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COMPUTATIONAL ASPECTS OF BULK-SERVICE QUEUEING SYSTEM WITH VARIABLE CAPACITY AND FINITE WAITING SPACE: $M/G^Y/1/N+B$

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Abstract In this paper, a comparative analysis of the computational aspects of a bulk-service queueing system with variable capacity and finite waiting space is carried out using the Jacobi method of iteration and the rootfinding method. Steady-state probabilities and moments of the number of customers in system at post-departure epochs have been obtained. In the special case when service is performed in batches of fixed size B, a set of relations among post-departure, random and pre-arrival epochs probabilities has also been obtained. For this special case, we present various performance measures such as moments of number in system at random epoch, probability of blocking, probability that server is busy, etcetera. A variety of numerical results have been obtained for several service-time distributions: Erlang, deterministic and hyperexponential. It was finally observed that the Jacobi method is less cumbersome than the rootfinding method.

1. Introduction

The analytic and computational aspects of single-server infinite-waiting-space queueing systems with bulk-service have been discussed by several authors, see e.g., Neuts [14], and Brière and Chaudhry [4]. However, in real life, we often encounter queues with finite waiting space which cannot always be approximated by queues with infinite waiting space. Besides, queues run most efficiently when the traffic intensity is unity. Although numerical results for batch-arrival queueing systems with finite waiting space are available, see e.g., Manfield [13], Baba [1] and Nobel [16], unfortunately, not much seems to have been done computationally on the corresponding bulk-service finite-waiting-space queueing models despite the fact that analytical results on such models have been available for quite some time and they have a wide range of applications in transportation, computers, communications, production process, finite dam, and other areas.

The first analytic study related to bulk-service, finite-waiting-space queues seems to have been carried out by Finch [9] who analyzes the $M/M^B/1/NB+B$ queueing system. Singh [18] discusses the more general model by allowing variable service capacity and arbitrary service-time distribution. Bagchi and Templeton [2,3] have carried out the analytic analysis of the more general model such as $M^X/G^Y/1/N+1$ queueing system.

In recent years, there has been a lot of criticism about the applicability of these available analytic results. A practitioner who often finds the solution of many important queueing models in terms of probability generating function (p.g.f.) or in some other complicated mathematical form, has no easy way of implementing the models in practical situations and therefore he ends up with the question: what good is the model, if its implementation is difficult? Amongst others, Cohen [8, p. 640] states that a rather neglected area in queueing theory is the development of algorithmic and numerical methods, though some systematic work, in this direction, has now been started, see e.g. the work of Neuts [15] using the phase technique, Brière and Chaudhry [4] using roots.

The objective of this paper is to do computational analysis of the bulk-service queueing system with variable capacity and finite waiting space discussed by Singh [18,19] who obtains the p.g.f. of the number in system left behind by a departing batch. It was thought that one could obtain the complete probability distribution and moments from the probability generating function. But numerical experiments have shown that it is difficult, if not impossible, to obtain any information regarding various queue characteristics from the given probability generating function. Similar remarks also apply to the model proposed by Lwin and Ghosal [12].

We use the Jacobi iterative method to solve simultaneous equations of the model under discussion. It not only gives accurate results, but they can be obtained within a reasonable amount of time, even on a PC. We also tried the Gauss-Seidel iterative method. For small values of N (see later for its definition), it generally gave good results at a faster rate. Unfortunately, as N gets large, it took too long and too many iterations to converge. This observation has also been made by Chu [7] in another application. In view of this difficulty, we preferred to use the Jacobi iterative method which converged in all the cases that we tested. Further, a set of relationships among post-departure epoch (p.d.e.), pre-arrival epoch (p.a.e.) and random epoch (r.e.) probabilities has been obtained in the special case when service is performed in batches of fixed size B. The model with batches of fixed size is analytically more tractable than the more general model of variable service capacity. From the p.d.e. probabilities of $M/G^{[B]}/1/N+B$, where service is rendered in batches of fixed size B, we also obtain p.d.e. probabilities of the number of phases for the model $E_B/G/1/N+B$ which has been studied analytically by Truslove [20] and Hokstad [11]. They, however, do not give any numerical results. In the special case when waiting space is equal to batch size, $M/G^B/1/B+B$, an exact analytical result has been obtained. For numerical purposes (see tables), the service-time distribution has been taken as exponential (M), Erlang (E_k), deterministic (D) and hyperexponential (HE_2) which cover a wide range of distributions that may arise in practice. The major contribution of this paper is numerical with some important analytical results interspersed.

2. The Model $M/G^Y/1/N+B$

Analytically, the model is well described in Chaudhry and Templeton [5, p. 186] and we follow their notations. For the sake of completeness, necessary details are also given here. Customers arrive one at a time according to a Poisson process with parameter λ. The customers are served in batches of variable capacity with maximum capacity B, i.e., not more than B customers can be served at any time. Let $\sigma_0 = 0, \sigma_1, \sigma_2, \ldots, \sigma_n, \ldots$ be the epochs of departure of the successive batches. The service times, V_n, are independently identically distributed random variables (i.i.d.r.vs.) with distribution function (D.F.) $B(v)$. The sequence $\{V_n\}$ is independent of the arrival process. If Y_n customers are already present with the server at epoch σ_n, then the server takes $min(B - Y_n, \mathit{whole\ \ queue\ \ length})$ at σ_n. Suppose that Y_n are i.i.d.r.vs. with distribution given by

$$P(Y_n = m) = \begin{cases} b_m, & 0 \le m \le B \\ 0, & m > B, \end{cases}$$

where $\sum_0^B b_m = 1$.

