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ON THE JOINT DISTRIBUTION OF THE DEPARTURE INTERVALS IN AN *M/G/1/N* QUEUE

Akihiko Ishikawa
Iwate University

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Abstract This paper discusses a stationary departure process from the $M/G/1/N$ queue. Using a Markov renewal process, we examine the joint density function f_k of the k-successive departure intervals. In Section 2, we discuss the covariance of departure intervals. The departure intervals are statistically independent in case of $N = 0$ or $N = 1$, but not in case of $N = 2$ or $N = 3$. In Section 3, f_k in the $M/M/1/N$ is shown to be a symmetric function of arrival and service rates, and we find that $cov(d_1, d_k)$ is not dependent on lag k, for $k \leq N + 1$. Further, we prove that the covariance of departure intervals in the dual (reversed) system is equal to one in the original system, for any lag k.

1. Introduction

In this paper, we discuss the departure process of a queueing system. In order to examine the covariance of departure process, we consider the joint density function of the k-successive departure intervals in the $M/G/1/N$ queue.

Many papers have been published on this subject. Burke [1] and Finch [6] have proved that the departure process in the $M/M/1$ queue is again a Poisson process. Jenkins [9] has discussed the covariance of departure process in the $M/E_\lambda/1$ queue. For the $M/G/1/N$ queue, Daley [2] and Daley & Shanbhag [3] have analyzed the departure process. Disney et al. [5], Magalhaes & Disney [11], and Simon & Disney [15] have studied the joint distribution of departure intervals by using a Markov process. Moreover, King [10] has shown that: (1) $cov(d_1, d_k) = 0$ for $k \geq 2$, in the $M/G/1/0$ and $M/D/1/1$ queue; (2) $cov(d_1, d_k) = 0$ for $k \geq 3$, in the $M/G/1/1$ queue. For the $M/M/1$ queue, Hubbard et al. [8] have noted that a probability $P(j,t)$ is a symmetrical expression with regard to an arrival rate λ and a service rate μ, where $P(j,t)$ =Pr{exactly j customers depart from the system during a time interval $[0,t)$}. Saito [14] has analyzed the departure process in an $M/G/s/0$ queue. Makino [12] has discussed a loss probability for the $M/M/1/N \to /M/1/1$ tandem queue. Furthermore, Daley [4] and Reynolds [13] have surveyed the departure process.

Using a Markov process, we examine the joint density function $f_k(x_1, x_2, \cdots, x_k)$ of the k-successive departure intervals in the $M/G/1/N$ queue. In Section 2, we discuss the covariance of departure intervals of lag k. The departure intervals are statistically independent in case of $N = 0$ or $N = 1$, but not in case of $N = 2$ or $N = 3$. Especially, in the $M/G/1/2$ queue, we find that the covariance of departure intervals of lag k are represented as a geometric progression: $cov(d_1, d_k) = \beta_1^{k-3} cov(d_1, d_3)$, for $k \geq 3$, where β_1 =Pr{exactly one customer arrives at the system during a service time}.

Furthermore in the $M/M/1/N$ queue, for $k \leq N+1$, we can see that: (1) f_k is symmetrical with regard to an arrival rate λ and a service rate μ; (2) $cov(d_1, d_k)$ does not depend on lag k. Lastly, for any lag k, we find that the covariance in the dual (reversed) system is equal to one in the original system.

2. The Stationary $M/G/1/N$ Queue
2.1 The joint density function

We consider a single server queueing system with a Poisson arrival process with rate λ, i.e. the density function $a(x) = \lambda e^{-\lambda x}$. The service times are independently and identically distributed random variables with an arbitrary probability distribution $B(x)$. Suppose that the distribution $B(x)$ has density function $b(x)$; $b(x) = dB(x)/dx$. Let us denote the mean service time by: $\frac{1}{\mu} = \int_0^\infty xb(x)dx$. Further, let $\rho = \frac{\lambda}{\mu}$ be the traffic intensity, and the queue has a capacity N (excluding in service), that is, $M/G/1/N$. If an arriving customer finds the queue is full, then he does not enter into the system. The steady state exists either for $N < \infty (0 < \rho < \infty)$ or for $N = \infty (0 < \rho < 1)$.

Let τ_ξ denote the epoch of departure of the ξ-th customer and let d_ξ denote the departure interval: $d_\xi = \tau_\xi - \tau_{\xi-1} (\xi = \cdots, -1, 0, 1, \cdots)$. Further, let Q_ξ denote the number of customers in the system just after the ξ-th customer departs.

Let us introduce some notations as follows:

$$\begin{aligned}
\Pr\{x \le d_\xi < x + dx, Q_\xi = Q_{\xi-1} + j - 2 < N \mid Q_{\xi-1} > 0\}/dx &= \frac{1}{(j-1)!}(\lambda x)^{j-1} e^{-\lambda x} b(x) \\
&= b_j(x) \qquad (j = 1, 2, \cdots, N), \\
\Pr\{x \le d_\xi < x + dx, Q_\xi = N \mid Q_{\xi-1} = N + 2 - j\}/dx &= b(x) - \sum_{k=1}^{j-1} b_k(x) \\
&= \bar{b}_j(x) \qquad (j = 2, 3, \cdots, N+1), \\
\Pr\{x \le d_\xi < x + dx \mid Q_{\xi-1} = 0\}/dx &= \int_0^x a(x-t)b(t)dt \\
&= c(x) \\
\Pr\{x \le d_\xi < x + dx, Q_\xi = j - 1 \mid Q_{\xi-1} = 0\}/dx &= \int_0^x a(x-t)b_j(t)dt \\
&= c_j(x) \qquad (j = 1, 2, \cdots, N),
\end{aligned}$$

and

$$\begin{aligned}
\Pr\{x \le d_\xi < x + dx, Q_\xi = N \mid Q_{\xi-1} = 0\}/dx &= c(x) - \sum_{k=1}^{N} c_k(x) \\
&= \bar{c}_{N+1}(x).
\end{aligned}$$

