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A SEARCH GAME WITH TRAVELING COST

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Abstract There are n neighboring cells in a straight line. A man selects a cell, hides in it and stays there. The searcher examines each cell until he finds the hider. Associated with the examination are a traveling cost dependent on the distance from the last cell examined and an examination cost which varies from cell to cell. The searcher wishes to minimize the expectation of cost of finding the hider, whereas the hider wishes to maximize it. The problem is formulated as a two-person zero-sum game and it is solved.

1. Introduction

In this paper, it is pointed out that the results in [6] can be generalized to the case where the examination cost of each cell depends on the location of the cell and the distances between cells are different but still additive (See (2.1) below). An optimal strategy for Player 1 is unique in spite of the varieties of examination costs and distances. If the examination cost of each cell is a rational number and if each cell can be subdivided in any way, then our model can be reduced to the model such that the examination cost of each cell does not depend on the location. In such case, the significance of the first generalization, i.e., the dependence of the examination cost, may decrease. But in the case that each cell cannot be subdivided for physical or other reasons, it is still meaningful. This is discussed in Case 1 of Section 3. The proof of the main theorem has become shorter and more elegant than that of [6], by the uses of the induction on the number of cells and a recursive relation between the values of games.

Gluss [4] analyzed a model in which there are $n+1$ neighboring cells in a straight line, labeled from 0 to n in that order. An object is in one of them except Cell 0 with a priori probabilities $p_1, \cdots, p_n$. At the beginning of the search the searcher is at Cell 0 that is next to Cell 1. It is required to determine a strategy that minimizes the statistical expectation of the cost of finding the object. Associated with the examination of each cell is the examination cost. The only difference between his model and the previous one (See [1]) is that while the cost is constant in the latter, it varies through time, that is, a traveling cost is added in the model by Gluss (See [4]). Gluss treated two cases : $p_1 \leq \cdots \leq p_n$ and $p_1 \geq \cdots \geq p_n$. He showed that the latter case is trivial, the searcher should examine each cell in the order of $1, 2, \cdots, n$, and in the other case he found approximately optimal search strategies when p_i is proportional to i. These strategies are represented by a parameter.

While Gluss treats a one decision-maker problem, in Kikuta [6], we assumed a hider with his will in place of an object and took the game theoretical point of view. Thus, there are a hider and a searcher. While the searcher wishes to minimize the cost of finding the hider, the hider chooses a cell so as to maximize it. We have a two-person zero-sum game.

The unique optimal strategy for the hider obtained in [6] can be compared with an a priori probability in Gluss [4]. The latter is proportionally increasing and the former is hyperbolically increasing as the label of a cell becomes large.

Another variant of the model of Gluss is in Kikuta [5] (Also see [7]), where the searcher is at the central cell at the beginning of the search. But, it is, still, a one decision-maker problem. In Chapter 5 of [8] search games on graphs are treated. The game in this paper is, in a sense, a search game on a complete graph. A point which must be indicated seems to be how are the cost functions. Search games in [9] consider overlooking probabilities. Again, the cost functions must be compared.

In the next section our model is stated as a two-person zero-sum game in detail. In Section 3 some strategies are defined and our main theorem is given, which states they are optimal. Section 4 is devoted to proving the theorem. In Section 5 some remarks are given.

2. The Model and Notation

There are $n+1$ neighboring cells in a straight line, labeled from 0 to n in that order. Player 1 (the hider) selects a cell except Cell 0, hides in it, and stays there. Player 2 (the searcher) examines each cell until he finds Player 1.

0	1		i		n

Figure 1.

Associated with the examination of Cell $i(1 \leq i \leq n)$ is the examination cost that consists of two parts: (i) a traveling cost $d(i,j) > 0$ of examining Cell i after having examined Cell j, and (ii) an examination cost $c(i) > 0$. (i) means that the examination cost varies through the search and is a function of the cell which was last examined. We assume

$$\begin{aligned} &d(i,j) + d(j,k) = d(i,k) \text{ for all } i,j,k \text{ such that } 1 \leq i < j < k \leq n, \\ &d(i,j) = d(j,i) \text{ for all } i,j \text{ such that } 1 \leq i,j \leq n, i \neq j, \text{and} \\ &d(i,j) > 0 \text{ for all } i,j \text{ such that } 1 \leq i,j \leq n, i \neq j. \end{aligned} \tag{2.1}$$

For convenience, we let $d(i,i) = 0$ for all $i : 1 \leq i \leq n$. There is no possibility of overlooking Player 1, given that the right cell is searched. It is assumed that at the beginning of the search Player 2 is at Cell 0. Before searching (hiding) Player 2 (Player 1) must determine a strategy so as to make the cost of finding Player 1 as small (large) as possible.

A *(pure) strategy* for Player 1 is to choose an element, say i, of $N \equiv \{1, 2, \cdots, n\}$, which means he determines on hiding in Cell i. This is denoted by $\underline{i}(i \in N)$. The set of all strategies for Player 1 is denoted by $\underline{N} \equiv \{\underline{1}, \underline{2}, \cdots, \underline{n}\}$. A strategy for Player 2 is defined by a permutation on N. The set of all permutations on N is denoted by $\underline{M} \equiv \{\underline{1}, \underline{2}, \cdots, \underline{m}\}$, where $m = n!$. Thus under a strategy $\underline{j}$, Player 2 examined Cells $\underline{j}(1), \underline{j}(2), \cdots, \underline{j}(n)$ in this order. In particular, $\underline{1}$ expresses the identity and $\underline{m}$ expresses the permutation $\underline{m}(i) = n - i + 1$ for $i = 1, \cdots, n$. We let $\underline{j}(0) = 0$ for all $\underline{j} \in \underline{M}$.

For a strategy pair $(\underline{i}, \underline{j})(\underline{i} \in \underline{N}, \underline{j} \in \underline{M})$, let $k = \underline{j}^{-1}(i)$. Then the cost of finding Player 1, written as $f(\underline{i}, \underline{j})$, is :

$$f(\underline{i}, \underline{j}) = \sum_{r=1}^{k} [d(\underline{j}(r), \underline{j}(r-1)) + c(\underline{j}(r))]. \tag{2.2}$$

Thus we have a two-person zero-sum game, which is denoted by $(f; \underline{N}, \underline{M})$. Since both $\underline{N}$ and $\underline{M}$ are finite sets this game is expressed by a matrix whose (i,j)-component is $f(\underline{i}, \underline{j})(\underline{i} \in \underline{N}, \underline{j} \in \underline{M})$. The numbers of rows and columns are n and $n!$ respectively. We find that this matrix does not always have a saddle point, by seeing the case of 2-cell. Let $n = 2$. Let $\underline{1}$ and $\underline{2}$ be the identity permutation and the other one. Then $f(\underline{1},\underline{1}) = c(1)+d(0,1), f(\underline{1},\underline{2}) = d(0,2)+d(1,2)+c(1)+c(2), f(\underline{2},\underline{1}) = d(0,2)+c(1)+c(2)$, and $f(\underline{2},\underline{2}) = d(0,2)+c(2)$. This 2×2 matrix has no saddle point. Thus, we need to have the mixed extension of $(f;\underline{N},\underline{M})$ in order to have a solution for any game. Let $\Gamma \equiv (f;P,Q)$ be the mixed extension. The elements of P and Q are called *mixed* strategies, or simply, strategies, without confusion. For a strategy pair $(p,q)\in P\times Q, f(p,q)$ is the expected cost of finding Player 1.

In this note we fix $c(1),\cdots,c(n-1)$ and $c(n)$. For $k = 1,\cdots,n$, consider a case that there are $k+1$ neighboring cells in a straight line, labeled $n-k, n-k+1, n-k+2,\cdots,n$ in this order. Player 2 is at Cell $n-k$ at the beginning. The examination-cost function and the distance function are naturally restricted. In the same way as in the n-cell case, we have a two person game, which is referred to as Γ^k. The value of Γ^k is denoted by v^k. Unless otherwise specified, the superscript, k, corresponds to Γ^k. The payoff function of Γ^k is denoted by f^k. The sets of strategies are denoted by P^k and Q^k. $\underline{N}^k = \{\underline{n-k+1},\underline{n-k+2},\cdots,\underline{n}\}$ and $\underline{M}^k$ is the set of pure strategies of Player 2 in Γ^k. Thus $f^n = f, P^n = P, Q^n = Q$, and $\Gamma^n = \Gamma$. We let $c^k \equiv (c(n-k+1),\cdots,c(n)), s^k \equiv c(n-k+1)+\cdots+c(n)$, and $b^k \equiv c(n-k)/[2d(n-k,n)+s^k]$ for $k = 1,\cdots,n-1$. For convenience, we let $b^0 = +\infty$. For convenience and simplicity, we often use the notation $r' \equiv n-k+r$ for $r = 0,\cdots,k$ in Γ^k. Note that $(r-\ell)' = r'-\ell$, and $(r+\ell)' = r'+\ell$ as long as these make sense.