The waiting room has a fixed capacity N for customers who wait for service while another B customers are being served so that $N + B$ customers can be present in the system at any time. Furthermore, the maximum number of customers in the system at a departure epoch of a batch is N.

As stated earlier, this paper starts where many others stopped, i.e., for the model under consideration the equations giving the p.d.e. probabilities are (for details, see Chaudhry and Templeton [5, pp. 187-189]):

$$P_j^+ = k_j \sum_{i=0}^{B-1} P_i^+ \phi_{B-i} + \sum_{i=1}^{B-1} P_i^+ \sum_{r=B-i+1}^{B} k_{j-i+B-r} b_r$$

(2.1) $$+ \sum_{i=B}^{N} P_i^+ \sum_{r=0}^{B} k_{j-i+B-r} b_r, \quad j = 0, 1, 2, \ldots, N-1$$

$$P_N^+ = l_N \sum_{i=0}^{B-1} P_i^+ \phi_{B-i} + \sum_{i=1}^{B-1} P_i^+ \sum_{r=B-i+1}^{B} l_{N-i+B-r} b_r$$

(2.2) $$+ \sum_{i=B}^{N} P_i^+ \sum_{r=0}^{B} l_{N-i+B-r} b_r$$

with

(2.3) $$Q(z) = \frac{\sum_{i=0}^{B-1} P_i^+ \{z^B \phi_{B-i} - z^i \Phi_{B-i}(z)\}}{z^B / K(z) - \Phi_B(z)},$$

where N_n^+ = number of customers in the system immediately after the n^{th} batch departure, and

$$P_j^+ = \lim_{n\to\infty} P_r(N_n^+ = j), \quad P^+(z) = \sum_{i=0}^{N} P_j^+ z^j;$$

$$\phi_j = P(Y_n \le j) = \sum_{i=0}^{j} b_i, \quad \Phi_j(z) = \sum_{i=0}^{j} b_i z^i; \quad \Phi_B(1) = \phi_B = 1, \quad \phi_0 = b_0;$$

$$k_j = \int_0^\infty \frac{e^{-\lambda v}(\lambda v)^j}{j!} dB(v), \quad K(z) = \sum_{i=0}^{\infty} k_i z^i = \bar{b}(\lambda - \lambda z), \quad \bar{b}(s) = \int_0^\infty e^{-sv} dB(v);$$

$$l_r = \sum_{i=r}^{\infty} k_i, \quad \rho = \frac{-\lambda \bar{b}^{(1)}(0)}{\overline{B}}, \quad \overline{B} = B - \Phi_B^{(1)}(1) = B - \sum_{i=0}^{B} i b_i.$$

It should be remarked that the actual expression of the p.g.f. $P^+(z)$ involves two parts, the first part gives the required terms $\{z^j\}_0^N$ and the second the terms $\{z^j\}_{N+B}^\infty$ which are not required. The first part of the p.g.f. $P^+(z)$ is denoted as $Q(z)$ and given in (2.3). If the expression (2.3) is expanded in a power series in some suitable region of convergence, then P_j^+, $j = 0, 1, 2, \ldots, N$, the coefficient of z^j can be obtained. However, this is not simple except in the case of exponential service times or single service.

Obtaining the roots of the denominator of (2.3) is not a problem and they can be easily found by the Chaudhry QROOT [6] software package for values of $\rho \lesseqgtr 1$. But the problem is with the numerator of (2.3) which involves the unknown probabilities $P_0^+, P_1^+, P_2^+, \ldots, P_{B-1}^+$. This difficultly may be explained as follows. Suppose we keep the unknowns and use partial fractions of (2.3). This not only leads to solving all the $N+1$ equations, but makes the first B equations redundant. This is explained in the appendix by means of a simple example, i.e., $M/M^3/1/7$. All this implies that even after obtaining the zeros of the denominator of (2.3) and then expanding $Q(z)$ using partial fractions, the problem reduces to a set of $N+1$ simultaneous equations whose solution, in turn, requires a numerical procedure if N is quite large. In fact, in the case of the finite-waiting-space bulk-service general model, the p.g.f. of the number in system at a departure epoch leads to a lengthier solution procedure. Besides, it does not even give moments as are generally obtained very easily using the infinite-waiting-space model, see, e.g., Brière and Chaudhry [4]. So the problem can , in general, be best dealt with using the Jacobi method (or the Gauss-Seidel method which, as stated earlier, works better for small values of N) on the equations given in (2.1) and (2.2) which, in fact, lead to very accurate results that are given in the attached tables.