Here, it is well known that the bivariate process $\{(d_\xi, Q_\xi)\}$ is a Markov renewal process with a kernel: $U(x) = [u_{ij}(x)] \quad (1 \le i \le N+1 \text{ and } 1 \le j \le N+1)$,

where for $i = 1$;

$$\begin{aligned}
u_{1j}(x) = c_j(x) &\qquad (j = 1, 2, \cdots, N), \\
\bar{c}_{N+1}(x) &\qquad (j = N+1),
\end{aligned}$$

for $i = 2, 3, \cdots, N+1$;

$$\begin{aligned}
u_{ij}(x) = b_{j-i+2}(x) &\qquad (j = i-1, i, \cdots, N), \\
\bar{b}_{N+3-i}(x) &\qquad (j = N+1),
\end{aligned}$$

and for $i = 3, 4, \cdots, N+1$;

$$u_{ij}(x) = 0 \qquad (j = 1, 2, \cdots, i-2).$$

So we have a transition probability matrix: $U = [u_{ij}](1 \le i \le N+1 \text{ and } 1 \le j \le N+1)$,

where $\quad u_{ij} = \int_0^\infty u_{ij}(x)dx.$

Let q_n denote the probability of n customers in the system at the departure epoch in the steady state, i.e., $q_n = \lim_{\xi \to \infty} \Pr\{Q_\xi = n\}$. That is, the imbedded probability distribution $\{q_n\}$ (the stationary system-size of the departure process) is the stationary distribution of $\{Q_\xi\}$. The imbedded distribution $\{q_n\}$ is a solution of the equilibrium equation:

$$ {}^T\underline{q}U = {}^T\underline{q}, \quad (\text{where } {}^T\underline{q} = [q_0, q_1, \cdots, q_N]). $$

In general, the above imbedded distribution $\{q_n\}$ is not equal to the stationary distribution $\{p_n\}$ at an arbitrary point of time. The relation between q_n and p_n is given by:

$$ \begin{aligned} p_n &= Cq_n && (n = 0, 1, \cdots, N), \\ & 1 - C && (n = N+1), \end{aligned} $$

where $C = \dfrac{1}{q_0 + \rho}$, (see §5.1.8 in Gross & Harris [7]).

Moreover, the matrices $U(x)$ and U have the following properties:

$$ \begin{aligned} U(x)\underline{e} &= \underline{e}_1 c(x) + \sum_{i=2}^{N+1} \underline{e}_i b(x) \\ &= \underline{e}_1[c(x) - b(x)] + \underline{e}b(x), \\ U\underline{e} &= \underline{e}, \end{aligned} \tag{2.1} $$

$$ U(x)\underline{e}_j = \underline{e}_1 c_j(x) + \sum_{i=2}^{j+1} \underline{e}_i b_{j+2-i}(x) \qquad (j = 1, 2, \cdots, N), \tag{2.2} $$

and

$$ U(x)\underline{e}_{N+1} = \underline{e}_1 \bar{c}_{N+1}(x) + \sum_{i=2}^{N+1} \underline{e}_i \bar{b}_{N+3-i}(x), \tag{2.3} $$

where $\underline{e}$ is a column vector each of whose elements are unity; $\underline{e} = {}^T[1, 1, \cdots, 1]$ and $\underline{e}_j$ is the j-th fundamental vector; $\underline{e}_j = {}^T[0, 0, \cdots, 0, \underset{(j)}{1}, 0, \cdots, 0]$.

Let $f_k(x_1, x_2, \cdots, x_k)$ be the joint density function of the k-successive departure intervals, at the stationary:

$$ \begin{aligned} & f_k(x_1, x_2, \cdots, x_k)dx_1 dx_2 \cdots dx_k \\ & \quad = \lim_{\xi \to \infty} \Pr\{x_1 \le d_{\xi+1} < x_1 + dx_1, x_2 \le d_{\xi+2} < x_2 + dx_2, \cdots, x_k \le d_{\xi+k} < x_k + dx_k\}. \end{aligned} $$

In the stationary $M/G/1/N$ queue, f_k can be expressed using a matrix form:

$$ f_k(x_1, x_2, \cdots, x_k) = {}^T\underline{q}U(x_1)U(x_2)\cdots U(x_k)\underline{e}, $$

(see Disney et al. [5] and Simon & Disney [15]).

2.2 The Covariance of lag k

Let us denote the marginal density functions by:

$$f_{k+1}(\bullet, x_1, x_2, \cdots, x_k) = \int_0^\infty f_{k+1}(y, x_1, x_2, \cdots, x_k)dy$$

and

$$f_{k+1}(x_1, x_2, \cdots, x_k, \bullet) = \int_0^\infty f_{k+1}(x_1, x_2, \cdots, x_k, y)dy$$

By the stationary assumption we can write:

$$f_{k+n}(\bullet\bullet\bullet, x_1, x_2, \cdots, x_k, \bullet\bullet\bullet) = f_k(x_1, x_2, \cdots, x_k).$$

In the stationary, we abbreviate $d_\xi, d_{\xi+1}, d_{\xi+2}$ and $d_{\xi+k}$, to d, d_1, d_2 and d_k, respectively. Let $f_1(x), E(d)$ and $V(d)$ denote the density function, the expectation and the variance of d, respectively:

$$\begin{aligned} f_1(x) &= {}^T\underline{q}U(x)\underline{e} \\ &= b(x) + q_0\alpha(x), \\ E(d) &= \frac{1}{\mu} + q_0\frac{1}{\lambda} \text{ and } V(d) = V_b + \frac{1}{\lambda^2}q_0(2-q_0) \end{aligned}$$

where $\alpha(x) = c(x) - b(x)$ and $V_b = \int_0^\infty x^2 b(x)dx - \frac{1}{\mu^2}$.