Our problem is to solve Γ^n.

3. Optimal Strategies.

The purpose of this section is to give optimal strategies and examine their properties. Define a k-vector p^k for $k = 1,\cdots,n$ inductively as follows:

$$p^k = 1/[1+b^{k-1}](b^{k-1},p^{k-1}), k = 2,\cdots,n, \text{and } p^1 = (1). \tag{3.1}$$

For each $k, 1\le k\le n, p^k$ is a probability vector. All components of p^n are positive. The following proposition gives properties of p^n, which are referred to later.

Proposition 1.

(i) $b^{n-i-1}p_i^n/[1+b^{n-i-1}] = b^{n-i}p_{i+1}^n$ for $i = 1,\cdots,n-2$, and $p_{n-1}^n = b^1 p_n^n$.

(ii) Suppose $\underline{j}\in\underline{M}^k$, and $\underline{j}(x') = 1'$ for some x. Then

$$p_{1'}^k[\sum_{i=1}^{x} d(\underline{j}(i'-1),\underline{j}(i')) + \sum_{i=1}^{x-1} c(\underline{j}(i'))] \ge c(1')\sum_{i=1}^{x-1} p_{\underline{j}(i')}^k,$$

where we let $\underline{j}(0') = 1'$ for convenience.

(iii) $p_1^n/c(1) < p_2^n/c(2) < \cdots < p_n^n/c(n)$.

Proposition 1 (iii) means the probability to the examination cost ratio increases as the distance from Cell 0 becomes large. Proposition 1 (ii) is, in a sense, an inverse inequality. For example, let $k = n$, and $\underline{j}$ be such that $\underline{j}(1) = 2$ and $\underline{j}(2) = 1$. Then from Proposition 1(ii), we have $p_1^n/c(1) \ge p_2^n/[2d(1,2)+c(2)]$.

Proof: (i) From (3.1), for $i = 1,\cdots,n-1$,

$$p_i^n = b^{n-i}p_n^n\prod_{r=2}^{n-i}[1+b^{r-1}] \text{ and } p_n^n = 1/\Pi_{r=2}^n[1+b^{r-1}]. \tag{3.2}$$

From these, $b^{n-i-1}p_i^n = b^{n-i}[1 + b^{n-i-1}]p_{i+1}^n$ for $i = 1, \cdots, n-2$, and $p_{n-1}^n = b^1 p_n^n$. These are just (i).

(ii) By induction on k. When $k = 2$, it suffices to check

$$p_{1'}^2/c(1') = p_{2'}^2/[2d(1',2') + c(2')].$$

Let $k = 3$. It suffices to check

$$\begin{aligned} p_{1'}^3/c(1') \geq \max\{&[p_{2'}^3 + p_{3'}^3]/[2d(1',3') + c(2') + c(3')],\\ &p_{2'}^3/[2d(1',2') + c(2')], p_{3'}^3/[2d(1',3') + c(3')]\}. \end{aligned}$$

These are shown by (3.1) for $k = 2, 3$ and the definition of b^k. Assume the inequalities are true for $2, \cdots, k-1$. We check the case of k. Suppose $\underline{j}(y') = 2'$. First assume $y' < x'$.

$$\begin{aligned} E &\equiv p_{1'}^k[\sum_{i=1}^{x} d(\underline{j}(i'-1), \underline{j}(i')) + \sum_{i=1}^{x-1} c(\underline{j}(i'))] - c(1')\sum_{i=1}^{x-1} p_{\underline{j}(i')}^k \\ &= p_{1'}^k[d(1', \underline{j}(1')) + \sum_{i=2}^{x-1} d(\underline{j}(i'-1), \underline{j}(i')) + d(\underline{j}(x'-1), 1') + \sum_{\underline{j}(i')\neq 2'} c(\underline{j}(i'))] \\ &\quad - c(1') \sum_{\underline{j}(i')\neq 2'} p_{\underline{j}(i')}^k + c(2')p_{1'}^k - c(1')p_{2'}^k \\ &= p_{1'}^k[d(2', \underline{j}(1')) + \sum_{i=2}^{x-1} d(\underline{j}(i'-1), \underline{j}(i')) + d(\underline{j}(x'-1), 2') + \sum_{\underline{j}(i')\neq 2'} c(\underline{j}(i'))] \\ &\quad - c(1') \sum_{\underline{j}(i')\neq 2'} p_{\underline{j}(i')}^k + c(2')p_{1'}^k - c(1')p_{2'}^k + 2d(1',2')p_{1'}^k. \end{aligned}$$

Here from $y' < x'$, we have

$$d(2', \underline{j}(1')) + \sum_{i=2}^{x-1} d(\underline{j}(i'-1), \underline{j}(i')) + d(\underline{j}(x'-1), 2') \geq \sum_{i=1}^{x-1} d(\underline{j}'(i'-1), \underline{j}'(i')),$$

where $\underline{j}'$ is defined by : $\underline{j}'(0') = 2', \underline{j}'(x'-1) = 2', \underline{j}'(i') = \underline{j}(i')$ for $1 \leq i \leq y-1$ and $\underline{j}'(i') = \underline{j}(i'+1)$ for $y \leq i \leq x-2$. Furthermore, by (3.1), $p_{1'}^k = b^{k-1}(1 + b^{k-2})p_{2'}^k/b^{k-2} = \dfrac{c(1')}{c(2')}\dfrac{2d(2',k') + s^{k-1}}{2d(1',k') + s^{k-1}}p_{2'}^k$. Hence

$$\begin{aligned} E \geq \frac{c(1')}{c(2')}\{&\frac{2d(2',k') + s^{k-1}}{2d(1',k') + s^{k-1}}p_{2'}^k[\sum_{i=1}^{x-1} d(\underline{j}'(i'-1), \underline{j}'(i')) + \sum_{\underline{j}(i')\neq 2'} c(\underline{j}(i'))] \\ &+ 2d(1',2')\frac{2d(2',k') + s^{k-1}}{2d(1',k') + s^{k-1}}p_{2'}^k + c(2')\frac{2d(2',k') + s^{k-1}}{2d(1',k') + s^{k-1}}p_{2'}^k \\ &- c(2') \sum_{\underline{j}(i')\neq 2'} p_{\underline{j}(i')}^k - c(2')p_{2'}^k\}. \end{aligned}$$

By the induction hypothesis,

$$\geq \frac{c(1')}{c(2')}\{\frac{2d(2',k')+s^{k-1}}{2d(1',k')+s^{k-1}}c(2')\sum_{\underline{j}(i')\neq 2'} p^k_{\underline{j}(i')}$$
$$+2d(1',2')\frac{2d(2',k')+s^{k-1}}{2d(1',k')+s^{k-1}}p^k_{2'}+c(2')\frac{2d(2',k')+s^{k-1}}{2d(1',k')+s^{k-1}}p^k_{2'}$$
$$-c(2')\sum_{\underline{j}(i')\neq 2'} p^k_{\underline{j}(i')}-c(2')p^k_{2'}\}$$
$$=\frac{c(1')}{c(2')}\{-\frac{2d(1',2')}{2d(1',k')+s^{k-1}}c(2')\sum_{\underline{j}(i')\neq 2'} p^k_{\underline{j}(i')}$$
$$+2d(1',2')\frac{2d(2',k')+s^{k-1}}{2d(1',k')+s^{k-1}}p^k_{2'}+c(2')\frac{2d(2',k')+s^{k-1}}{2d(1',k')+s^{k-1}}p^k_{2'}-c(2')p^k_{2'}\}$$
$$=\frac{c(1')}{c(2')}\{\frac{2d(1',2')}{2d(1',k')+s^{k-1}}[(2d(2',k')+s^{k-1})p^k_{2'}-c(2')\sum_{\underline{j}(i')\neq 2'} p^k_{\underline{j}(i')}]$$
$$-\frac{2d(1',2')}{2d(1',k')+s^{k-1}}c(2')p^k_{2'}\}.$$