3. Special Cases of the Model $M/G^Y/1/N+B$

(i) $M/G^B/1/N+B$

Here the service is rendered in batches of size less than or equal to B.
Assuming

$$b_r = \begin{cases} 0, & r \neq 0 \\ 1, & r = 0 \end{cases}$$

so that $\phi_{B-i} = 1 = \Phi_{B-i}(z) = \Phi_B(z)$, the model $M/G^Y/1/N+B$ reduces to $M/G^B/1/N+B$.

(ii) $M/G^{[B]}/1/N+B$

If we assume that the service is performed in batches of fixed size B, then the new model is denoted by $M/G^{[B]}/1/N+B$. As in the infinite-waiting-space model, the p.d.e. probabilities for cases (i) and (ii) are the same. Besides, in this case, it is easy to get the relationship among p.d.e, p.a.e. and r.e. probabilities. Let P_j^-, P_j and P_j^+ be the p.a.e., r.e. and p.d.e. probabilities, respectively. Now, first note that since the arrivals follow the Poisson process, $\{P_j\}_0^{N+B} = \{P_j^-\}_0^{N+B}$. This well known fact is used in the derivation of the result given in (3.5) below. The relationship between the probabilities P_j^- and P_j^+ is given in the following theorem.

Theorem: The probability distributions $\{P_j^-\}_0^{N+B}$ and $\{P_j^+\}_0^{N+B}$ are related to one another by

$$P_j^- = \frac{\mu}{\lambda - \mu P^+ + B\mu} \sum_{i=j-B+1}^{j} P_i^+, \quad 0 \le j \le N+B-1$$

and

$$P_{N+B}^- = \frac{\lambda - \mu P^+}{\lambda - \mu P^+ + B\mu},$$

where

$$P^+ = \sum_{j=B}^{N+B-1} \sum_{i=0}^{B-1} P_{j-i}^+ = \sum_{j=1}^{B-1} j P_j^+ + \sum_{j=B}^{N} B P_j^+.$$

Proof: Define

$$N_s = \# \text{ in the system}, \quad N_{svc} = \# \text{ in service, and} \quad N_q = \# \text{ in queue}$$

so that

$$N_s = N_{svc} + N_q, \tag{3.1}$$

where

$$N_{svc} = \begin{cases} 0, & 0 \le N_s \le B-1 \\ B, & N_s \ge B. \end{cases}$$

In steady state, let N_A and N_D be the total number of steps in the arrival and departure processes, respectively, over a given long period of time. Using the level crossing method due to Foster and Perera [10] or Shanthikumar and Chandra [17], in the steady state, the number of times a given level is crossed in the arrival process can differ at most by one from the number of times it is crossed in the departure or service process, i.e., the rate at which it is crossed from above and the rate at which it is crossed from below must be equal. It is the product of the p.a.e. probabilities with N_A which gives the rate up; the rate down is the product of N_D with p.d.e. probabilities, i.e., the general relationship for the rates of crossing level j is:

$$N_A P_j^- = N_D \sum_{i=0}^{B-1} P_{j-i}^+, \quad 0 \le j \le N+B-1 \tag{3.2}$$

with boundary conditions:

$$P_j^-,\ P_j^+ = 0, \quad j < 0; \quad P_j^- = 0, \quad j > N+B; \text{ and } \quad P_j^+ = 0, \quad j > N.$$

Summing both sides of (3.2) over j from 0 to $N+B-1$, we get

$$N_A \sum_{j=0}^{N+B-1} P_j^- = N_D \sum_{j=0}^{N+B-1} \sum_{i=0}^{B-1} P_{j-i}^+$$

or

$$N_A(1 - P_{N+B}^-) = N_D B$$

or

$$\frac{N_D}{N_A} = \frac{1 - P_{N+B}^-}{B}. \tag{3.3}$$

So from (3.2), the p.a.e. probabilities can be written as

$$P_j^- = \frac{1 - P_{N+B}^-}{B} \sum_{i=0}^{B-1} P_{j-i}^+, \quad 0 \le j \le N+B-1. \tag{3.4}$$

Now, using the argument that, in steady-state, customers must depart the system at the same rate at which they join it,

$$\lambda(1 - P_{N+B}^-) = \mu B \sum_{j=B}^{N+B} P_j^-. \tag{3.5}$$

While the left-hand side of (3.5) represents the effective arrival rate, the summation on the right-hand side gives the probability that the server is busy at arrival epoch and hence, in this model, at random epoch. Substituting P_j^- from (3.4) into (3.5),

$$\lambda(1 - P_{N+B}^-) = \mu(1 - P_{N+B}^-) \sum_{j=B}^{N+B-1} \sum_{i=0}^{B-1} P_{j-i}^+ + \mu B P_{N+B}^-. \tag{3.6}$$

Letting

$$\sum_{j=B}^{N+B-1} \sum_{i=0}^{B-1} P_{j-i}^+ = \sum_{j=1}^{B-1} j P_j^+ + \sum_{j=B}^{N} B P_j^+ \equiv P^+ \tag{3.7}$$

and using (3.7) in (3.6), we have

$$P_{N+B}^- = \frac{\lambda - \mu P^+}{\lambda - \mu P^+ + \mu B}.$$

Interestingly, the service time distribution only enters into this through the value of P^+. The remaining p.a.e./r.e. probabilities can be expressed as

$$P_j^- = \frac{\mu}{\lambda - \mu P^+ + \mu B} \sum_{i=j-B+1}^{j} P_i^+, \quad 0 \le j \le N+B-1.$$

Hence the stated result.