Let us dentoe the degenerative density function by:

$$\begin{aligned} f_{1,k}(x_1, x_k) &= \int_0^\infty \int_0^\infty \cdots \int_0^\infty f_k(x_1, x_2, \cdots, x_k)dx_2dx_3\cdots dx_{k-1} \\ &= {}^T\underline{q}U(x_1)U^{k-2}U(x_k)\underline{e} \qquad (k \geq 2). \end{aligned}$$

Furthermore, let us denote the covariance of lag k by:

$$\begin{aligned} cov(d_1, d_k) &= E(d_1, d_k) - E(d_1)E(d_k) \\ &= \int_0^\infty \int_0^\infty x_1 x_k d\gamma_{1,k}(x_1, x_k) \qquad (k \geq 2), \end{aligned}$$

where $d\gamma_{1,k}(x_1, x_k) = f_{1,k}(x_1, x_k)dx_1dx_k - f_1(x_1)f_1(x_k)dx_1dx_k$.

In this paper, we have assumed $B(x)$ has a density. However, this assumption is not essential. If $B(x)$ has not density, then we can derive similar results for the joint cumulative distribution function F_k:

$$F_k(x_1, x_2, \cdots, x_k) = \lim_{\xi\to\infty} \Pr\{0 \leq d_{\xi+1} \leq x_1, 0 \leq d_{\xi+2} \leq x_2, \cdots, 0 \leq d_{\xi+k} \leq x_k\}.$$

2.3 Examples

(1) $N = 0$. Clearly we have: $q_0 = 1,\ U(x) = c(x),\ f_1(x) = c(x)$,

$$f_k(x_1, x_2, \cdots, x_k) = \prod_{i=1}^k f_1(x_i) \text{ and } d\gamma_{1,k}(x_1, x_k) = 0 \qquad (k \geq 2).$$

In this $M/G/1/0$ case, the departure intervals are statistically independent and the interdeparture process is a renewal process.

(2) $N = 1$. It is easily to see that:

$$q_0 = \beta_0,\ q_1 = 1 - \beta_0,\ U(x) = \begin{bmatrix} c_1(x) & \bar{c}_2(x) \\ b_1(x) & \bar{b}_2(x) \end{bmatrix},\ U = \begin{bmatrix} \beta_0 & 1-\beta_0 \\ \beta_0 & 1-\beta_0 \end{bmatrix},$$

$$d\gamma_{1,2}(x_1, x_2) = [q_0 c_1(x_1) + q_1 b_1(x_1) - q_0^2 \alpha(x_1) - q_0 b(x_1)]\alpha(x_2) dx_1 dx_2,$$

and

$$cov(d_1, d_2) = -\frac{1}{\lambda^2}(\rho\beta_0 - \beta_1),$$

where $\beta_j = \Pr\{\text{exactly } j \text{ customers arrive at the system during a service time}\}$

$$= \int_0^\infty b_{j+1}(x)dx$$
$$= \int_0^\infty c_{j+1}(x)dx \qquad (j = 0, 1, \cdots).$$

In this case the matrix U satisfies $U = \underline{e}^T \underline{q}$, so that we have:

$$\begin{aligned} f_{1,3}(x_1, x_3) &= {}^T \underline{q} U(x_1) U U(x_3) \underline{e} \\ &= {}^T \underline{q} U(x_1)(\underline{e}^T \underline{q}) U(x_3) \underline{e} \\ &= f_1(x_1) f_1(x_3) \end{aligned}$$

and

$$d\gamma_{1,3}(x_1, x_3) = 0.$$

In the same manner we have:

$$f_{1,k}(x_1, x_k) = f_1(x_1) f_1(x_k) \qquad (k \geq 3), \tag{2.4}$$

and

$$d\gamma_{1,k}(x_1, x_k) = 0 \qquad (k \geq 3). \tag{2.5}$$

The above (2.4) and (2.5) imply that the separate intervals are statistically independent in the $M/G/1/1$ queue.

If the service times are constant, then $\beta_0 = e^{-\rho}, \beta_1 = \rho e^{-\rho}$ and $cov(d_1, d_2) = 0$. So the departure intervals are statistically independent and the interdeparture process is a renewal process in the $M/D/1/1$. The above conclusion of $N = 0$ and $N = 1$ have been already shown by King [10].

(3) $N = 2$. In this part we have:

$$q_0 = \frac{1}{1-\beta_1}\beta_0^2,\ q_1 = \frac{1}{1-\beta_1}\beta_0(1-\beta_0),\ q_2 = \frac{1}{1-\beta_1}(1-\beta_0-\beta_1),$$

$$U(x) = \begin{bmatrix} c_1(x) & c_2(x) & \bar{c}_3(x) \\ b_1(x) & b_2(x) & \bar{b}_3(x) \\ 0 & b_1(x) & \bar{b}_2(x) \end{bmatrix}, U = \begin{bmatrix} \beta_0 & \beta_1 & 1-\beta_0-\beta_1 \\ \beta_0 & \beta_1 & 1-\beta_0-\beta_1 \\ 0 & \beta_0 & 1-\beta_0 \end{bmatrix},$$

$$d\gamma_{1,2}(x_1, x_2) = -[q_0^2 \alpha(x_1) + q_0 b(x_1) - q_0 c_1(x_1) - q_1 b_1(x_1)]\alpha(x_2) dx_1 dx_2,$$

and

$$cov(d_1, d_2) = -\frac{1}{\lambda^2}[q_0^2 + \rho q_0 - q_0(\beta_0 + \beta_1) - q_1\beta_1].$$

