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# TRANSIENT DIFFUSION APPROXIMATION FOR $M/G/m$ SYSTEM

Bong Dae Choi  Yang Woo Shin
*Korea Advanced Institute of Science and Technology*

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*Abstract* We investigate a transient diffusion approximation by diffusion process with elementary return boundary for the number of customers in the $M/G/m$ system. We formulate and solve the forward diffusion equation with variable coefficient, whose solution is a transient approximation to queue size distribution. Numerical examples show that these diffusion approximation results are quite accurate for all traffic cases. It is shown that stationary approximation by Kimura is obtained from our transient diffusion approximation.

## 1. Introduction

The purpose of this paper is to provide a transient diffusion approximation of the number of customers for $M/G/m$ queueing system. The advantage of diffusion approximation over other technique is that explicit though approximate solutions with a high degree of accuracy are obtainable for relatively complex situation where the only possible alternative lies in numerical methods or simulation experiments. Considerable works on an approximation method using a diffusion model have been done in the stationary state (Chiamsiri and Leonard [6], Gaver ([11],[12]), Halachmi and Franta [14], Heyman [16], Iglehart [17], Kimura ([20],[21]), Kobayashi [22], et al.). However, there are many practical situations where we need to know the transient behavior of queueing system. It is often important to know how long it takes an ergodic queueing system to reach steady-state and the rate of convergence in which the system approaches to steady-state. Such an application appears in the problem of deciding when transient phenomena ends, and how many data points are discarded in the course of using a computer simulation to estimate the steady-state characteristics of a queueing system. A brief survey of possible applications is given in Duda [9]. But time-dependent theory of queueing system is much more difficult mathematically than the steady-state theory. Even for $M/M/1$ system the transient exact solution is given in terms of infinite sum of Bessel function and is far from the practical use. The problem of the diffusion approximation is reduced to that of solving ordinary differential equations for stationary behavior and that of solving partial differential equations for transient behavior. In the single server case, the transient approximation was investigated by diffusion process with reflected boundary (Kobayashi [23], Abate and Whitt ([1],[2],[3])) and with elementary return boundary (Duda ([8],[9])). It is known that a diffusion process with elementary return boundary gives more accurate approximation for light traffic conditions in which the system are more frequently empty

(Gelenbe [12]). The multiserver case is treated in this paper. In detail, we investigate a transient approximation by diffusion process with elementary return boundary for the number of customers in the $M/G/m$ system. For a multi-server queueing system we need to solve a partial differential equation with state variable coefficients. In section 2, we first present forward diffusion equation on the real positive line for a diffusion process (strictly speaking, elementary return process) which approximates the number of customers in the system and then derive the explicite solution of the forward diffusion equation. In section 3 we obtain the stationary solution from the transient solution by letting $t \to \infty$ and show that the result coincides with Kimura's result [21]. Numerical examples are then presented in order to evaluate the accuracy in section 4.

## 2. Transient diffusion approximation for M/G/m system

For the $M/G/m$ queueing system, we assume the followings. Suppose that customers arrive at the queueing system at the instants $t_1, t_2, \cdots$, where interarrival times $t_{k+1} - t_k (k = 0, 1, 2, \cdots, t_0 = 0)$ are independent identically distributed random variables with exponentinal distribution with parameter $\lambda$. Assume that there are $m$ identical servers acting in parallel. It is assumed that the service times have general distribution with mean $\frac{1}{\mu}$ and variance ${\sigma_b}^2$, which are independent of the interarrival times and the number of customers in the system. To approximate the number of customers in $M/G/m$ system, we take an elementary return process $\{X(t), t \geq 0\}$ with state space $[0, \infty)$ and with the elementary return boundary at $x = 0$. The elementary return process can be explained as follows. When the trajectory of $X(t)$ reaches the boundary, it remains there for a random interval of time called a holding time. After the sojourn at the boundary the trajectory jumps into the interior of the region and starts from scratch. In the queueing context the holding time at $x = 0$ represents the time interval during which the system is empty. Since arrival process is Poisson, the holding time has exponential distribution with parameter $\lambda$. The elementary return process was fully investigated by Feller [10].