Here by (3.1),

$$(2d(2',k')+s^{k-1})p^k_{2'}=c(2')/[1+b^{k-1}], \text{and}$$

$$\sum_{\underline{j}(i')\neq 2'} p^k_{\underline{j}(i')}\leq \frac{1}{1+b^{k-1}}\frac{1}{1+b^{k-2}}.$$

Hence

$$(2d(2',k')+s^{k-1})p^k_{2'}-c(2')\sum_{\underline{j}(i')\neq 2'} p^k_{\underline{j}(i')}$$

$$\geq \frac{c(2')}{1+b^{k-1}}-c(2')\frac{1}{1+b^{k-1}}\frac{1}{1+b^{k-2}}=c(2')p^k_{2'}.$$

Thus,

$$E\geq \frac{c(1')}{c(2')}\{\frac{2d(1',2')}{2d(1',k')+s^{k-1}}c(2')p^k_{2'}-\frac{2d(1',2')}{2d(1',k')+s^{k-1}}c(2')p^k_{2'}\}=0.$$

Second assume $y'>x'$.

$$E=p^k_{1'}[\sum_{i=1}^{x} d(\underline{j}(i'-1),\underline{j}(i'))+\sum_{i=1}^{x-1} c(\underline{j}(i'))]-c(1')\sum_{i=1}^{x-1} p^k_{\underline{j}(i')}$$
$$=p^k_{1'}[d(2',\underline{j}(1'))+\sum_{i=2}^{x-1} d(\underline{j}(i'-1),\underline{j}(i'))+d(\underline{j}(x'-1),2')+\sum_{i=1}^{x-1} c(\underline{j}(i'))]$$
$$-c(1')\sum_{i=1}^{x-1} p^k_{\underline{j}(i')}+2d(1',2')p^k_{1'}.$$

By (3.1), $p_{1'}^k = b^{k-1}(1+b^{k-2})p_{2'}^k/b^{k-2} = \dfrac{c(1')}{c(2')}\dfrac{2d(2',k')+s^{k-1}}{2d(1',k')+s^{k-1}}p_{2'}^k$. Hence

$$\begin{aligned} E = {} & \frac{c(1')}{c(2')}\Big\{\frac{2d(2',k')+s^{k-1}}{2d(1',k')+s^{k-1}}p_{2'}^k[d(2',\underline{j}(1')) + \sum_{i=2}^{x-1} d(\underline{j}(i'-1),\underline{j}(i')) \\ & + d(\underline{j}(x'-1),2') + \sum_{i=1}^{x-1} c(\underline{j}(i'))] - c(2')\sum_{i=1}^{x-1} p_{\underline{j}(i')}^i \\ & + 2d(1',2')\frac{2d(2',k')+s^{k-1}}{2d(1',k')+s^{k-1}}p_{2'}^k\Big\}. \end{aligned}$$

Applying the induction hypothesis,

$$\begin{aligned} E \geq {} & \frac{c(1')}{c(2')}\Big\{\frac{2d(2',k')+s^{k-1}}{2d(1',k')+s^{k-1}}c(2')\sum_{i=1}^{x-1} p_{\underline{j}(i')}^k - c(2')\sum_{i=1}^{x-1} p_{\underline{j}(i')}^k \\ & + 2d(1',2')\frac{2d(2',k')+s^{k-1}}{2d(1',k')+s^{k-1}}p_{2'}^k\Big\} \\ = {} & \frac{c(1')}{c(2')}\Big\{-\frac{2d(1',2')}{2d(1',k')+s^{k-1}}c(2')\sum_{i=1}^{x-1} p_{\underline{j}(i')}^k \\ & + 2d(1',2')\frac{2d(2',k')+s^{k-1}}{2d(1',k')+s^{k-1}}p_{2'}^k\Big\} \\ = {} & \frac{c(1')}{c(2')}\frac{2d(1',2')}{2d(1',k')+s^{k-1}}\{(2d(2',k')+s^{k-1})p_{2'}^k - c(2')\sum_{i=1}^{x-1} p_{\underline{j}(i')}^k\}. \end{aligned}$$

Again applying (3.1), we have

$$\geq \frac{c(1')}{c(2')}\frac{2d(1',2')}{2d(1',k')+s^{k-1}}c(2')p_{2'}^k \geq 0.$$

(iii) Since (i) holds and since $b^1 = c(n-1)/[2d(n-1,n)+c(n)]$, we have $p_{n-1}^n/c(n-1) = b^1p_n^n/c(n-1) = p_n^n/[2d(n-1,n)+c(n)] < p_n^n/c(n)$. Furthermore, $p_i^n/c(i) = [1+b^{n-i-1}]b^{n-i}p_{i+1}^n/[c(i)b^{n-i-1}] = \dfrac{2d(i+1,n)+s^{n-i}}{2d(i,n)+s^{n-i}}p_{i+1}^n/c(i+1) < p_{i+1}^n/c(i+1)$ for $i = 1,\cdots,n-2$. **Q.E.D.**

For any $\underline{j} \in \underline{M}$, define $\underline{\rho j} \in \underline{M}$ by

$$\underline{\rho j}(i) = \underline{j}(n+1-i) \text{ for all } i = 1,\cdots,n. \tag{3.3}$$

$\underline{\rho j}$ reverses the order of examination under $\underline{j}$. Thus, if $\underline{j}$ is expressed as an n-vector, that is, $\underline{j} = [\underline{j}(1),\underline{j}(2),\cdots,\underline{j}(n)]$, then $\underline{\rho j} = [\underline{j}(n),\cdots,\underline{j}(1)]$. We can assume $\underline{j}$ is as follows :

$$\begin{aligned} & \underline{j}(1) < \underline{j}(2) < \cdots < \underline{j}(i_1), \\ & \underline{j}(i_1) > \underline{j}(i_1+1) > \cdots > \underline{j}(i_2), \\ & \underline{j}(i_2) < \underline{j}(i_2+1) < \cdots < \underline{j}(i_3), \\ & \qquad \vdots \\ & \underline{j}(i_{2h-1}) > \underline{j}(i_{2h-1}+1) > \cdots > \underline{j}(n). \end{aligned}$$

Thus, $\underline{j}$ has h peaks and we say $\underline{j}$ is a *h-peaked* strategy. We set $\underline{j}(n+1) = \underline{j}(0) = 0$ for convenience.

Example 2. The next illustration is a 3-peaked strategy when $n = 14$. $\underline{j} = [1,2,5,10,6,3,8,4,9,11,14,13,12,7]$.

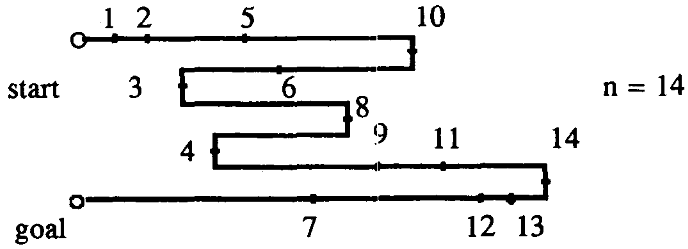

Figure 2.

If $\underline{j} \in \underline{M}$ is h-peaked then $\rho\underline{j}$ is also h-peaked. In particular 1-peaked strategies are interesting since less traveling costs are required under them. Thus let $\underline{M}_1$ be the set of all 1-peaked strategies of Player 2. The number of all elements of $\underline{M}_1$ is 2^{n-1}. Define a 2^{k-1}-vector q^k for $k = 1, \cdots, n$ inductively as follows : For $k = 2, \cdots, n$,

$$q^k = (1/[1 + a^{k-1}])(a^{k-1}q^{k-1}, q^{k-1}), \text{ and } q^1 = (1), \tag{3.4}$$

and for $k = 1, \cdots, n-1$,

$$\frac{2d(0', k') + s^{k+1}}{1 + a^k} = \frac{2d(1', k') + s^k}{1 + a^{k-1}} + d(0', 1') + c(1')\frac{a^{k-1}}{1 + a^{k-1}},$$
$$\text{and } a^0 = +\infty. \tag{3.5}$$

Extend a 2^{k-1}-vector q^k to a $k!$-vector q^{*k} by adding the zero-vector of dimension $k!$-2^{k-1}. That is, $q^{*k} = (q^k, 0)$. Here the components in the first half and in the second half of q^k correspond to 1-peaked strategies such that $\underline{j}(1') = 1'$ and $\rho\underline{j}(1') = 1'$ respectively. The zero-vector corresponds to other strategies (See (3.8) below).