The matrix U can be represented as $U = R\Lambda R^{-1}$ where

$$R = \begin{bmatrix} 1-\beta_0-\beta_1 & 1 & \beta_1-\beta_0+\beta_0^2 \\ 1-\beta_0-\beta_1 & 1 & -\beta_0+\beta_0^2 \\ -\beta_0 & 1 & \beta_0^2 \end{bmatrix} \text{ and } \Lambda = \begin{bmatrix} \beta_1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

So that we obtain:

$$\begin{aligned} f_{1,k}(x_1,x_k) &= {}^T\underline{q}U(x_1)U^{k-2}U(x_k)\underline{e} \\ &= {}^T\underline{q}U(x_1)R\Lambda^{k-2}R^{-1}U(x_k)\underline{e} \qquad (k \geq 3), \\ d\gamma_{1,k}(x_1,x_k) &= -\beta_1^{k-3}dK(x_1)\alpha(x_k)dx_k \qquad (k \geq 3), \end{aligned}$$

and

$$cov(d_1,d_k) = \beta_1^{k-3}cov(d_1,d_3) \qquad (k \geq 3),$$

where

$$dK(x) = \{q_0^2\alpha(x) + q_0b(x) - \beta_0[q_0(c_1(x)+c_2(x)) + q_1(b_1(x)+b_2(x)) + q_2b_1(x)]\}dx,$$

and

$$cov(d_1,d_3) = -\frac{1}{\lambda^2}\{q_0^2 + \rho q_0 - \beta_0[q_0(\beta_0+2\beta_1+2\beta_2) + q_1(\beta_1+2\beta_2) + q_2\beta_1]\}.$$

In general, we see that the departure intervals are generally not independent. The results are summarized as the following theorem.

THEOREM 1. In the stationary $M/G/1/2$ queue, the covariances of lag k are given as follows:

$$\begin{aligned} & cov(d_1,d_2) = -\frac{1}{\lambda^2}[q_0^2 + \rho q_0 - q_0(\beta_0+\beta_1) - q_1\beta_1], \\ & cov(d_1,d_3) = -\frac{1}{\lambda^2}\{q_0^2 + \rho q_0 - \beta_0[q_0(\beta_0+2\beta_1+2\beta_2) + q_1(\beta_1+2\beta_2) + q_2\beta_1]\} \\ \text{and} \quad & cov(d_1,d_k) = \beta_1^{k-3}cov(d_1,d_3) \qquad (k \geq 4). \end{aligned}$$

(4) $N = 3$. In this case we have:

$$\delta_1 = \beta_1 + \sqrt{\beta_0\beta_2},\ \delta_2 = \beta_1 - \sqrt{\beta_0\beta_2},\ H = \frac{1}{(\delta_1-1)(\delta_2-1)}$$

$$q_0 = H\beta_0^3,\ q_1 = H\beta_0^2(1-\beta_0),\ q_2 = H\beta_0(1-\beta_0-\beta_1),\ q_3 = 1 - H\beta_0(1-\beta_1),$$

$$d\gamma_{1,2}(x_1,x_2) = [q_0c_1(x_1) + q_1b_1(x_1) - q_0f_1(x_1)]\alpha(x_2)dx_1dx_2,$$

$$cov(d_1,d_2) = \frac{1}{\lambda^2}[q_0(\beta_0+\beta_1) + q_1\beta_1 - q_0(q_0+\rho)]$$

and for $k \geq 3$,

$$d\gamma_{1,k}(x_1,x_k) = [\delta_1^{k-2}H_1v_1(x_1) + \delta_2^{k-2}H_2v_2(x_1)]\alpha(x_k)dx_1dx_k,$$

$$cov(d_1,d_k) = \frac{1}{\lambda}[\delta_1^{k-2}H_1E(v_1) + \delta_2^{k-2}H_2E(v_2)]$$

where

$$H_i = \frac{1}{2(\delta_i - 1)(\delta_i - \beta_1)\delta_i}$$

$$\begin{aligned}
v_i(x) &= \beta_0^3 f_1(x) + q_0[h_{i,1}c_1(x) + h_{i,2}c_2(x) + h_{i,3}c_3(x)] \\
&\quad + b_1(x)[q_1 h_{i,1} + q_2 h_{i,2} + q_3 h_{i,3}] + b_2(x)[q_1 h_{i,2} + q_2 h_{i,3}] + b_3(x) q_1 h_{i,3}, \\
E(v_i) &= \frac{1}{\lambda}\{\beta_0^3(q_0 + \rho) + q_0[h_{i,1}(\beta_0 + \beta_1) + h_{i,2}(\beta_1 + 2\beta_2) + h_{i,3}(\beta_2 + 3\beta_3)] \\
&\quad + \beta_1[q_1 h_{i,1} + q_2 h_{i,2} + q_3 h_{i,3}] + \beta_2[q_1 h_{i,2} + q_2 h_{i,3}] + \beta_3 q_1 h_{i,3}\} \\
h_{i,1} &= h_{i,2} = \beta_0(\delta_i - 1)(\beta_0 - \beta_1 + \delta_i), \; h_{i,3} = \beta_0^2(\delta_i - 1) \quad (\text{for } i = 1, 2).
\end{aligned}$$

3. The Stationary $M/M/1/N$ Queue

3.1 The joint density function

In this section we derive a closed form for $f_k(x_1, x_2, \cdots, x_k)$ in the case of exponential service times i.e. $b(x) = \mu e^{-\mu x}$. We temporally suppose that $\rho \neq 1$, but this assumption is not essential. Here, we have:

(3.1)
$$q_n = (1-\rho) H_N \rho^n \qquad (n = 0, 1, \cdots, N),$$
$$b_j(x) = \frac{1}{(j-1)!}(\lambda x)^{j-1} \mu e^{-(\lambda+\mu)x} \qquad (j = 1, 2, \cdots, N),$$

