we have

$$\int_{[a,b]} (t-\mu)^2 dF(t) \le (a-\mu)^2 \int_{[a,\mu)} dF(t) + (b-\mu)^2 \int_{(\mu,b]} dF(t).$$

The equality holds iff F satisfies $\int_{[a,\mu)} dF(t) = \int_{\{a\}} dF(t)$ and $\int_{(\mu,b]} dF(t) = \int_{\{b\}} dF(t)$. Such a cdf is uniquely determined by (4.6). This completes the proof. □

Lemma 8. *For any strategy $(x,F) \in (\mathbf{x}^M, \mathcal{F})$, if F has a positive probability on the interval (x, M) and satisfies $\phi(x,F) \ge -c$, then there exists a cdf $F'' \in \mathcal{F}$ such that F'' has no probability on the interval $(x+\varepsilon, M)$ for an arbitrary small $\varepsilon > 0$, and it satisfies $\phi(x,F'') \le \phi(x,F)$.*

Proof. Let $x' = x + \varepsilon$, $p := \int_{(x',M)} dF(t) > 0$ and $y = p^{-1}\int_{(x',M)} t dF(t)$, $x' < y < M$. From Lemma 7, the probability p of F is allotted to the points x' and M without changing its mean 0. Let a cdf F' be

$$F'(t) = \begin{cases} F(t) & ,\quad t < x' \\ F(x') + (< x', M >< p\frac{M-y}{M-x'}, p\frac{y-x'}{M-x'} >) & ,\quad x' \le t < M \\ 1 & ,\quad M \le t \end{cases} \tag{4.7}$$

which has the variance greater than σ^2. The difference γ^2 of the variances is given by

$$\begin{aligned} \gamma^2 &= \int_A t^2 dF'(t) - \int_A t^2 dF(t) \\ &= p[(x')^2 \frac{M-y}{M-x'} + M^2 \frac{y-x'}{M-x'}] - \int_{(x',M)} t^2 dF(t) > 0. \end{aligned}$$

Copyright © by ORSJ. Unauthorized reproduction of this article is prohibited.

Putting $\lambda = \sigma^2/(\sigma^2+\gamma^2)$, let us define

$$F''(t) = \lambda F'(t) + (1-\lambda)G_1(t),\ 0<\lambda<1, \tag{4.8}$$

where G_1 is the cdf degenerated at the origin 0. Then we find F'' is in $\mathcal{F}$, i.e., F'' has the mean 0 and the variance σ^2. From Lemma 4 and (4.8) it holds that

$$\phi(x,F'') = \frac{\lambda \overline{F}'(x)}{\overline{F}''(x)}\phi(x,F') + \frac{(1-\lambda)\overline{G}_1(x)}{\overline{F}''(x)}\phi(x,G_1), \tag{4.9}$$

where $\overline{G}_1(x) = 1$ and $\phi(x,G_1) = -c$ for $x<0$, $\overline{G}_1(x) = 0$ for $x \geq 0$. Since $\overline{F}'(x) = \overline{F}(x)$ and $\phi(x,F') = \phi(x,F)$ from (4.7) and (1.1), we have $\phi(x,F'') \leq \phi(x,F)$ because $\phi(x,F'')$ in (4.9) is a convex combination of $\phi(x,F)(\geq -c)$ and $-c$.

Now, we shall redefine the cdf F'' which has no probability on the interval(x',M).

When $x \geq 0$, F'' in (4.8) has no probability on (x',M) and the result holds. When $x<0$, let $x' = x+\varepsilon < 0$ and $F''_{(1)} \stackrel{\text{def}}{=} F''$ in (4.8) has a probability mass $p_{(1)} = F''_{(1)}(0) - F''_{(1)}(0-0)$ at the origin 0 on (x',M). If the mass $p_{(1)}$ of $F''_{(1)}$ is allotted to the points x' and M, then we can define $F'_{(2)}$ and $F''_{(2)}$ similarly as (4.7) and (4.8):

$$F'_{(2)}(t) = F''_{(1)}(t) + (< x', 0, M >< p_{(1)}\frac{M}{M+|x'|}, -p_{(1)}, p_{(1)}\frac{|x'|}{M+|x'|} >), \tag{4.7'}$$

$$F''_{(2)}(t) = \lambda_{(2)}F'_{(2)}(t) + (1-\lambda_{(2)})G_1(t),\ \text{where } \lambda_{(2)} = \frac{\sigma^2}{\sigma^2 + p_{(1)}M|x'|}. \tag{4.8'}$$