Our main result is :

Theorem 3. p^n is the unique optimal strategy for Player 1. q^{*n} is an optimal strategy for Player 2. The value v^n can be calculated by the recursive relation : For $k = 1, \cdots, n$,

$$v^k = d(0', 1') + c(1') + v^{k-1}/[1 + b^{k-1}], \tag{3.6}$$

or alternatively,

$$v^k = v^{k-1} + d(0', 1') + c(1')a^{k-1}/[1 + a^{k-1}]. \tag{3.7}$$

Theorem 3 gives only an optimal strategy for player 2, and it says nothing about the set of all optimal strategies for Player 2. Let $n = 3$. Let $\underline{1} = [1,2,3]$ and $\underline{2} = [1,3,2]$. Assume $c(1) = c(2) = c(3)$ and $d(i, i+1) = 1$ for $i = 0,1,2$. All optimal strategies for Player 2 are $t\underline{1} + (1/2 - t)\underline{2} + (1/2 - t)\rho\underline{2} + t\rho\underline{1}$ for $0 \le t \le 1/2$.

It is interesting to give properties of p^n and q^{*n}. From (3.4) we have for $\underline{j} \in \underline{M}_1^k$ and $k = 2, \cdots, n$,

$$q_{\underline{j}}^k = \prod_{\underline{j}^{-1}(i)<\underline{j}^{-1}(k)} a^{k-i} / \prod_{i=1}^{k-1} (1 + a^i). \tag{3.8}$$

Case 1. $c(i) = c$ for all $i = 1, \cdots, n$.

From (3.5) we have $a^k = 1$ for $k = 1, \cdots, n-1$. Hence $q_{\underline{j}}^n = 1/2^{n-1}$ for all $\underline{j} \in \underline{M}_1$. By (3.2), if $c \to 0$, then $p^n \to (0, 0, \cdots, 0, 1)$. $p_i^n/c \to 1/[2\bar{d}(i, n)]$ for $i = 1, \cdots, n-1$, and $p_n^n/c \to +\infty$ as $c \to 0$. If $c \to +\infty$, then $b^k \to 1/k$ for all k, and $p^n \to (1/n, \cdots, 1/n)$. Further if $d(i, i+1) = d$ for $i = 0, \cdots, n-1$, then the model reduces to the case in [6]. From (3.6), $v^k = d(0', k') + (k+1)c/2$ for $k = 1, \cdots, n$.

Let $n = 3$. $p_1^3 = \dfrac{c}{2d(1,3)+3c}, p_2^3 = \dfrac{c}{2d(1,3)+3c} \cdot \dfrac{2d(1,3)+2c}{2d(2,3)+2c}, p_3^3 = 1 - p_1^3 - p_2^3$. Suppose Cells 1 and 2 merge into one cell. Renumber two cells from $1'$ to $2'$. how should we evaluate the examination cost of Cell $1'$ in order that $p_{1'}^2 = p_1^3 + p_2^3$?

Conversely let $n = 2, c(1) = 2c, c(2) = c, c > 0$. Suppose Cell 1 is divided into two cells, $1'$ and $2'$ so that $c(1') = c(2') = c$. Renumber Cell 2 as $3'$. Then by letting $d(1', 2') = 0$, we have $p_1^2 = p_{1'}^3 + p_{2'}^3$, and $p_2^2 = p_{3'}^3$. The last argument extends to the n-cell case.

Case 2. $c(i) \to +\infty$ for some i.

If $1 \leq i \leq n-1$, then $b^k \to 0$ for $k \geq n-i+1$ and $b^{n-i} \to +\infty$. Thus by (3.2), $p_{i'}^n \to 0$ if $i' \neq i$, and $p_i^n \to 1$. If $i = n$, then $b^k \to 0$ for $k \geq 1$. By (3.2), $p^n \to (0, \cdots, 0, 1)$. By (3.5), $a^{n-i} \to +\infty$ and $a^k \to 0$ for all $k \geq n-i+1$. By (3.7), $q_{\underline{j}}^n > 0$ implies $\underline{j}^{-1}(i-1) > \underline{j}^{-1}(n), \cdots, \underline{j}^{-1}(1) > \underline{j}^{-1}(n)$ and $\underline{j}^{-1}(i) < \underline{j}^{-1}(n)$. Consequently $q_{\underline{j}}^n > 0$ implies $j(1) = i$ since $\underline{j}$ is 1-peaked. From (3.6), $v^k \to +\infty$ for $k \geq n-i+1$.

Case 3. $d(i, i+1) \to +\infty$ for some i.

$b^k \to 0$ for $k \geq n-i$. Hence $p^n \to (\mathbf{0}, p^{n-i})$, where $\mathbf{0}$ is the i-dim. zero-vector. By (3.5), $a^k \to 1$ for $k \geq n-i$. Hence by (3.4), $q_{\underline{j}}^n = q_{\underline{j}'}^n$ if $\underline{j}(z) = \underline{j}'(z)$ for $z = i+1, \cdots, n$. From (3.6), $v^k \to +\infty$ for $k \geq n-i$.

Case 4. $c(i) \to 0$ for some $i \neq n$.

$b^{n-i} \to 0$. Hence $p_i^n \to 0$ by (3.2).

4. Proof of the theorem.

In this section we prove the theorem by the induction on k.

Lemma 4. For $k = 1, \cdots, n$, and for all $p \in P^k$,

$$f^k(p, q^{*k}) = d(0', 1') + c(1')a^{k-1}/[1 + a^{k-1}] + \frac{2d(1', k') + s^k}{1 + a^{k-1}}. \tag{4.1}$$

Proof: First we note that (4.1) holds when $k = 1$, observing the right-hand side of (4.1) is $d(n-1, n) + c(n)$ by (3.5). We show (4.1) for k, assuming (4.1) for $k-1$. For $\underline{i} \in \underline{N}$ such that $2 \le i \le k$,

$$
\begin{aligned}
f^k(\underline{i}', q^{*k}) &= \sum_{\underline{j}\in \underline{M}_1^k} f^k(\underline{i}', \underline{j}) q_{\underline{j}}^k \\
&= \sum_{\underline{j}\in \underline{M}_1^k : \underline{j}(1')=1'} f^k(\underline{i}', \underline{j}) a^{k-1} q_{\underline{j}'}^{k-1}/[1+a^{k-1}] \\
&+ \sum_{\underline{j}\in \underline{M}_1^k : \underline{j}(k')=1'} f^k(\underline{i}', \underline{j}) q_{\underline{j}'}^{k-1}/[1+a^{k-1}] \\
&= a^{k-1}\{d(0',1') + c(1') + \sum_{\underline{j}\in \underline{M}_1^k : \underline{j}(1')=1'} f^{k-1}(\underline{i}', \underline{j}') q_{\underline{j}'}^{k-1}\}/[1+a^{k-1}] \\
&+ \{d(0',1') + \sum_{\underline{j}\in \underline{M}_1^k : \underline{j}(k')=1'} f^{k-1}(\underline{i}', \underline{j}') q_{\underline{j}'}^{k-1}\}/[1+a^{k-1}] \\
&= a^{k-1}\{d(0',1') + c(1') + f^{k-1}(\underline{i}', q^{*k-1})\}/[1+a^{k-1}] + \{d(0',1') \\
&+ f^{k-1}(\underline{i}', q^{*k-1})\}/[1+a^{k-1}] \\
&= d(0',1') + a^{k-1}c(1')/[1+a^{k-1}] + [2d(1',k') + s^k]/[1+a^{k-1}]
\end{aligned}
$$

by the induction hypothesis and (3.5). Here $\underline{j}' \in \underline{M}_1^{k-1}$ is defined by : $\underline{j}'(z') = \underline{j}(z')$ for all $2 \le z \le k$ when $\underline{j}(1') = 1'$, and $\underline{j}'(z') = \underline{j}((z-1)')$ for all $2 \le z \le k$ when $\underline{j}(k') = 1'$. Next let $i = 1$.

$$
\begin{aligned}
f^k(\underline{1}', q^{*k}) &= \sum_{\underline{j}\in \underline{M}_1^k : \underline{j}(1')=1'} a^{k-1} q_{\underline{j}'}^{k-1}[d(0',1') + c(1')]/[1+a^{k-1}] \\
&+ \sum_{\underline{j}\in \underline{M}_1^k : \underline{j}(k')=1'} q_{\underline{j}'}^{k-1}[d(0',k') + d(k',1') + c(1') + \cdots + c(k')] \\
&/[1+a^{k-1}] \\
&= d(0',1') + c(1') + [2d(1',k') + c(2') + \cdots + c(k')]/[1+a^{k-1}] \\
&= d(0',1') + a^{k-1}c(1')/[1+a^{k-1}] + [2d(1',k') + s^k]/[1+a^{k-1}].
\end{aligned}
$$

Q.E.D.