Once we have the p.a.e./r.e. probabilities $\{P_j^-\}_0^{N+B}$, the performance measures such as probability of blocking (PBL) and probability server is busy (PB) are given by P_{N+B}^- and

$1-P_0$, respectively. Also, the expected value (L) and the variance of the number in system can be easily obtained using

$$L \equiv E(N_s) = \sum_{n=0}^{N+B} nP_n$$
$$Var(N_s) = \sum_{n=0}^{N+B} n^2 P_n - E(N_s)^2.$$

(iii) $E_B/G/1/N+B$ from $M/G^{[B]}/1/N+B$

Whereas the model $M/G^{[B]}/1/N+B$ gives the distribution of the number in system at a p.d.e., $E_B/G/1/N+B$ gives the number of phases at the corresponding epoch. The relationship

$$P_n^+ = \sum_{j=nB}^{nB+B-1} P_j^+, \quad n \geq 0$$

gives the probability of n in the system for $E_B/G/1/N+1$ at a departure epoch.

(iv) $M/G^{[B]}/1/B+B$

In this case, the number of waiting spaces, i.e., the number in queue is equal to the fixed capacity of the service batch. It can be easily seen that from equation (2.1) we get an explicit analytical expression for P_j^+:

$$P_j^+ = k_j, \quad 0 \leq j \leq B-1$$
$$P_B^+ = 1 - \sum_{j=0}^{B-1} k_j.$$

4. Service Time Distributions

In this section, we give the expression for k_n for various service-time distributions.

(i) Erlang (E_k)

Here, the service-time distribution with k-exponential phases is such that each phase has a mean $1/\mu$. As the mean service time for E_k is k/μ, the utilization factor ρ becomes $\frac{\lambda k}{\mu B}$. This gives

$$K(z) = \left(1 + \frac{\rho \overline{B}}{k}(1-z)\right)^{-k},$$

which is the p.g.f. of a random variable following a negative binomial distribution with parameters k and $k/(k+\overline{B}\rho)$. This yields

$$k_0 = \left(k/(k+\overline{B}\rho)\right)^k$$
$$k_n = k_{n-1}(n+k-1)\overline{B}\rho \Big/ n(\overline{B}\rho + k), \quad n \geq 1.$$

Clearly, $k=1$ in the above case gives the results for exponential service-time distribution.

(ii) Deterministic (D)

In this case, the service time has a constant value, $1/\mu$, which implies that

$$K(z) = e^{-\overline{B}\rho(1-z)},$$

whence

$$k_0 = e^{-\overline{B}\rho}$$
$$k_n = k_{n-1}\overline{B}\rho/n, \quad n \geq 1.$$

(iii) Hyperexponential (HE_2)

In this case, the service time follows a two-phase hyperexponential where one phase has a mean of $1/\mu_1$ and the other of $1/\mu_2$. Also, two parameters σ_1 and σ_2 are such that $\sigma_1+\sigma_2=1$. If we define $\rho_1=\frac{\lambda}{\mu_1 B}$ and $\rho_2=\frac{\lambda}{\mu_2 B}$, then $\rho=\sigma_1\rho_1+\sigma_2\rho_2$. Given this information, we have

$$K(z)=\sum_{i=1}^{2}\frac{\mu_i\sigma_i}{\mu_i+\lambda(1-z)}.$$

This gives

$$k_0=\frac{1+\overline{B}\rho'}{1+(\rho_1+\rho_2)\overline{B}+\rho_1\rho_2\overline{B}^2}$$

$$k_1=\frac{-\overline{B}\rho'+k_0\{\overline{B}(\rho_1+\rho_2)+2\rho_1\rho_2\overline{B}^2\}}{1+(\rho_1+\rho_2)\overline{B}+\rho_1\rho_2\overline{B}^2}$$

$$k_n=\frac{-k_{n-2}\overline{B}\rho_1\rho_2+k_{n-1}\left\{\overline{B}(\rho_1+\rho_2)+2\rho_1\rho_2\overline{B}^2\right\}}{1+(\rho_1+\rho_2)\overline{B}+\rho_1\rho_2\overline{B}^2}\qquad n\geq 2,$$

where $\rho'=\rho_1\sigma_2+\rho_2\sigma_1$.

5. Numerical Results and Comments

Extensive numerical work has been carried out for the model under discussion. It has been observed that when traffic intensity $\rho\lesseqgtr 1$ and the waiting space N is moderate, say <20, the method converges very fast. But when $\rho=1$ and N is large, the convergence is slow, as one would expect. All the calculations were performed on a COMPAQ 286 PC in double precision. Though a large number of tables have been produced, only a few are presented here. The selection has been done in such a way that by looking at them one gets a feel and appreciation of the general applicability of the numerical procedure discussed in this paper.