Let us define $F'_{(n+1)} \in \mathcal{F}$ and $F''_{(n+1)} \in \mathcal{F}$ recursively by (4.7′) and (4.8′) with the subindexes (2) and (1) replaced by $(n+1)$ and (n) respectively $(n \geq 1)$. The mass $p_{(n+1)}$ of F''_{n+1} is given successively by

$$p_{(n+1)} = \frac{p_{(n)}}{k+p_{(n)}},\ k = \frac{\sigma^2}{M|x'|} > 0,\ n \geq 1, \tag{4.10}$$

where $\lim_{n\to\infty} p_{(n)} = 0$ for any $p_{(1)}$. By the similar argument about $F'' = F''_{(1)}$ in (4.9), $F''_{(n)} \in \mathcal{F}, n \geq 1$, $F'' \stackrel{\text{def}}{=} \lim_{n\to\infty} F''_{(n)} \in \mathcal{F}$ and $\phi(x,F'') \leq \phi(x,F''_{(n)}) \leq \phi(x,F)$, $n \geq 1$. This completes the proof. □

Lemma 9. *For any strategy $(x,F) \in (\mathbf{x}^M, \mathcal{F})$, if F has a positive probability on the interval $(-M,x)$ and it satisfies $\phi(x,F) \geq -c$, then there exists a cdf $F'' \in \mathcal{F}$ such that F'' has no probability on the interval $(-M,x)$, and it satisfies $\phi(x,F'') \leq \phi(x,F)$.*

Proof. Let $p = \int_{(-M,x)} dF(t) > 0$ and $y = p^{-1}\int_{(-M,x)} t\,dF(t)$, $-M<y<x$. From Lemma 7, the probability p of F is allotted to the point $-M$ and x without changing its mean 0. Let a cdf F' be

$$F'(t) = \begin{cases} F(-M) + (< -M, x >< p\frac{x-y}{M+x}, p\frac{M+y}{M+x} >) &, \ -M \leq t \leq x \\ F(t) &, \ x < t \end{cases}$$

which has the variance greater than σ^2. The difference γ^2 of the variances is given by

$$\gamma^2 = p[x^2\frac{M+y}{M+x} + M^2\frac{x-y}{M+x}] - \int_{(-M,x)} t^2 dF(t) > 0\ .$$

Copyright © by ORSJ. Unauthorized reproduction of this article is prohibited.

Putting $\lambda = \sigma^2/(\sigma^2+\gamma^2)$, let a cdf F'' be defined similarly by (4.8). Then, in the same fashion as in Lemma 8, the result holds. □

Let us define for any $x \in [-\sigma^2/M, \sigma^2/M]$, a three-point cdf $G_3^M(\cdot;x) \in \mathcal{F}$ which has all the mass at three points $-M, x, M$ with the mean 0 and the variance σ^2. This cdf is uniquely determined by

$$(4.11) \qquad G_3^M(t;x) = (< -M, x, M >< \frac{Mx+\sigma^2}{2M(M+x)}, \frac{M^2-\sigma^2}{M^2-x^2}, \frac{\sigma^2-Mx}{2M(M-x)} >) ,$$

and let $\mathcal{G}_3^M = \{G_3^M(t;x) \mid -\sigma^2/M \le x \le \sigma^2/M\}$. Note that if $x = \sigma^2/M$ or $-\sigma^2/M$, $G_3^M(t;x)$ becomes the two-point cdf $G_2(t;\sigma/M)$ or $G_2(t;M/\sigma)$ respectively.

Now, we shall derive a saddle point. It holds, from Theorem 3, that

$$(4.12) \qquad -c = \sup_{x\in R} \inf_{F\in\mathcal{F}(0,\sigma^2)} \phi(x,F) \le \sup_{x\in R} \inf_{F\in\mathcal{F}} \phi(x,F) .$$

Then the player 1 would decide a stopping level x in the following set

$$(4.13) \qquad \{x \mid \phi(x,F) \ge -c \text{ for all } F \in \mathcal{F}\} \cap \mathbf{x}^M \stackrel{\text{def}}{=} \mathbf{x}_c^M .$$

This set is not empty because $x = -c$ is contained in it.

Remark on Lemmas 8 and 9. Lemmas mean the following : for any $x \in \mathbf{x}_c^M$, if the player 1(he) decides his stopping level x, the player 2(she) may choose her cdf which has no probability on the interval $(-M,x) \cup (x+\varepsilon, M)$ for an arbitrary small $\varepsilon > 0$, where her cdf converges to $G_3^M(\cdot;x)$ of (4.11) as ε goes to 0. In this situation, his stopping set becomes $\{X \in (x, x+\varepsilon] \cup \{M\}\}$ for an arbitrary small $\varepsilon > 0$, i.e., $\{X \in \{M\}\}$ or $\{X \in \{x\} \cup \{M\}\}$.