Lemma 5. For $k = 1, \cdots, n$, and for any $\underline{j} \in \underline{M}_1^k$,

$$
\begin{aligned}
f^k(p^k, \underline{j}) &= d(0',1') + c(1') \\
&+ \sum_{z=1}^{k-1}[d(z', z'+1) + c(z'+1)] \prod_{r=1}^{z} \frac{1}{1+b^{k-r}}.
\end{aligned} \tag{4.2}
$$

Proof: First we note that (4.2) holds when $k = 1$, observing the right-hand side of (4.2) is $d(n-1, n) + c(n)$. We show (4.2) for k, assuming (4.2) holds for $k-1$. Let $\underline{j} = [i_1, \cdots, i_y, i_{y+1}, \cdots]$ and $i_{y+1} = k'$. $f^k(p^k, \underline{j}) = \sum_{i=1}^{k} p_{i'}^k f^k(\underline{i}', \underline{j})$. Suppose $i_h < i' < i_{h+1}$

for some $h, 0 \leq h \leq y$. Here $i_0 = 0'$. Assume first $i_1 > 1'$ and $i' \geq 2'$. Then

$$\begin{aligned} f^k(\underline{i}', \underline{j}) &= d(0',k') + d(i',k') + s^{n+1-i'} + \sum_{r=1}^{h} c(i_r) \\ &= d(0',1') + d(1',k') + d(i',k') + s^{n+1-i'} + \sum_{r=1}^{h} c(i_r) \\ &= d(0',1') + f^{k-1}(\underline{i}', \underline{j}'), \end{aligned}$$

where $\underline{j}'(z') = \underline{j}((z-1)')$ for $2 \leq z \leq k$, and $\underline{j}' \in \underline{M}_1^{k-1}$. Furthermore, it is easy to see for $r = 0, \cdots, y+1$,

$$f^k(\underline{i}_r, \underline{j}) = d(0',1') + f^{k-1}(\underline{i}_r, \underline{j}').$$

Thus

$$\begin{aligned} f^k(p^k, \underline{j}) &= p_{1'}^k f^k(\underline{1}', \underline{j}) + \sum_{i=2}^{k} p_{i'}^{k-1}[d(0',1') + f^{k-1}(\underline{i}', \underline{j}')]/[1 + b^{k-1}] \\ &= p_{1'}^k f^k(\underline{1}', \underline{j}) = \{d(0',1') + \sum_{i=2}^{k} p_{i'}^{k-1} f^{k-1}(\underline{i}', \underline{j}')\}/[1 + b^{k-1}] \\ &= b^{k-1}[d(0',k') + d(1',k') + s^k]/[1 + b^{k-1}] + \{d(0',1') + f^{k-1}(p^{k-1}, \underline{j}')\} \\ &\quad /[1 + b^{k-1}] \\ &= d(0',1') + c(1') + f^{k-1}(p^{k-1}, \underline{j}')/[1 + b^{k-1}]. \end{aligned}$$

By the induction hypothesis,

$$\begin{aligned} f^{k-1}(p^{k-1}, \underline{j}') &= d(1',2') + c(2') \\ &\quad + \sum_{z=2}^{k-1}[d(z', z'+1) + c(z'+1)] \prod_{r=1}^{z} \frac{1}{1 + b^{k-1-r}}. \end{aligned}$$

Hence

$$\begin{aligned} f^k(p^k, \underline{j}) &= d(0',1') + c(1') + \{d(1',2') + c(2') \\ &\quad + \sum_{z=2}^{k-1}[d(z', z'+1) + c(z'+1)] \prod_{r=1}^{z} \frac{1}{1 + b^{k-1-r}}\}/[1 + b^{k-1}] \\ &= d(0',1') + c(1') + [d(1',2') + c(2')]/[1 + b^{k-1}] \\ &\quad + \{\sum_{z=2}^{k-1}[d(z', z'+1) + c(z'+1)] \prod_{r=1}^{z} \frac{1}{1 + b^{k-1-r}}\}/[1 + b^{k-1}] \\ &= d(0',1') + c(1') + \sum_{z=1}^{k-1}[d(z', z'+1) + c(z'+1)] \prod_{r=1}^{z} \frac{1}{1 + b^{k-r}}. \end{aligned}$$

Next assume $i_1 = 1', i \geq 2$ and $i_h < i' < i_{h+1}$. Then

$$\begin{aligned} f^k(\underline{i}', \underline{j}) &= d(0',k') + d(i',k') + \sum_{z=i}^{k} c(z') + \sum_{r=2}^{h} c(i_r) + c(1') \\ &= d(0',1') + c(1') + f^{k-1}(\underline{i}', \underline{j}'), \end{aligned}$$

where $\underline{j}'(z') = \underline{j}(z')$ for $2 \leq z \leq k$, and $\underline{j}' \in \underline{M}_1^{k-1}$.

Furthermore, for $r = 1, \cdots, y+1$,

$$f^k(\underline{i}_r, \underline{j}) = d(0', 1') + c(1') + f^{k-1}(\underline{i}_r, \underline{j}').$$

Thus

$$\begin{aligned} f^k(p^k, \underline{j}) &= p^k_{1'} f^k(\underline{1}', \underline{j}) + \sum_{i=2}^{k} p^{k-1}_{i'}[d(0',1') + c(1') \\ &+ f^{k-1}(\underline{i}', \underline{j}')]/[1 + b^{k-1}] \\ &= p^k_{1'} f^k(\underline{1}', \underline{j}) + \{d(0',1') + c(1') + f^{k-1}(p^{k-1}, \underline{j}')\}/[1 + b^{k-1}] \\ &= d(0',1') + c(1') + \sum_{z=1}^{k-1}[d(z', z'+1) + c(z'+1)] \prod_{r=1}^{z} \frac{1}{1 + b^{k-r}}. \end{aligned}$$

Q.E.D.

Lemma 6. For $k = 1, \cdots, n$, and for all $\underline{j} \in \underline{M}^k$,

$$\begin{aligned} f^k(p^k, \underline{j}) \geq\ & d(0', 1') + c(1') \\ &+ \sum_{z=1}^{k-1}[d(z', z'+1) + c(z'+1)] \prod_{r=1}^{z} \frac{1}{1 + b^{k-r}}. \end{aligned}$$

Proof: First we note that Lemma 6 holds when $k = 1$, observing both sides become $d(n-1, n) + c(n)$. We show Lemma 6 for k, assuming it holds for $k-1$. For $\underline{j} \in \underline{M}^k$, let $\underline{j}(x') = 1'$.

$$\begin{aligned} f^k(p^k, \underline{j}) &= \sum_{i=1}^{k} p^k_{i'} f^k(\underline{i}', \underline{j}) \\ &= p^k_{1'} f^k(\underline{1}', \underline{j}) + \sum_{i=2}^{k} p^{k-1}_{i'} f^k(\underline{i}', \underline{j})/[1 + b^{k-1}] \\ &= p^k_{1'} f^k(\underline{1}', \underline{j}) + \{\sum_{z=1}^{x-1} p^{k-1}_{\underline{j}(z')}[d(0',1') + f^{k-1}(\underline{i}', \underline{j}')] \\ &+ \sum_{z=x+1}^{k} p^{k-1}_{\underline{j}(z')}[d(0',1') + d(\underline{j}(x'-1), \underline{j}(x')) + d(\underline{j}(x'), \underline{j}(x'+1)) \\ &- d(\underline{j}(x'-1), \underline{j}(x'+1)) + c(1') + f^{k-1}(\underline{i}', \underline{j}')]\}/[1 + b^{k-1}], \end{aligned}$$

where $\underline{j}'(z') = \underline{j}(z')$ if $z \leq x-1$, and $= \underline{j}(z'+1)$ if $z \geq x$.
Here, note that

$$f^k(\underline{1}', \underline{j}) = d(0',1') + c(1') + f^{k-1}(\underline{j}(x'-1), \underline{j}') + d(\underline{j}(x'-1), \underline{j}(x')),$$

$$\sum_{z=1}^{x-1} p^{k-1}_{\underline{j}(z')} f^{k-1}(\underline{i}', \underline{j}') + \sum_{z=x+1}^{k} p^{k-1}_{\underline{j}(z')} f^{k-1}(\underline{i}', \underline{j}') = f^{k-1}(p^{k-1}, \underline{j}'),$$