(3.2)
$$\bar{b}_j(x) = b(x) - \sum_{i=1}^{j-1} b_i(x) \qquad (j = 2, 3, \cdots, N+1),$$

(3.3)
$$c(x) = \frac{1}{1-\rho}[a(x) - \rho b(x)],$$

(3.4)
$$c_j(x) = a(x)\rho^{j-1} - \sum_{i=1}^{j} b_i(x)\rho^{j+1-i} \qquad (j = 1, 2, \cdots, N),$$

(3.5)
$$\begin{aligned}
\bar{c}_{N+1}(x) &= c(x) - \sum_{j=1}^{N} c_j(x) \\
&= c(x) - \frac{a(x)}{1-\rho}(1-\rho^N) + \sum_{j=1}^{N}\sum_{k=1}^{j} \rho^k b_{j+1-k}(x),
\end{aligned}$$

$$\alpha(x) = \frac{1}{1-\rho}[a(x) - b(x)],$$

(3.6)
$$\begin{aligned}
f_1(x) &= {}^T\underline{q}U(x)\underline{e} \\
&= H_N[a(x) - \rho^{N+1}b(x)], \\
E(d) &= \frac{1}{\lambda}H_N(1-\rho^{N+2}), \\
V(d) &= \frac{1}{\lambda^2}H_N^2[(1-\rho^{N+2})^2 - 2\rho^{N+1}(1-\rho)^2]
\end{aligned}$$

and

(3.7)
$$\begin{aligned}
f_2(x_1, x_2) &= {}^T\underline{q}U(x_1)U(x_2)\underline{e} \\
&= H_N[a(x_1)a(x_2) - \rho^{N+1}b(x_1)b(x_2)],
\end{aligned}$$

where

$$H_N = \frac{1}{1-\rho^{N+1}} = \frac{\mu^{N+1}}{\mu^{N+1}-\lambda^{N+1}}.$$

Now, we prepare two lemmas.

Lemma 1. For $j = 1, 2, \cdots, N$, we have:

$$^T\underline{q}U(x)\underline{e}_j = q_{j-1}a(x). \tag{3.8}$$

Proof: Using (2.2), (3.1) and (3.4), it is easy to show:

$$\begin{aligned}
^T\underline{q}U(x)\underline{e}_j &= {}^T\underline{q}[\underline{e}_1 c_j(x) + \sum_{i=2}^{j+1} \underline{e}_i b_{j+2-i}(x)] \\
&= q_0 c_j(x) + \sum_{i=2}^{j+1} q_{i-1} b_{j+2-i}(x) \\
&= (1-\rho)H_N[a(x)\rho^{j-1} - \sum_{i=2}^{j+1} \rho^{i-1} b_{j+2-i}(x)] + \sum_{i=2}^{j+1} q_{i-1} b_{j+2-i}(x) \\
&= q_{i-1}a(x) \qquad\qquad (j = 1, 2, \cdots, N).
\end{aligned}$$

Lemma 2. For $j = N+1$, we have:

$$^T\underline{q}U(x)\underline{e}_{N+1} = q_N c(x). \tag{3.9}$$

Proof: Using (2.2), (2.3) and (3.1) - (3.5) we have:

$$\begin{aligned}
^T\underline{q}U(x)\underline{e}_{N+1} &= {}^T\underline{q}[\underline{e}_1 \overline{c}_{N+1}(x) + \sum_{i=2}^{N+1} \underline{e}_i \overline{b}_{N+3-i}(x)] \\
&= q_0 \overline{c}_{N+1}(x) + \sum_{i=2}^{N+1} q_{i-1} \overline{b}_{N+3-i}(x) \\
&= (1-\rho)H_N\{[c(x) - \frac{1}{1-\rho}(1-\rho^N)a(x) + \sum_{j=1}^{N}\sum_{k=1}^{j} b_{j+1-k}(x)\rho^k] \\
&\qquad + \sum_{i=2}^{N+1} \rho^{i-1}[b(x) - \sum_{j=1}^{N+2-i} b_j(x)]\} \\
&= (1-\rho)H_N\{c(x) - \frac{1}{1-\rho}(1-\rho^N)[a(x) - \rho b(x)] \\
&\qquad + \sum_{k=1}^{N} \rho^k [\sum_{j=k}^{N} b_{j+1-k}(x) - \sum_{j=1}^{N+1-k} b_j(x)]\} \\
&= (1-\rho)H_N[c(x) - (1-\rho^N)c(x)] \\
&= q_N c(x).
\end{aligned}$$

From the above lemmas we obtain the following theorem.

THEOREM 2. In the stationary $M/M/1/N$ queue, for $k \leq N+1$, the joint density function can be expressed as follows:

$$(3.10) \qquad f_k(x_1, x_2, \cdots, x_k) = H_N[\prod_{i=1}^{k} a(x_i) - \rho^{N+1} \prod_{i=1}^{k} b(x_i)]$$
$$= \frac{1}{\mu^{N+1} - \lambda^{N+1}}[\mu^{N+1} \prod_{i=1}^{k} a(x_i) - \lambda^{N+1} \prod_{i=1}^{k} b(x_i)].$$

Proof: We prove (3.10) by the induction on k. From (3.6) and (3.7), for $k = 1$ and $k = 2$, (3.10) holds. Assuming its validity for $k = h(\leq N)$, we shall show (3.10) for $k = h+1$. From (2.1) and (2.2) we have:

$$U(y)\underline{e} = \underline{e}b(y) + \underline{e}_1\alpha(y)$$

and

$$U(y_1)U(y_2)\underline{e} = \underline{e}b(y_1)b(y_2) + \underline{e}_1[\alpha(y_1)b(y_2) + c_1(y_1)\alpha(y_2)] + \underline{e}_2 b_1(y_1)\alpha(y_2).$$