From the above remark, we have for any $x \in \mathbf{x}_c^M$,

$$(4.14) \qquad \inf_{F\in\mathcal{F}} \phi(x,F) = \min[\phi(x, G_3^M(\cdot;x)), \phi(x-0, G_3^M(\cdot;x))].$$

If there exists a point $x^s \in \mathbf{x}_c^M$ such that

$$(4.15) \qquad \phi(x^s, G_3^M(\cdot;x^s)) = (T_{\mathcal{G}_3^M}^{\ell})^{-1}(c)(= \phi(x^s-0, G_3^M(\cdot;x^s))) \ge -c ,$$

the strategy $(x^s, G_3^M(\cdot;x^s))$, $x \in \mathbf{x}_c^M$, $G_3^M(\cdot;x^s) \in \mathcal{G}_3^M \subset \mathcal{F}$, is the saddle point and $\phi^s = (T_{\mathcal{G}_3^M}^{\ell})^{-1}(c)$ is the saddle value. Because, from (4.14) and (2.10), the following relation is satisfied.

$$\min[\phi(x^s, G_3^M(\cdot;x^s)), \phi(x^s-0, G_3^M(\cdot;x^{\prime}s))] \le \sup_{x\in\mathbf{x}_c^M} \inf_{F\in\mathcal{F}} \phi(x,F)$$

$$\le \inf_{F\in\mathcal{F}} \sup_{x\in\mathbf{x}_c^M} \phi(x,F) \le \inf_{F\in\mathcal{G}_3^M} \sup_{x\in\mathbf{x}_c^M} \phi(x,F) = (T_{\mathcal{G}_3^M}^{\ell})^{-1}(c) \; = \; \phi(x^s, G_3^M(\cdot;x^s)).$$

For any $x, y \in [-\sigma^2/M, \sigma^2/M]$, it is shown, by the graph of $T_{G_3^M(\cdot;y)}(x)$, that

$$(4.16) \quad T_{G_3^M(\cdot;y)}(x) = \int_x^M [1-G_3^M(t;y)]dt \ge T_{G_3^M(\cdot;x)}(x) = \int_x^M [1-G_3^M(t;x)]dt = \frac{\sigma^2-Mx}{2M} .$$

This line segment $T = (\sigma^2 - Mx)/2M$ connects the points $(-\sigma^2/M, \sigma^2/M)$ and $(\sigma^2/M, 0)$ in (x,T) -plane ,and the equality in (4.16) holds only when $y = x$.

Copyright © by ORSJ. Unauthorized reproduction of this article is prohibited.

Then, we have proved (4.15).

$$(4.17)\qquad (T^{\ell}_{G_3^M})^{-1}(c) = \frac{\sigma^2}{M} - 2c = (\frac{\sigma^2}{M} - c)^+ - c \geq -c\ .$$

$$(4.18)\qquad x^s = (\frac{\sigma^2}{M} - c)^+ - c,\ F^s = G_3^M(\cdot; x^s)\ .$$

Hence, we have the main theorem of our problem.

Theorem 5. *For a class $\mathcal{F}(0,\sigma^2,M)$ of cdf's, $\sigma \leq M$, we have the following solution:*
(i) When $\sigma^2/M \leq c \leq M$, the same result as Theorem 3 holds, that is, the saddle value ϕ^s and all the saddle points $(x^s, F^s) \in (\mathbf{x}^s, \mathcal{F}^s)$ are given by

$$\phi^s = -c,\ \mathbf{x}^s = [-M, -c] \text{ and } \mathcal{F}^s = \{F \mid \int_{[-c,M]} dF(t) = 1, F \in \mathcal{F}(0,\sigma^2,M)\}\ .$$

(ii) When $0 < c < \sigma^2/M$, the saddle value ϕ^s and the saddle point (x^s, F^s) are given by

$$\phi^s = (\sigma^2/M - c)^+ - c,\quad x^s = (\sigma^2/M - c)^+ - c \text{ and } F^s(t) = G_3^M(t; x^s) \text{ defined by } (4.11).$$

Unfortunately, we could not show the uniqueness of the saddle point (x^s, F^s) in Case (ii). Finally, we conclude this paper by illustrating ϕ^u of Theorem 4 and ϕ^s of Theorem 5 in Figure 1.

Acknowledgments The author would like to thank two anonymous referees for their helpful comments.

References

[1] Chow,Y.S., H.Robbins and D.Siegmund, *Great Expectations : The Theory of Optimal Stopping,* Houghton-Mifflin, New York, 1971.
[2] Ekeland,I. and R.Teman, *Convex Analysis and Variational Problems*, North-Holland, 1976.
[3] Feller,W., *An Introduction to Probability Theory and its Applications, Vol. II,* Wiley, New York, 1968.
[4] Nakagami,J. and M.Yasuda, "*Saddle Point of an Inventory Problem* ", *J. Information and Optimization Sciences,* **2** (1981), pp. 181-191.
[5] Pickands,J.III, "*Extreme Order Statistics with Cost of Sampling*", *Adv. Appl. Prob.*, **15** (1983), pp. 783-797.