$$\sum_{z=1}^{x-1} p^{k-1}_{\underline{j}(z')} + \sum_{z=x+1}^{k} p^{k-1}_{\underline{j}(z')} = 1, \text{ and}$$

$$p^k_{1'} = b^{k-1}/[1 + b^{k-1}].$$

Thus,

$$
\begin{aligned}
f^k(p^k,\underline{j}) &= d(0',1') + f^{k-1}(p^{k-1},\underline{j}')/[1+b^{k-1}] + c(1') \\
&+ p^k_{1'}[f^{k-1}(\underline{j}(x'-1),\underline{j}') + d(\underline{j}(x'-1),\underline{j}(x'))] \\
&- c(1')\sum_{z=1}^{x-1} p^{k-1}_{\underline{j}(z')}/[1+b^{k-1}] \\
&+ \sum_{z=x+1}^{k} p^{k-1}_{\underline{j}(z')}[d(\underline{j}(x'-1),\underline{j}(x')) + d(\underline{j}(x'),\underline{j}(x'+1)) \\
&- d(\underline{j}(x'-1),\underline{j}(x'+1))]/[1+b^{k-1}].
\end{aligned}
$$

By Proposition 1(ii),

$$
p^k_{1'}[f^{k-1}(\underline{j}(x'-1),\underline{j}') + d(\underline{j}(x'-1),\underline{j}(x'))] - c(1')\sum_{z=1}^{x-1} p^{k-1}_{\underline{j}(z')}/[1+b^{k-1}] \geq 0.
$$

Hence, noting (2.1),

$$
\begin{aligned}
f^k(p^k,\underline{j}) &\geq d(0',1') + f^{k-1}(p^{k-1},\underline{j}')/[1+b^{k-1}] + c(1') \\
&+ \sum_{z=x+1}^{k} p^{k-1}_{\underline{j}(z')}[d(\underline{j}(x-1),\underline{j}(x)) + d(\underline{j}(x),\underline{j}(x+1)) \\
&- d(\underline{j}(x-1),\underline{j}(x+1))]/[1+b^{k-1}] \\
&\geq d(0',1') + f^{k-1}(p^{k-1},\underline{j}')/[1+b^{k-1}] + c(1').
\end{aligned}
$$

By the induction hypothesis,

$$
\begin{aligned}
f^{k-1}(p^{k-1},\underline{j}') &\geq d(1',2') + c(2') \\
&+ \sum_{z=1}^{k-2}[d(z'+1,z'+2) + c(z'+2)]\prod_{r=1}^{z}\frac{1}{1+b^{k-r-1}}.
\end{aligned}
$$

Hence,

$$
\geq d(0',1') + c(1') + + \sum_{z=1}^{k-1}[d(z',z'+1) + c(z'+1)]\prod_{r=1}^{z}\frac{1}{1+b^{k-r}}.
$$

Q.E.D.

Corollary 7. For $k = 1,\cdots,n$,

$$
v^k = d(0',1') + c(1') + \frac{v^{k-1}}{1+b^{k-1}}, \text{ and}
$$

$$
v^k = d(0',1') + \frac{a^{k-1}c(1')}{1+a^{k-1}} + v^{k-1}.
$$

Proof: From Lemma 4 and Lemma 5, letting $p = p^k$,

$$
\begin{aligned}
&d(0',1') + c(1')a^{k-1}/[1+a^{k-1}] + \frac{2d(1',k') + s^k}{1+a^{k-1}} \\
&= f^k(p^k,q^{*k}) = \sum_{\underline{j}\in \underline{M}^k_1} q^k_{\underline{j}} f^k(p^k,\underline{j}) \\
&= d(0',1') + c(1') + \sum_{z=1}^{k-1}[d(z',z'+1) + c(z'+1)]\prod_{r=1}^{z}\frac{1}{1+b^{k-r}}.
\end{aligned}
$$

From this and Lemmas 4, 5, and 6, we see p^k and q^{*k} are optimal strategies. The value of the game is :

$$v^k = d(0',1') + c(1') + \sum_{z=1}^{k-1}[d(z',z'+1) + c(z'+1)]\prod_{r=1}^{z}\frac{1}{1+b^{k-r}}.$$
$$= d(0',1') + c(1')a^{k-1}/[1+a^{k-1}] + \frac{2d(1',k') + s^k}{1+a^{k-1}}.$$

From these we have recursive relations on v^k. **Q.E.D.**

Lemma 8. Let $\underline{j[r]} = [1,\cdots,r-1,n,n-1,\cdots,r](1 \le r \le n)$ be 1-peaked strategies for Player 2. Let A_n be an n-by-n matrix whose (s,t)-component is $f(\underline{t},\underline{j[n-s+1]})$ for $s = 1,\cdots,n$ and $t = 1,\cdots,n$. Then the rank of A_n is equal to n.

Proof : We show the rank of A_n is n, by basic transformations of matrices. For each $x = 2,\cdots,n$, subtract the first column-vector from the xth column-vector of A_n. Then make a matrix A_{n-1} by sweeping out the first column and the nth row by the pivot element $f(\underline{1},\underline{j[1]})$. The (s,t)-component of A_{n-1} is :

$$f^{n-1}(\underline{t},\underline{j[n-s+1]}) \equiv f(\underline{t},\underline{j[n-s+1]}) - f(\underline{1},\underline{j[n-s+1]}) + a_t^{n-1}$$

for $s = 1,\cdots,n-1$ and $t = 2,\cdots,n$. Here for $t = 2,\cdots,n$,

$$a_t^{n-1} = [d(1,t) + c(1) + \cdots + c(t-1)]\frac{f(\underline{1},\underline{j[2]})}{f(\underline{1},\underline{j[1]})}.$$

Here, note that $f(\underline{1},\underline{j[2]}) = \cdots = f(\underline{1},\underline{j[n]})$. The $(n,1)$-component of A_{n-1} is 1 and the others are 0's. For each $x = 3,\cdots,n$, subtract the second column-vector from the x-th column-vector of A_{n-1}. Then make a matrix A_{n-2} by sweeping out the second column and the $(n-1)$st row by the pivot element $f^{n-1}(\underline{2},\underline{j[2]})$. The (s,t)-component of A_{n-2} is : For $s = 1,\cdots,n-2$ and $t = 3,\cdots,n$,

$$f^{n-2}(\underline{t},\underline{j[n-s+1]}) \equiv f^{n-1}(\underline{t},\underline{j[n-s+1]}) - f^{n-1}(\underline{2},\underline{j[n-s+1]}) + a_t^{n-2}$$
$$= f(\underline{t},\underline{j[n-s+1]}) - [d(0,2) + c(1) + c(2)] + [a_t^{n-1} - a_2^{n-1}] + a_t^{n-2}.$$

Here for $t = 3,\cdots,n$,

$$a_t^{n-2} = [f^{n-1}(\underline{2},\underline{j[2]}) - f^{n-1}(\underline{t},\underline{j[2]})]\frac{f^{n-1}(\underline{2},\underline{j[3]})}{f^{n-1}(\underline{2},\underline{j[2]})}.$$

Note that $f^{n-1}(\underline{2},\underline{j[3]}) = \cdots = f^{n-1}(\underline{2},\underline{j[n]})$. The $(n,1)$ and $(n-1,2)$-components of A_{n-2} are 1's and the others are 0's. Continue with this process. We will finally have for $s = 1,2$ and $t = n-1,n$,

$$f^2(\underline{t},\underline{j[n-s+1]}) \equiv f^3(\underline{t},\underline{j[n-s+1]}) - f^3(\underline{n-2},\underline{j[n-s+1]}) + a_t^2$$
$$= f(\underline{t},\underline{j[n-s+1]}) - [d(0,n-2) + c(1) + \cdots + c(n-2)]$$
$$+ [a_t^{n-1} - a_{n-2}^{n-1}] + [a_t^{n-2} - a_{n-2}^{n-2}] + \cdots + [a_t^3 - a_{n-2}^3] + a_t^2. \tag{4.3}$$

Furthermore for $u = 2,\cdots,n-1$ and $t = n+1-u,\cdots,n$,

$$a_t^u = [f^{u+1}(\underline{n-u},\underline{j[n-u]}) - f^{u+1}(\underline{t},\underline{j[n-u]})]\frac{f^{u+1}(\underline{n-u},\underline{j[n-u+1]})}{f^{u+1}(\underline{n-u},\underline{j[n-u]})}.$$