Thus we can write recursively,

$$U(y_1)U(y_2)\cdots U(y_h)\underline{e} = \underline{e}b(y_1)b(y_2)\cdots b(y_h) + \sum_{j=1}^{h} \underline{e}_j L_{h,j}$$

and

$$f_h(y_1, y_2, \cdots, y_h) = {}^T\underline{q}U(y_1)u(y_2)\cdots U(y_h)\underline{e}$$
$$= {}^T\underline{q}\underline{e}b(y_1)b(y_2)\cdots b(y_h) + {}^T\underline{q}\sum_{j=1}^{h} \underline{e}_j L_{h,j}$$
$$= \prod_{i=1}^{h} b(y_i) + \sum_{j=1}^{h} q_{j-1}L_{h,j},$$

where each $L_{h,j}$ is a certain unknown function.
For $h \leq N$, using the inductive hypothesis we have:

$$(3.11) \qquad \sum_{j=1}^{h} q_{j-1}L_{h,j} = f_h(y_1, y_2, \cdots, y_h) - \prod_{i=1}^{h} b(y_i)$$
$$= H_N[\prod_{i=1}^{h} a(x_i) - \rho^{N+1}\prod_{i=1}^{h} b(x_i)] - \prod_{i=1}^{h} b(x_i)$$
$$= H_N[\prod_{i=1}^{h} a(x_i) - \prod_{i=1}^{h} b(x_i)] \qquad (h \leq N).$$

Considering the function of the $(h+1)$-successive departure intervals $x, y_1, y_2, \cdots, y_h$, only $h \leq N$, from (2.1), (3.8) and (3.11) we obtain:

$$(3.12) \qquad f_{h+1}(x, y_1, y_2, \cdots, y_h) = {}^T\underline{q}U(x)U(y_1)U(y_2)\cdots U(y_h)\underline{e}$$
$$= {}^T\underline{q}U(x)\{\underline{e}\prod_{i=1}^{h} b(y_i) + \sum_{j=1}^{h} \underline{e}_j L_{h,j}\}$$

$$
\begin{aligned}
&=^T \underline{q}[\underline{e}b(x) + \underline{e}_1\alpha(x)] \prod_{i=1}^{h} b(y_i) + \sum_{j=1}^{h} q_{j-1}a(x)L_{h,j} \\
&= [b(x) + q_0\alpha(x)] \prod_{i=1}^{h} b(y_i) + a(x)H_N[\prod_{i=1}^{h} a(y_i) - \prod_{i=1}^{h} b(y_i)] \\
&= H_N\{a(x)\prod_{i=1}^{h} a(y_i) + [(1-\rho^{N+1})b(x) - b(x)] \prod_{i=1}^{h} b(y_i)\} \\
&= H_N[a(x)\prod_{i=1}^{h} a(y_i) - \rho^{N+1}b(x)\prod_{i=1}^{h} b(y_i)].
\end{aligned}
$$

Obviously, (3.12) is just the same as (3.10) for $k = h + 1$.

In the above calculations, we need not refer to (3.9) because of $k \leq N+1$. In case of $k \geq N+2$, we must use (3.9), and for $k = N+2$ we have:

$$
\begin{aligned}
f_{N+2}(x_1, x_2, \cdots, x_{N+2}) &=^T \underline{q}U(x_1)U(x_2)\cdots U(x_{N+2})\underline{e} \\
&=^T \underline{q}U(x_1)[\underline{e}\prod_{i=2}^{N+2}(x_i) + \sum_{j=1}^{N+1} \underline{e}_j L_{N+1,j}(x_2, x_3, \cdots, x_{N+2})] \\
&=^T \underline{q}[\underline{e}b(x_1) + \underline{e}_1\alpha(x_1)] \prod_{i=2}^{N+2} b(x_i) + \sum_{j=1}^{N} q_{j-1}a(x_1)L_{N+1,j} \\
&\qquad + q_N c(x_1)L_{N+1,N+1} \\
&= [b(x_1) + q_0\alpha(x_1)] \prod_{i=2}^{N+2} b(x_i) + a(x_1)H_N[\prod_{i=2}^{N+2} a(x_i) - \prod_{i=2}^{N+2} b(x_i)] \\
&\qquad + q_N[c(x_1) - a(x_1)]L_{N+1,N+1} \\
&= H_N[\prod_{i=1}^{N+2} a(x_i) - \rho^{N+1}\prod_{i=1}^{N+2} b(x_i)] + R_{N+2},
\end{aligned}
$$

where

$$
R_{N+2}(x_1, x_2, \cdots, x_{N+2}) = q_N[c(x_1) - a(x_1)]L_{N+1,N+1}(x_2, x_3, \cdots, x_{N+2}).
$$

REMARK 1. In case of $k \geq N+2$, we have:

$$
f_k(x_1, x_2, \cdots, x_k) = H_N[\prod_{i=1}^{k} a(x_i) - \rho^{N+1}\prod_{i=1}^{k} b(x_i)] + R_k(x_1, x_2, \cdots, x_k)
$$

where R_k is some unknown function.

3.2 The covariance of the departure intervals

On the basis of Theorem 2, we obtain the following results.