Jun-ichi NAKAGAMI
Department of Mathematics
Faculty of Science, Chiba University
1-33 Yayoi-cho, Chiba-city, 260 JAPAN

Copyright © by ORSJ. Unauthorized reproduction of this article is prohibited.

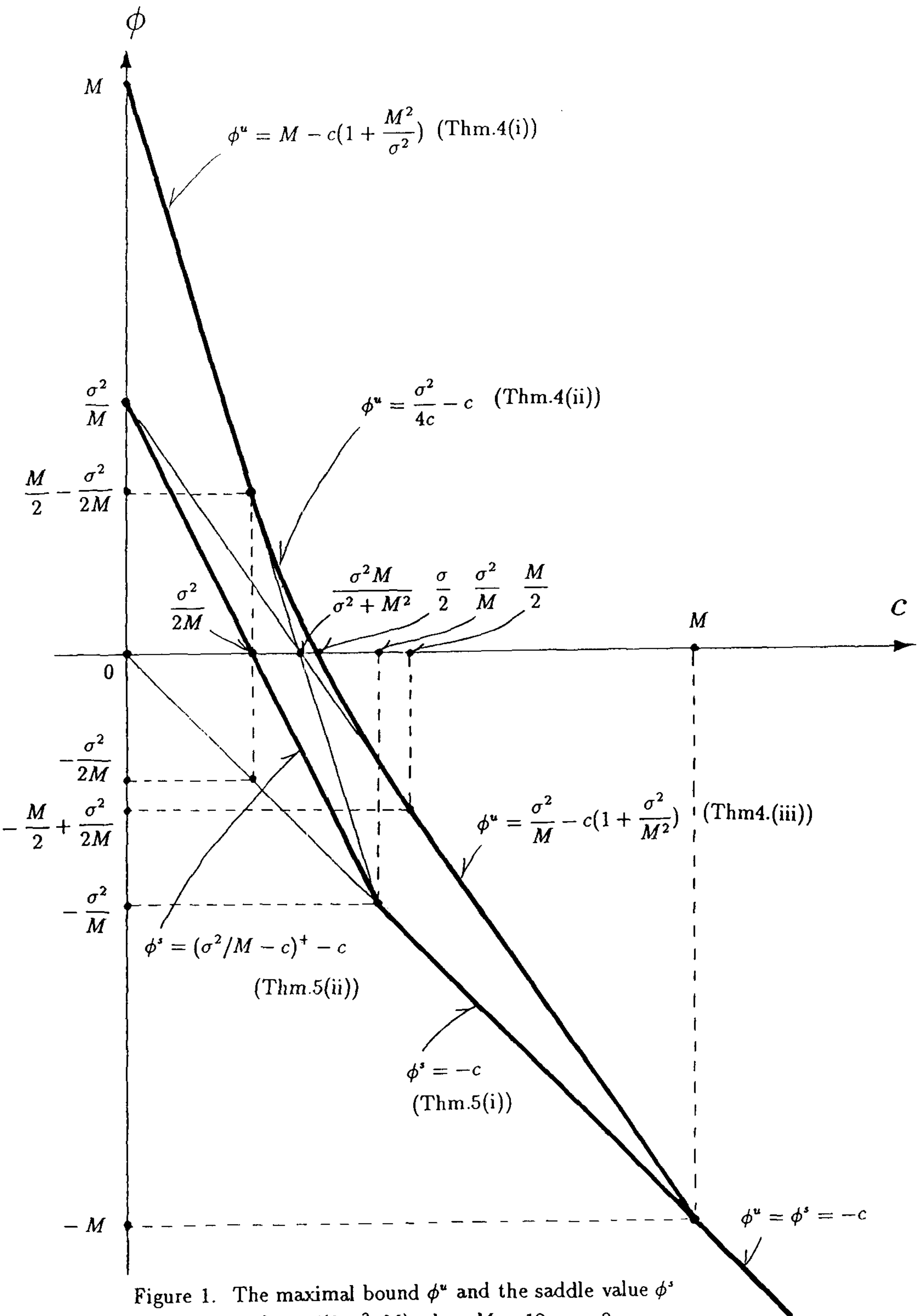

Figure 1. The maximal bound ϕ^u and the saddle value ϕ^s in the class $\mathcal{F}(0, \sigma^2, M)$ when $M = 12$, $\sigma = 8$.

Copyright © by ORSJ. Unauthorized reproduction of this article is prohibited.