Table 1 gives the probability distribution of the number in system at p.d.e. for the case when $B=10$, $b_0=0.1$, $b_5=0.2$, $b_8=0.2$ and $b_{10}=0.5$. The service time distribution is taken as Erlang with three phases (E_3). Results have been obtained for $\rho=0.5$, 1 and 2. At the bottom of the table, mean (μ) and standard deviation (σ) of the number in system are also given. In the special case when service is performed in batches of fixed size, performance measures such as probability of blocking (PBL), probability server is busy (PB), and average number in system (L) at r.e. are given in table 2 for $\rho=0.5$, 1, 2, and 5, $N=10$, 20, 50, 100, and 200 and service distribution E_3. The PBL in cases when the service-time distribution is exponential, deterministic or hyperexponential is given in table 3. It can be easily seen from tables 2 and 3 that, for a given ρ, the PBL decreases as waiting space increases irrespective of the service time distribution. Also, when $\rho<1$, the decay is faster as N increases and tends to 0 for large N. In the case of $\rho=1$, the decay in PBL is very slow. But for $\rho\gg 1$, $PBL\to 1-1/\rho$ for large N. This can be easily seen to follow from equation (3.5). Since for $\rho\gg 1$, the server is unlikely to be idle, the result follows. In table 4, we present the effect of batch size on PBL for waiting space $N=20$. It is observed that PBL increases as B increases. That is, to reduce the PBL the server should serve batches of smaller size. Finally, in table 5, for $\rho<1$, we present the comparison of the results obtained using the present method and those obtained through the truncation of the $M/E_3^B/1/\infty$ model. The first column gives the probability of number in system at p.d.e. when $N=20$ and the second column gives the results of $N=\infty$ truncated and normalized at $N=20$. This is presented for $\rho=0.2$ and 0.99. It can be seen that when traffic is light, the finite-waiting-space probabilities can be approximated from those of the infinite waiting space by truncating and normalizing at N. However, in heavy traffic, the approximation, as one would expect, is not very good. The cumulative distribution functions (CDFs) of the number in system for these models are shown in figure 1. Finally, in table 6, we obtain the distribution of number in system for $E_B/G/1/N+1$ at p.d.e. using the result obtained through phases. These results again match those of infinite waiting space by truncating and normalizing at N. Once again, in light traffic, the approximation through the infinite-waiting-space model is good.

6. Conclusions

We have successfully studied the behavior of the model under discussion. In short, we can say that though Singh gives analytically closed-form results for the model under discussion, they are computationally non-tractable. It is shown here that for both $M/G^B/1/N+B$ and $E_B/G/1/N+B$, the results can easily be dealt with numerically. Though we have considered commonly used service-time distributions, the method can be used for more general service-time distributions, discrete or continuous. Finally, a remark may be made about the future work related to this model. We expect to obtain, though it does not appear to be feasible at this stage, the relationship among the p.d.e., p.a.e. and r.e. probabilities in case of variable service capacity which may be helpful in obtaining various performance measures such as PBL, PB, etcetera.

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Appendix

It can be easily shown from (2.1) and (2.2) that for $M/M^B/1/N+B$ the following relationships between the ratios of the last B probabilities hold:

$$(A1)\qquad \frac{P_j}{P_{j-1}} = \frac{B\rho}{B\rho+1}, \quad N-B+1 \le j \le N-1$$

$$(A2)\qquad \frac{P_N}{P_{N-1}} = B\rho.$$

We discuss later how these relationships can be used fruitfully.

$\underline{M/M^3/1/7,\ b_0=1}$

Here, the denominator of (2.3) is $z^3[1+3\rho(1-z)]-1$. In this case, the zeros are

$$z_0 = 1, \quad z_1 = 1.446, \quad z_{2,3} = -0.3898 \pm 0.5559i, \text{ for } \quad \rho = 0.5.$$

From (2.3), we have

$$(A3)\qquad Q(z) = \frac{P_0^+(z^2+z+1) + P_1^+(z^2+z) + P_2^+ z^2}{-3\rho(z-z_1)(z-z_2)(z-z_3)}$$

or by making partial fractions and picking coefficients of z^n

$$(A4)\qquad P_n^+ = -P_0^+ \sum_{i=1}^{3} \frac{A_i}{z_i^{n+1}} - P_1^+ \sum_{i=1}^{3} \frac{B_i}{z_i^{n+1}} - P_2^+ \sum_{i=1}^{3} \frac{C_i}{z_i^{n+1}}, \quad n = 0, 1, 2, 3, 4,$$

where

$$A_i = \frac{z_i^2+z_i+1}{-3\rho\prod_{\substack{j=1\\ i\neq j}}^{3}(z_i-z_j)}, \quad B_i = \frac{z_i^2+z_i}{-3\rho\prod_{\substack{j=1\\ i\neq j}}^{3}(z_i-z_j)}, \quad C_i = \frac{z_i^2}{-3\rho\prod_{\substack{j=1\\ i\neq j}}^{3}(z_i-z_j)}.$$