Proposition 1. In the stationary $M/M/1/N$ queue, for $k \leq N+1$ we have:

$$
f_{1,k}(x_1, x_k) = H_N[a(x_1)a(x_k) - \rho^{N+1}b(x_1)b(x_k)]
$$

and

$$\gamma_{1,k}(x_1, x_k) = -H_N^2 \rho^{N+1}[a(x_1) - b(x_1)][a(x_k) - b(x_k)].$$

Proposition 2. In the stationary $M/M/1/N$ queue, for $k \leq N+1$ the covariance $cov(d_1, d_k)$ is independent of lag k, as follows:

$$\begin{aligned} cov(d_1, d_k) &= -\frac{1}{\lambda^2} H_N^2 (1-\rho)^2 \rho^{N+1} \\ &= -[\frac{\lambda - \mu}{\lambda^{N+1} - \mu^{N+1}}]^2 (\lambda\mu)^{N-1}. \end{aligned}$$

REMARK 2. Supposing that $\rho < 1$. Then for arbitrary k, the limit exists:

$$\begin{aligned} \lim_{N\to\infty} f_k(x_1, x_2, \cdots, x_k) &= \lim_{N\to\infty} \frac{1}{1-\rho^{N+1}} [\prod_{i=1}^{k} a(x_i) - \rho^{N+1} \prod_{i=1}^{k} b(x_i)] \\ &= \prod_{i=1}^{k} a(x_i). \end{aligned}$$

This conclusion has been already discussed in [1], [6] and [10].

Noting that (3.10) is a symmetrical expression with regard to λ and μ, we consider $M(\lambda)/M(\mu)/1/N$ and its dual $M(\mu)/M(\lambda)/1/N$. In the dual (reversed) system, let $f_k^*(x_1, x_2, \cdots, x_k)$ denote the joint density function corresponding to $f_k(x_1, x_2, \cdots, x_k)$. Similarly, let $cov(d_1^*, d_k^*)$ denote the covariance of lag k in the dual system. The relations between the dual systems are given as follows:

THEOREM 3. In the dual systems, for any k we have:

$$f_k^*(x_1, x_2, \cdots, x_k) = f_k(x_k, x_{k-1} \cdots, x_1)$$

and

$$cov(d_1^*, d_k^*) = cov(d_1, d_k).$$

Proof: See Appendix.

For $k \leq N+1$, from (3.10) we obtain the following relation.

Proposition 3. For $k \leq N+1$, the two functions f_k and f_k^* equal to one another: $f_k(x_1, x_2, \cdots, x_k) = f_k^*(x_1, x_2, \cdots, x_k)$.

3.3 In case of $\rho = 1$

In this case, the results are given as follows:

$$\begin{aligned} q_n &= \frac{1}{N+1} \qquad (n = 0, 1, \cdots, N), \\ c(x) &= \lambda^2 x e^{-\lambda x}, \qquad \alpha(x) = \lambda(\lambda x - 1)e^{-\lambda x}, \\ f_1(x) &= q_0 \lambda(\lambda x + N)e^{-\lambda x}, \\ E(d) &= \frac{1}{\lambda} q_0 (N+2), \qquad V(d) = \frac{1}{\lambda^2} q_0^2 (N^2 + 4N + 2). \end{aligned}$$

And for $k \leq N+1$:

$$f_k(x_1, x_2, \cdots, x_k) = q_0[N+1-k+\lambda(x_1+x_2+\cdots+x_k)] \prod_{i=1}^{k} a(x_i),$$

$$d\gamma_{1.k}(x_1, x_k) = -q_0^2[\lambda x_1 - 1]a(x_1)[\lambda x_k - 1]a(x_k)dx_1dx_k,$$

and

$$cov(d_1, d_k) = -\frac{1}{\lambda^2}q_0^2.$$

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References

[1] P.J. Burke: The output of a queuing system. *Operations Research*, Vol. 4 (1956), 699–704.

[2] D.J. Daley: The correlation structure of the output processes of some single server queuing systems. *The Annals of Mathematical Statistics*, Vol. 39 (1968), 1007–1019.

[3] D.J. Daley and D.N. Shanbhag: Independent inter-departure times in $M/G/1/N$ queues. *J.R. Statist. Soc. B*, Vol. 37 (1975), 259-263.

[4] D.J. Daley: Queueing output processes. *Adv. Appl. Prob.*, Vol. 8 (1976), 395–415.

[5] R.L. Disney, R.L. Farrell and P.R. de Morais: A characterization of $M/G/1$ queues with renewal departure processes. *Management Science*, Vol. 19 (1973), 1222–1228.

[6] P.D. Finch: The output process of the queueing system $M/G/1$. *J. R. Statist. Soc. B*, Vol. 21 (1959), 375–380.

[7] D. Gross and C.M. Harris: FUNDAMENTALS OF QUEUEING THEORY. John Wiley, New York, 1985.

[8] J.R. Hubbard, C.D. Pegden and M. Rosenshine: The departure process for the $M/M/1$ queue. *J. Appl. Prob.*, Vol. 23 (1986), 249–255.

[9] J.H. Jenkins: On the correlation structure of the departure process of the $M/E_\lambda/1$ queue. *J. R. Statist. Soc. B*, Vol. 28 (1966), 336–344.

[10] R.A. King: The covariance structure of the departure process from $M/G/1$ queues with finite waiting lines. *J. R. Statist. Soc. B*, Vol. 33 (1971), 401–405.

[11] M.N. Magalhaes and R.L. Disney: Departures from queues with changeover times. *Queueing Systems*, Vol. 5 (1989), 295–312.

[12] T. Makino: Quasi loss probability and quasi throughput of the system $M/M/1/N \rightarrow /M/1/1$. *SUT Journal of Mathematics*, Vol. 26, No. 1 (1990), 101–109.

[13] J.F. Reynolds: The covariance structure of queues and related processes — a survey of recent work. *Adv. Appl. Prob.*, Vol. 7 (1975), 383–415.

[14] H. Saito: The output of loss systems with general service time distributions. *Operations Research Letters*, Vol. 7 (1988), 321–324.

[15] B. Simon and R.L. Disney: Markov renewal processes and renewal processes: Some conditions for equivalence. *New Zealand Oper. Res.*, Vol. 12 (1984), 19–29.