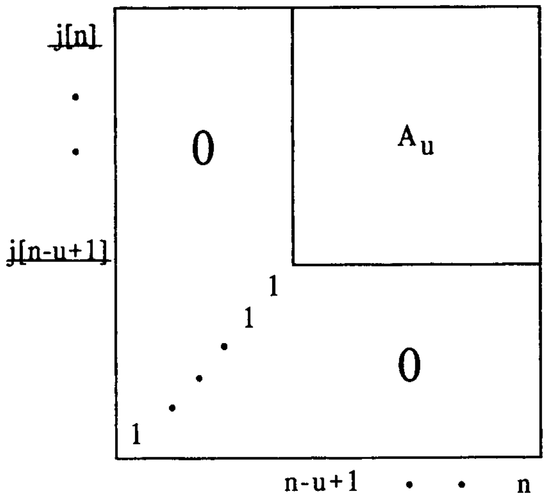

Figure 3

We show inductively that for $u = 2, \cdots, n-1$,

(I) $f^{u+1}(\underline{n-u}, \underline{j[n-u]} > f^{u+1}(\underline{n-u}, \underline{j[n-u+1]}) = \cdots = f^{u+1}(\underline{n-u}, \underline{j[n]})$.

(II) $a_t^u = [d(n-u,t) + c(n-u) + \cdots + c(t-1)]\dfrac{f^{u+1}(\underline{n-u}, \underline{j[n-u+1]})}{f^{u+1}(\underline{n-u}, \underline{j[n-u]})}$

$\times \prod_{z=u+2}^{n} (1 - \dfrac{f^z(\underline{n+1-z}, \underline{j[n+2-z]})}{f^z(\underline{n+1-z}, \underline{j[n+1-z]})})$,

for $t = n-u+1, \cdots, n$.

(III) $f^u(\underline{t}, \underline{j[n-s+1]}) = f(\underline{t}, \underline{j[n-s+1]}) - [d(0, n-u) + c(1) + \cdots + c(n-u)]$

$+ \sum_{z=u+1}^{n-1} [a_t^z - a_{n-u}^z] + a_t^u > 0$

for $s = 1, \cdots, u$ and $t = n-u+1, \cdots, n$.

First, let $u = n-1$. (I) By the definition of f^n, $f^n(\underline{1}, \underline{j[1]}) = d(0,n) + d(n,1) + c(1) + \cdots + c(n)$. $f^n(\underline{1}, \underline{j[2]}) = \cdots = f^n(\underline{1}, \underline{j[n]}) = d(0,1) + c(1)$.

$$\text{(II) } a_t^{n-1} = [f^n(\underline{1}, \underline{j[1]}) - f^n(\underline{t}, \underline{j[1]})]\frac{f^n(\underline{1}, \underline{j[2]})}{f^n(\underline{1}, \underline{j[1]})}$$
$$= [d(1,t) + c(1) + \cdots + c(t-1)]\frac{f^n(\underline{1}, \underline{j[2]})}{f^n(\underline{1}, \underline{j[1]})},$$

by the definitions of a_t^{n-1} and f^n.

(III) For $s = 1, \cdots, n-1$ and $t = 2, \cdots, n$,

$$f^{n-1}(\underline{t}, \underline{j[n-s+1]}) = f^n(\underline{t}, \underline{j[n-s+1]}) - [d(0,1) + c(1)] + a_t^{n-1} > 0$$

since $f^n(\underline{t}, \underline{j[n-s+1]}) > d(0,1) + c(1)$ and $a_t^{n-1} > 0$ by (2.2) and (II) for $u = n-1$.

Assume (I), (II), and (III) hold for $u+1,\cdots,n$. We check (I), (II), and (III) for u.
(I) From (III) for $u+1$, we have

$$f^{u+1}(\underline{n-u},\underline{j[n-s+1]}) = f^{n}(\underline{n-u},\underline{j[n-s+1]}) - [d(0,n-u-1)+c(1)+\cdots + c(n-u-1)] + \sum_{z=u+2}^{n-1}[a^{z}_{n-u}-a^{z}_{n-u-1}] + a^{u+1}_{n-u}$$

for $s=1,\cdots,u+1$. Here by the definitions of f^n and $\underline{j[n-s+1]}$,

$$\begin{aligned} &f^{n}(\underline{n-u},\underline{j[n-u+1]}) = \cdots = f^{n}(\underline{n-u},\underline{j[n]}) \\ &= d(0,n-u)+c(1)+\cdots+c(n-u), \text{ and} \\ &f^{n}(\underline{n-u},\underline{j[n-u]}) = d(0,n)+d(n,n-u)+c(1)+\cdots+c(n). \end{aligned}$$

These imply (I) for u.
(II) From the definition and (III) for $u+1$, we have

$$\begin{aligned} a^{u}_{t} &= [f^{u+1}(\underline{n-u},\underline{j[n-u]}) - f^{u+1}(\underline{t},\underline{j[n-u]})]\frac{f^{u+1}(\underline{n-u},\underline{j[n-u+1]})}{f^{u+1}(\underline{n-u},\underline{j[n-u]})} \\ &= \{f^{n}(\underline{n-u},\underline{j[n-u]}) - f^{n}(\underline{t},\underline{j[n-u]})] \\ &+ \sum_{z=u+1}^{n-1}[a^{z}_{n-u}-a^{z}_{t}]\}\frac{f^{u+1}(\underline{n-u},\underline{j[n-u+1]})}{f^{u+1}(\underline{n-u},\underline{j[n-u]})} \end{aligned}$$

From (II) for $z=u+1,\cdots,n-1$,

$$a^{z}_{n-u}-a^{z}_{t} = -\frac{f^{z+1}(\underline{n-z},\underline{j[n-z+1]})}{f^{z+1}(\underline{n-z},\underline{j[n-z]})} \times \prod_{r=z+2}^{n}(1-\frac{f^{r}(\underline{n+1-r},\underline{j[n+2-r]})}{f^{r}(\underline{n+1-r},\underline{j[n+1-r]})})[d(n-u,t)+c(n-u)+\cdots+c(t-1)].$$

Summing these for $z=u+1,\cdots,n-1$,

$$\begin{aligned} \sum_{z=u+1}^{n-1}[a^{z}_{n-u}-a^{z}_{t}] &= -[d(n-u,t)+c(n-u)+\cdots+c(t-1)] \\ &\times \sum_{z=u+1}^{n-1}\frac{f^{z+1}(\underline{n-z},\underline{j[n-z+1]})}{f^{z+1}(\underline{n-z},\underline{j[n-z]})}\prod_{r=z+2}^{n}(1-\frac{f^{r}(\underline{n+1-r},\underline{j[n+2-r]})}{f^{r}(\underline{n+1-r},\underline{j[n+1-r]})}). \end{aligned}$$

Noting that $f^{n}(\underline{n-u},\underline{j[n-u]}) - f^{n}(\underline{t},\underline{j[n-u]}) = d(n-u,t)+c(n-u)+\cdots+c(t-1)$ by the definition of f^n, and also by elementary algebra,

$$\begin{aligned} &1-\sum_{z=u+1}^{n-1}\frac{f^{z+1}(\underline{n-z},\underline{j[n-z+1]})}{f^{z+1}(\underline{n-z},\underline{j[n-z]})}\prod_{r=z+2}^{n}(1-\frac{f^{r}(\underline{n+1-r},\underline{j[n+2-r]})}{f^{r}(\underline{n+1-r},\underline{j[n+1-r]})}) \\ &= \prod_{r=u+2}^{n}(1-\frac{f^{r}(\underline{n+1-r},\underline{j[n+2-r]})}{f^{r}(\underline{n+1-r},\underline{j[n+1-r]})}), \end{aligned}$$

we have

$$a^{u}_{t} = [d(n-u,t)+c(n-u)+\cdots+c(t-1)]\frac{f^{u+1}(\underline{n-u},\underline{j[n-u+1]})}{f^{u+1}(\underline{n-u},\underline{j[n-u]})} \times \prod_{r=u+2}^{n}(1-\frac{f^{r}(\underline{n+1-r},\underline{j[n+2-r]})}{f^{r}(\underline{n+1-r},\underline{j[n+1-r]})}).$$