The numerical values of the A_i's, B_i's and C_i's are

$$\begin{array}{ll} A_1 = -0.6409 & A_{2,3} = -0.0129 \pm 0.1747i \\ B_1 = -0.4431 & B_{2,3} = 0.2216 \mp 0.1322i \\ C_1 = -0.2620 & C_{2,3} = 0.1310 \pm 0.1670i. \end{array}$$

Using $\sum_{n=0}^{4} P_n^+ = 1$, we have

$$(A5)\qquad \left\{ \begin{array}{l} 1 = P_0^+ T_0 + P_1^+ T_1 + P_2^+ T_2 \\ P_0^+ = P_0^+\alpha_0 + P_1^+\beta_0 + P_2^+\gamma_0 \\ P_1^+ = P_0^+\alpha_1 + P_1^+\beta_1 + P_2^+\gamma_1 \\ P_2^+ = P_0^+\alpha_2 + P_1^+\beta_2 + P_2^+\gamma_2 \\ P_3^+ = P_0^+\alpha_3 + P_1^+\beta_3 + P_2^+\gamma_3 \\ P_4^+ = P_0^+\alpha_4 + P_1^+\beta_4 + P_2^+\gamma_4 \end{array} \right\},$$

where

$$T_0 = \sum_{i=1}^{3} -\frac{A_i}{z_i}\cdot\frac{1-z_i^{-5}}{1-z_i^{-1}}, \quad T_1 = \sum_{i=1}^{3} -\frac{B_i}{z_i}\cdot\frac{1-z_i^{-5}}{1-z_i^{-1}}, \quad T_2 = \sum_{i=1}^{3} -\frac{C_i}{z_i}\cdot\frac{1-z_i^{-5}}{1-z_i^{-1}}$$

$$\alpha_n = \sum_{i=1}^{3} -\frac{A_i}{z_i^{n+1}}, \quad \beta_n = \sum_{i=1}^{3} -\frac{B_i}{z_i^{n+1}} \quad \text{and} \quad \gamma_n = \sum_{i=1}^{3} -\frac{C_i}{z_i^{n+1}}, \quad n = 0, 1, 2, 3, 4.$$

Solving for the various unknowns, we get

$$T_0 = 1 \quad \alpha_0 = 1 \quad \alpha_1 = 0 \quad \alpha_2 = 0 \quad \alpha_3 = \frac{3}{2} \quad \alpha_4 = -\frac{3}{2}$$

$$T_1 = \frac{5}{2} \quad \beta_0 = 0 \quad \beta_1 = 1 \quad \beta_2 = 0 \quad \beta_3 = -1 \quad \beta_4 = \frac{5}{2}$$

$$T_2 = 0 \quad \gamma_0 = 0 \quad \gamma_1 = 0 \quad \gamma_2 = 1 \quad \gamma_3 = -1 \quad \gamma_4 = 0.$$

From (A5), we have

$$(A6) \qquad \left\{ \begin{array}{l} 1 = P_0^+ + \frac{5}{2}P_1^+ \\ P_0^+ = P_0^+ \\ P_1^+ = P_1^+ \\ P_2^+ = P_2^+ \\ P_3^+ = \frac{3}{2}P_0^+ - P_1^+ - P_2^+ \\ P_4^+ = -\frac{3}{2}P_0^+ + \frac{5}{2}P_1^+ \end{array} \right\}.$$

It can be easily seen that three equations become redundant, and thus we have three equations in five unknowns. Consequently, these can be solved by using two equations from set (1). But testing shows that when the first two equations of set (1) are solved using the last two equations of set (A6), they become redundant. Consider now the last two equations of set (1), viz. the equations corresponding to P_2^+ and P_3^+. These equations give $P_2^+ = \frac{18}{125} + \frac{12}{125}P_4^+$ and $P_3^+ = \frac{54}{625} + \frac{36}{625}P_4^+$. Solving these alongwith the set (A6) gives the solution

$$P_0^+ = 0.3433,\ P_1^+ = 0.2627,\ P_2^+ = 0.1576,\ P_3^+ = 0.0946,\ P_4^+ = 0.1419.$$

We could have obtained the above solution using two equations from the equations in (A1) or (A1) and (A2). It may be remarked that though all the numerical calculations were done in double precision, they are given here only up to four decimal places. The solution to the system of equations in set (1) using the Jacobi method agrees with the solution obtained above using roots and the Jacobi method.

On the basis of the above example and several other cases which we tested, it may be remarked that in $M/M^B/1/N+B$, the above method produces B redundant equations. Since it also makes the first $N-B+1$ equations of set (1) redundant, we either use the last $B-1$ equations in set (1) or $B-1$ equations from set (A1) or sets (A1) and (A2) to solve this system completely. Furthermore, using this method, one first needs to find the roots and then solve the equations by borrowing from set (1) or (A1) or sets (A1) and (A2). Although we discuss one example when $G = M$ to show that B equations become redundant, the same thing was observed if we take $G = E_2$. In view of this, it is felt that the redundancy of equations will hold for an arbitrary G. As such, it is better to solve the original set directly for any service-time distribution than go through the above method, although the above method will generate more zero elements so that less calculations will have to be performed to solve the system of equations.