Appendix

(proof of Theorem 3)

For the dual system i.e. $M(\mu)/M(\lambda)/1/N$ queue, let $\underline{q}^*$ denote the vector of the imbeded probability, and $U^*(x)$ denote the kernel, corresponding to $\underline{q}$ and $U(x)$, respectively. Naturally, we suppose $\rho \neq 1$, then we have:

$$^T\underline{q}^* = H_N(1-\rho)[\rho^N, \rho^{N-1}, \cdots, \rho, 1]$$

and

$$U^*(x) = [u_{ij}^*(x)] \qquad (i = 1,2,\cdots,N+1 \text{ and } j = 1,2,\cdots,N+1),$$

where for $i = 1$;

$$\begin{aligned} u_{1j}^*(x) &= \rho^{1-j}\overline{b}_{j+1}(x) && (j = 1,2,\cdots,N), \\ &\rho^{-N}\overline{c}_{N+1}(x) && (j = N+1), \end{aligned}$$

for $i = 2,3,\cdots,N+1$;

$$\begin{aligned} u_{ij}^*(x) &= \rho^{i-j}b_{j-i+2}(x) && (j = i-1, i, \cdots, N), \\ &\rho^{i-N-1}c_{N+2-i}(x) && (j = N+1), \end{aligned}$$

and for $i = 3,4,\cdots,N+1$;

$$u_{ij}^*(x) = 0 \qquad (j = 1,2,\cdots,i-2).$$

Let us denote a transform matrix Z by:

$$Z = [z_{ij}] \qquad (i = 1,2,\cdots,N+1 \text{ and } j = 1,2,\cdots,N+1),$$

where

$$\begin{aligned} z_{ij} =& \rho^{i-1} && (i+j = N+2), \\ & 0 && (\text{otherwise}). \end{aligned}$$

It is easy to see that:

$$Z^{-1} = \rho^{-N}Z,\ (^T\underline{q}^*)Z = \{\rho^N H_N(1-\rho)\}^T\underline{e},\ \text{and } \rho^N H_N(1-\rho)Z^{-1}\underline{e} = \underline{q}.$$

Here, we note that:

$$\begin{aligned} [Z^TU(x)Z^{-1}]_{ij} &= \rho^{-N}[Z^TU(x)Z]_{ij} \\ &= \rho^{-N}z_{i,N+2-i}u_{N+2-j,N+2-i}(x)z_{N+2-j,j} \\ &= \rho^{i-j}u_{N+2-j,N+2-i}(x). \end{aligned}$$

Each element becomes as follows.

$$\begin{aligned} \text{For } i = 1 \text{ and } j = N+1 : [Z^TU(x)Z^{-1}]_{1,N+1} &= \rho^{-N}u_{1,N+1}(x) \\ &= \rho^{-N}\overline{c}_{N+1}(x) \\ &= u_{1,N+1}^*(x). \end{aligned}$$

$$\begin{aligned} \text{For } i = 1 \text{ and } 1 \leq j \leq N : [Z^TU(x)Z^{-1}]_{1j} &= \rho^{1-j}u_{N+2-j,N+1}(x) \\ &= \rho^{1-j}\overline{b}_{j+1}(x) \\ &= u_{1j}^*(x). \end{aligned}$$

$$\text{For } 2 \le i \le N+1 \text{ and } i-1 \le j \le N : [Z^T U(x) Z^{-1}]_{ij} = \rho^{i-j} u_{N+2-j,N+2-i}(x)$$
$$= \rho^{i-j} b_{j+2-i}(x)$$
$$= u^*_{ij}(x).$$

$$\text{For } 2 \le i \le N+1 \text{ and } j = N+1 : [Z^T U(x) Z^{-1}]_{i,N+1} = \rho^{i-N-1} u_{1,N+2-i}(x)$$
$$= \rho^{i-N-1} c_{N+2-i}(x)$$
$$= u^*_{i,N+1}(x).$$

$$\text{For } 3 \le i \le N+1 \text{ and } 1 \le j \le i-2 : [Z^T U(x) Z^{-1}]_{ij} = \rho^{i-j} u_{N+2-j,N+2-i}(x)$$
$$= 0$$
$$= u^*_{ij}(x).$$

Therefore, we get:

$$U^*(x) = Z^T U(x) Z^{-1}.$$

for any k, we obtain:

$$\begin{aligned} f^*_k(x_1, x_2, \cdots, x_k) &=^T \underline{q}^* U^*(x_1) U^*(x_2) \cdots U^*(x_k) \underline{e} \\ &=^T \underline{q}^* (Z^T U(x_1) Z^{-1})(Z^T U(x_2) Z^{-1} \cdots (Z^T U(x_k) Z^{-1} \underline{e} \\ &=^T \underline{q}^* Z \{^T U(x_1)^T U(x_2) \cdots {}^T U(x_k)\} Z^{-1} \underline{e} \\ &=^T \underline{e} \{^T U(x_1)^T U(x_2) \cdots {}^T U(x_k)\} \underline{q} \\ &=^T \{^T \underline{q} U(x_k) U(x_{k-1}) \cdots U(x_2) U(x_1) \underline{e}\} \\ &= f_k(x_k, x_{k-1}, \cdots, x_2, x_1). \end{aligned}$$

Using another expression, we have:

$$\Pr\{d^*_1 \le x_1, d^*_2, \le x_2, \cdots, d^*_k \le x_k\} = \Pr\{d_1 \le x_k, d_2 \le x_{k-1}, \cdots, d_k \le x_1\}$$

and

$$\begin{aligned} cov(d^*_1, d^*_k) &= cov(d_k, d_1) \\ &= cov(d_1, d_k). \end{aligned}$$

So the theorem is proved.

Akihiko Ishikawa
Fundamentals of Natural Science
College of Humanities
and Social Sciences
Iwate University
Ueda 3-18-34, Morioka-shi,
Iwate 020, JAPAN