(III) By the induction hypothesis,

$$
\begin{aligned}
f^u(\underline{t}, \underline{j[n-s+1]}) &= f^{u+1}(\underline{t}, \underline{j[n-s+1]}) - f^{u+1}(\underline{n-u}, \underline{j[n-s+1]}) + a_t^u \\
&= f^n(\underline{t}, \underline{j[n-s+1]}) - [d(0, n-u-1) + c(1) + \cdots + c(n-u-1)] \\
&+ \sum_{z=u+2}^{n-1} [a_t^z - a_{n-u-1}^z] + a_t^{u+1} \\
&- \{f^n(\underline{n-u}, \underline{j[n-s+1]}) - [d(0, n-u-1) + c(1) + \cdots + c(n-u-1)] \\
&+ \sum_{z=u+2}^{n-1} [a_{n-u}^z - a_{n-u-1}^z] + a_{n-u}^{u+1}\} + a_t^u \\
&= f^n(\underline{t}, \underline{j[n-s+1]}) - f^u(\underline{n-u}, \underline{j[n-s+1]}) + \sum_{z=u+1}^{n-1} [a_t^z - a_{n-u}^z] + a_t^u.
\end{aligned}
$$

Here note that $f^n(\underline{n-u}, \underline{j[n-s+1]}) = d(0, n-u) + c(1) + \cdots + c(n-u)$. Thus it remains to prove $f^u(\underline{t}, \underline{j[n-s+1]}) > 0$. By the definition of a_t^u,

$$
\begin{aligned}
a_t^u &= [f^{u+1}(\underline{n-u}, \underline{j[n-u]}) - f^{u+1}(\underline{t}, \underline{j[n-u]})] \frac{f^{u+1}(\underline{n-u}, \underline{j[n-u+1]})}{f^{u+1}(\underline{n-u}, \underline{j[n-u]})} \\
&= \{d(n-u, t) + c(n-u) + \cdots + c(t-1) \\
&+ \sum_{z=u+1}^{n-1} [a_{n-u}^z - a_t^z]\} \frac{f^{u+1}(\underline{n-u}, \underline{j[n-u+1]})}{f^{u+1}(\underline{n-u}, \underline{j[n-u]})}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
f^u(\underline{t}, \underline{j[n-s+1]}) &= f^n(\underline{t}, \underline{j[n-s+1]}) - f^n(\underline{n-u}, \underline{j[n-s+1]}) \\
&+ [d(n-u, t) + c(n-u) + \cdots + c(t-1)] \frac{f^{u+1}(\underline{n-u}, \underline{j[n-s+1]})}{f^{u+1}(\underline{n-u}, \underline{j[n-u]})} \\
&+ \sum_{z=u+1}^{n-1} [a_t^z - a_{n-u}^z](1 - \frac{f^{u+1}(\underline{n-u}, \underline{j[n-s+1]})}{f^{u+1}(\underline{n-u}, \underline{j[n-u]})}).
\end{aligned}
$$

Here $f^n(\underline{t}, \underline{j[n-s+1]}) = d(0, t) + c(1) + \cdots + c(t)$, if $n-u+1 \leq t \leq n-s$, and $f^n(\underline{t}, \underline{j[n-s+1]}) = d(0, n) + d(n, t) + c(1) + \cdots + c(n-s) + c(n) + \cdots + c(t)$, if $n-s+1 \leq t \leq n$. Thus we see $f^n(\underline{t}, \underline{j[n-s+1]}) > d(0, n-u) + c(1) + \cdots + c(n-u) = f^n(\underline{n-u}, \underline{j[n-s+1]})$. From (II) for $u+1, \cdots, n-1$, we have $a_t^z - a_{n-u}^z > 0$ for $z = u+1, \cdots, n-1$. From (I) for $u+1$, we have $1 - \dfrac{f^{u+1}(\underline{n-u}, \underline{j[n-u+1]})}{f^{u+1}(\underline{n-u}, \underline{j[n-u]})} > 0$. Consequently we have $f^u(\underline{t}, \underline{j[n-s+1]}) > 0$. This completes the proof of (I), (II) and (III).

From (4.3),

$$f^2(\underline{n-1}, \underline{j[n]}) = d(n-2, n-1) + c(n-1) + \sum_{z=3}^{n-1}[a_{n-1}^z - a_{n-2}^z] + a_{n-1}^2,$$

$$f^2(\underline{n}, \underline{j[n-1]}) = d(n-2, n) + c(n) + \sum_{z=3}^{n-1}[a_n^z - a_{n-2}^z] + a_n^2,$$

$$f^2(\underline{n-1}, \underline{j[n-1]}) = d(n-2, n) + d(n, n-1) + c(n-1) + c(n)$$
$$+ \sum_{z=3}^{n-1}[a_{n-1}^z - a_{n-2}^z] + a_{n-1}^2,$$

$$f^2(\underline{n}, \underline{j[n]}) = d(n-2, n) + c(n-1) + c(n) + \sum_{z=3}^{n-1}[a_n^z - a_{n-2}^z] + a_n^2.$$

From (III) for $u = 2$, we have $f^2(\underline{n-1}, \underline{j[n-1]}) > f^2(\underline{n-1}, \underline{j[n]}) > 0$ and $f^2(\underline{n}, \underline{j[n]}) > f^2(\underline{n}, \underline{j[n-1]}) > 0$. Hence $det(A_2) \neq 0$. **Q.E.D.**

Proof of the theorem: By Corollary 7, it remains to prove the uniqueness of the optimal strategy for Player 1. Suppose p' is an optimal strategy for Player 1. By Lemma 4 and Corollary 7, $f(p', q^{*n}) = \sum_{\underline{j} \in \underline{M}_1} q_j^{*n} f(p', \underline{j}) = v^n$ and $f(p', \underline{j}) \geq v^n$ for all $\underline{j} \in \underline{M}_1$. Hence $f(p', \underline{j}) = v^n$ for all $\underline{j} \in \underline{M}_1$. This, combined with Lemma 5 and Lemma 8, implies $p' = p^n$. **Q.E.D.**

5. Remarks.

(i) It is interesting to compare the value of the game with the minimum value of the one-person problem which is derived from the model here by replacing Player 1 by the nature. If an object is in one of n cells with a priori probabilities $p_1, \cdots, p_n$, the problem becomes : Minimize $f(p,q)$ subject to $q \in Q$. If a one decision-maker problem is uncertain (that is, not risky) the decision-maker may assume the uniform distribution. Thus, let p^u be the uniform distribution on N. That is, $p^u = (1/n, \cdots, 1/n)$. Then $\mathrm{Min}\{f(p^u, q) : q \in Q\} \leq f(p^u, q^{*n}) = v^n$ by Lemma 4 and Theorem 3. If $c(1) = c(2) = \cdots = c(n) = c$, then $\mathrm{Min}\{f(p^u, q) : q \in Q\} = \mathrm{Min}\{f(p^u, \underline{j}) : \underline{j} \in \underline{M}_1\} = \mathrm{Min}\{(n+1)c/2 + \frac{1}{n}\sum_{i=1}^{n} d(0,i) + \frac{2}{n} \sum_{\underline{j}^{-1}(i) \geq \underline{j}^{-1}(n)} d(i,n))\} = (n+1)c/2 + \frac{1}{n}\sum_{i=1}^{n} d(0,i) = f(p^u, \underline{1})$, where $\underline{1}$ is the identity permutation. On the other hand, $v^n = d(0,n) + (n+1)c/2$.

Gluss [4] considered the minimization problem : Minimize $f(p\#, \underline{j})$ subject to $\underline{j} \in \underline{M}_1$. Here $p\#_i = 2i/[n(n+1)]$ for $i = 1, \cdots, n$. But he assumed $c(1) = \cdots = c(n) = c$ and $d(i, i+1) = d$ for $i = 0, \cdots, n-1$ (See p. 279 of [4], and p. 185 of [6]).

(ii) It is interesting to consider a continuous version of the model dealt with in this note, in which the interval of [0,1] is given instead of the n-cells. Player 1 chooses one point in it and hides there. We must define strategies for Player 2 suitably before beginning the analysis (see [2] or [3]).

Both Fristedt [2] and Gal [3] treated linear search games, in which the cost is the time the searcher requires to discover the hider divided by the distance of the hider from the origin of the real line. Thus it is an interesting problem to solve the case where the cost, $f(\underline{i}, \underline{j})$, is replaced by $f(\underline{i}, \underline{j})/i$, and to compare with results by them.

(iii) The second variant is a model in which Player 2 is at the central cell at the beginning of the search. Kikuta [5] treated a one-person problem in which Player 1 is replaced by the nature. The analysis of this model has not been done yet.

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