Table 1

Probability distribution of number in system at post-departure epoch for $M/E_3^{10}/1/35$.

$b_0 = 0.1$, $b_5 = 0.2$, $b_8 = 0.2$, $b_{10} = 0.5$

n	ρ = 0.5	ρ = 1.0	ρ = 2.0
0	0.152983	0.016874	0.000206
1	0.176587	0.028005	0.000462
2	0.151794	0.033331	0.000724
3	0.120194	0.035512	0.000993
4	0.092931	0.036306	0.001282
5	0.071393	0.036596	0.001626
6	0.054764	0.036716	0.002055
7	0.041993	0.036749	0.002581
8	0.032199	0.036716	0.003202
9	0.024688	0.036622	0.003915
10	0.018933	0.036617	0.004791
11	0.014519	0.036654	0.005881
12	0.011131	0.036596	0.007182
13	0.008534	0.036504	0.008719
14	0.006541	0.036307	0.010485
15	0.005039	0.038050	0.015647
16	0.003857	0.038119	0.020364
17	0.002940	0.036909	0.023777
18	0.002244	0.035371	0.026163
19	0.001711	0.033798	0.027964
20	0.001350	0.036147	0.035782
21	0.001020	0.035449	0.042191
22	0.000752	0.032660	0.045712
23	0.000599	0.033139	0.053108
24	0.000432	0.030564	0.057583
25	0.000874	0.133690	0.597607
$\sum P_n^+$	1.000000	1.000000	1.000000
μ	3.728010	13.932439	22.686253
σ	3.753287	7.758390	3.983750

Table 2

Probability of blocking (PBL), Probability of server busy (PB) and Average number in system (L) for $M/E_k^B/1/N+B$, $k=3$, $B=10$.

N	ρ	0.5	1	2	5
10	PBL	0.026896	0.203159	0.518873	0.800381
	PB	0.994868	0.999020	0.999893	0.999996
	L	3.388008	5.938935	11.202922	16.263877
20		0.001042	0.106971	0.501088	0.800001
		0.995275	0.999627	0.999997	0.999999
		6.219504	7.704186	16.669771	24.495860
50		0.000000	0.043125	0.500000	0.800000
		0.995280	0.999849	0.999999	1.000000
		17.551189	17.881155	34.006643	49.213077
100		0.000000	0.021614	0.500000	0.800000
		0.995280	0.999924	1.000000	1.00004
		33.377778	40.983459	62.953323	90.408492
200		0.000000	0.010820	0.500000	0.800000
		0.995280	0.999962	1.000000	1.00000
		46.912520	92.672459	120.846699	172.799321

Table 3

Probability of blocking for $M/M^B/1/N+B$, $B = 10$.

N \ ρ	0.5	1	2	5
10	.074719	0.278261	0.551132	0.803988
20	.017532	0.186235	0.515778	0.800393
50	.000253	0.092437	0.500477	0.800000
100	.000000	0.050228	0.500001	0.800000
200	.000000	0.026253	0.500000	0.8000

Probability of blocking for $M/D^B/1/N+B$, $B = 10$.

N \ ρ	0.5	1	2	5
10	0.002213	0.111198	0.500205	0.80000
20	0.000000	0.035113	0.500000	0.80000
50	0.000000	0.011302	0.500000	0.80000
100	0.000000	0.0053005	0.500000	0.800000
200	0.000000	.002578	0.500000	0.800000

Table 4

Effect of batch size (B) on the probability of blocking for $M/E_3^3/1/20+B$.

B	ρ = 0.5	ρ = 1.0	ρ = 2.0
2	0.000000	0.039385	0.500000
4	0.000003	0.055667	0.500022
6	0.000049	0.072367	0.500049
8	0.000297	0.089482	0.500314
10	0.001042	0.106971	0.501088
12	0.002617	0.124881	0.502797
14	0.005278	0.124881	0.505704
16	0.009128	0.161145	0.509282
18	0.014125	0.178057	0.513048
20	0.039385	0.193631	0.516727

Table 5

Probability distribution of number in system at post-departure epoch for $M/E_3^4/1/24$ and $M/E_3^4/1/\infty$ truncated and normalized at N=20.

n	ρ = 0.2		ρ = 0.99	
	N	∞	N	∞
0	0.489004	0.489004	0.013630	0.014146
1	0.311037	0.311037	0.027272	0.028304
2	0.132049	0.132049	0.037403	0.038819
3	0.046777	0.046777	0.043989	0.045654
4	0.014934	0.014934	0.047903	0.049716
5	0.004456	0.004456	0.050039	0.051933
6	0.001268	0.001268	0.051073	0.053004
7	0.000349	0.000349	0.051447	0.053397
8	0.000093	0.000093	0.051450	0.053403
9	0.000024	0.000024	0.051264	0.053193
10	0.000006	0.000006	0.050958	0.052868
11	0.000002	0.000002	0.050494	0.052483
12	0.000000	0.000000	0.050192	0.052067
13	0.000000	0.000000	0.049949	0.051638
14	0.000000	0.000000	0.049290	0.051204
15	0.000000	0.000000	0.047733	0.050769
16	0.000000	0.000000	0.050582	0.050335
17	0.000000	0.000000	0.049060	0.049905
18	0.000000	0.000000	0.043409	0.049477
19	0.000000	0.000000	0.035766	0.049053
20	0.000000	0.000000	0.097097	0.048632
$\sum P_n^+$	1.000000	1.000000	1.000000	1.000000
μ	0.808589	0.808589	10.984300	10.628650
σ	1.014090	1.014090	5.827328	5.638382

Table 6

Probability distribution of number in system at post-departure epcoh for E_4 / E_3 /1/24 and E_4 / E_3 /1/∞ truncated and normalized at N=20.

n	ρ = 0.5		ρ = 0.99	
	N	∞	N	∞
0	0.767965	0.767965	0.038564	0.037841
1	0.206568	0.206568	0.063215	0.062028
2	0.023089	0.023089	0.064398	0.063190
3	0.002164	0.002164	0.062494	0.061321
4	0.000195	0.000195	0.060394	0.059261
5	0.000017	0.000017	0.058349	0.057254
6	0.000002	0.000002	0.056372	0.055315
7	0.000000	0.000000	0.054463	0.053441
8	0.000000	0.000000	0.052617	0.051630
9	0.000000	0.000000	0.050835	0.049881
10	0.000000	0.000000	0.049112	0.048191
11	0.000000	0.000000	0.047449	0.046558
12	0.000000	0.000000	0.045841	0.044981
13	0.000000	0.000000	0.044288	0.043457
14	0.000000	0.000000	0.042788	0.041985
15	0.000000	0.000000	0.041338	0.040562
16	0.000000	0.000000	0.039937	0.039188
17	0.000000	0.000000	0.038575	0.037860
18	0.000000	0.000000	0.037087	0.036578
19	0.000000	0.000000	0.033626	0.035338
20	0.000000	0.000000	0.018258	0.034141
$\sum P_n^+$	1.000000	1.000000	1.000000	1.000000
μ	0.260115	0.260115	8.835112	9.041883
σ	0.504340	0.504340	5.717369	5.857749

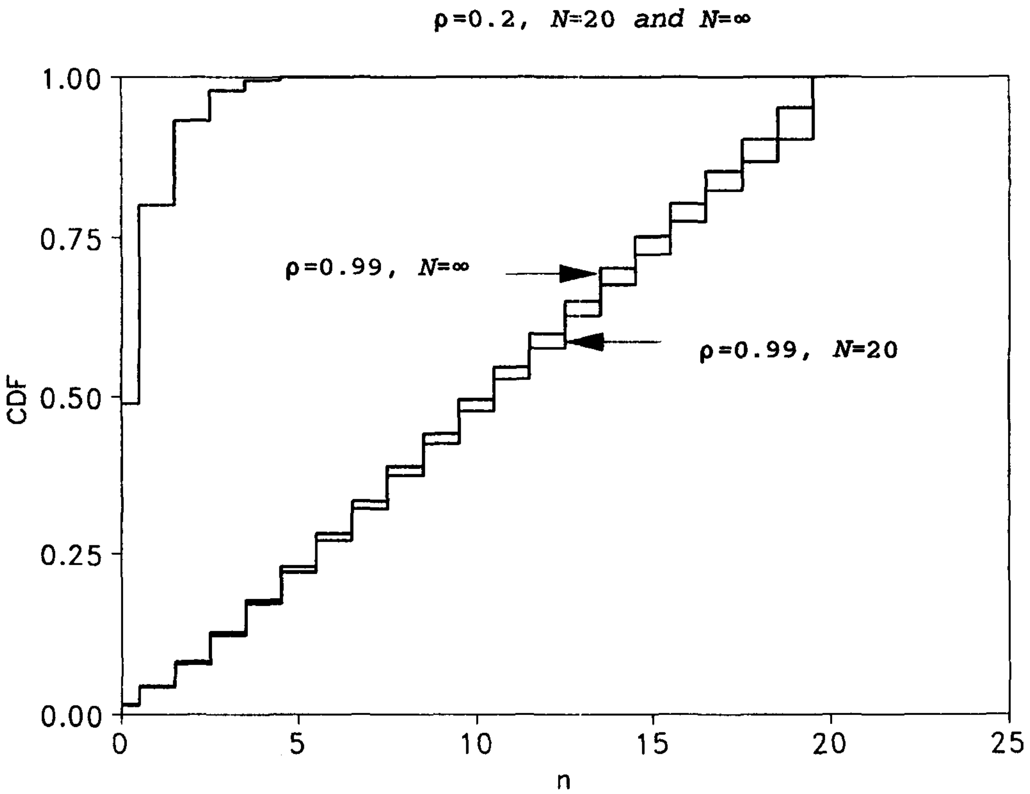

Fig. 1: CDFs of the number in system for the examples given in Table 5.