Let the elementary return process $\{X(t), t \geq 0\}$ with state space $[0, \infty)$ and $X(0) = x_0$ be an approximation of the number of customers in the $M/G/m$ system. Then the process $X(t)$ is specified by the diffusion parameters $a(x)$ and $b(x)$ called infinitesimal variance and infinitesimal mean and defined by

$$a(x) = \lim_{\Delta t \to 0} \frac{Var(X(t + \Delta t) - X(t) \mid X(t) = x)}{\Delta t} \tag{2.1}$$

$$b(x) = \lim_{\Delta t \to 0} \frac{E(X(t + \Delta t) - X(t) \mid X(t) = x)}{\Delta t} \tag{2.2}$$

Define the probability density function $f(x, t \mid x_0)$ of $X(t)$ given $X(0) = x_0$ by

$$f(x, t \mid x_0)dx = Pr(x \leq X(t) < x + dx \mid X(0) = x_0) \tag{2.3}$$

Since $X(t)$ approximates the number of customers in the system at time $t$, we assume the initial value $X(0) = x_0$ is nonnegative integer throughout this paper. Since the the holding time at origin has exponential distribution with mean $\frac{1}{\lambda}$, $f(x,t \mid x_0)$ satisfies the following partial differential equation, so called a forward equation (or Fokker-Planck equation) (Feller [10])

$$\frac{\partial f}{\partial t} = \frac{1}{2}\frac{\partial^2}{\partial x^2}\{a(x)f(x,t \mid x_0)\} - \frac{\partial}{\partial x}\{b(x)f(x,t \mid x_0)\} + \lambda P(t)\delta(x-1) \tag{2.4}$$

and

$$\frac{dP(t)}{dt} = -\lambda P(t) + \lim_{x \downarrow 0} C_{x,t} f \tag{2.5}$$

where

$$C_{x,t} f = \frac{1}{2}\frac{\partial}{\partial x}\{a(x)f(x,t \mid x_0)\} - b(x)f(x,t \mid x_0),$$

and $P(t)$ denotes the probabilities that the process $X(t)$ is at origin at time $t$ and $\delta(\cdot)$ is the Dirac's delta function. In addition to the above partial differential equations on $f(x,t \mid x_0)$, we must specify the boundary condition at $x = 0$ and the initial condition at $t = 0$. Since the boundary at $x = 0$ behaves as absorbing boundary during their holding time we set

$$\lim_{x \downarrow 0} f(x,t|x_0) = 0 \tag{2.7}$$

for all $t \geq 0$. We set

$$f(x,0 \mid x_0) = \delta(x - x_0) \tag{2.8}$$

and

$$P(0) = \begin{cases} 0 & \text{if } x_0 > 0 \\ 1 & \text{if } x_0 = 0. \end{cases} \tag{2.9}$$

We use the diffusion parameters proposed by Kimura [21] as follows;

$$a(x) = \lambda + \min(\lceil x \rceil, m)\mu^3 {\sigma_b}^2 \tag{2.10a}$$

$$b(x) = \lambda - \min(\lceil x \rceil, m)\mu \tag{2.10b}$$

where $\lceil x \rceil$ is the smallest integer not smaller than $x$. The main problems of diffusion approximation are to choose appropriate diffusion parameters and boundary conditions and then to solve the partial differential equation (2.4).

Now we present the solution of partial differential equation (2.4) which is a transient approximation of the number of customers in the $M/G/m$ system. The transient solutions derived in this section have the forms of Laplace transform and the corresponding time-dependent functions can be obtained numerically.

Let $a_k = a(k)$, $b_k = b(k)$, $k = 1, 2, \cdots, m$ and $f_k(x, t|x_0)$ be the restriction of $f(x, t|x_0)$ to $k-1 < x \leq k$, $t \geq 0$, $k = 1, 2, \cdots, m-1$ and $f_m(x, t|x_0)$ the restriction of $f(x, t|x_0)$ to $m-1 < x < \infty$, $t \geq 0$. Then the equation (2.4) becomes for $k-1 < x < k$, $t > 0$, $k = 1, 2, \cdots, m-1$,

$$\frac{\partial f_k}{\partial t} = \frac{1}{2} a_k \frac{\partial^2 f_k}{\partial x^2} - b_k \frac{\partial f_k}{\partial x} \tag{2.11}$$

and for $m-1 < x < \infty$, $t > 0$,

$$\frac{\partial f_m}{\partial t} = \frac{1}{2} a_m \frac{\partial^2 f_m}{\partial x^2} - b_m \frac{\partial f_m}{\partial x}. \tag{2.12}$$

It should be noted that there exists a continuous solution of the equation (2.4) even if $a(x)$ and $b(x)$ are piecewise continuous functions with a finite number of discontinuities (Mandl [25]). Hence we impose the following smooth conditions (see Kimura [21] for stationary case)

$$\lim_{x \downarrow k-1} f_k(x, t|x_0) = f_{k-1}(k-1, t|x_0) \quad k = 2, 3, \cdots, m. \tag{2.13}$$

For the convenience of solving partial differential equation, let $g_k(t|x_0) = f_k(k, t|x_0)$, $k = 1, 2, \cdots, m-1$, and $f_k(k-1, t|x_0) = \lim_{x \downarrow k-1} f_k(x, t|x_0) \quad k = 1, 2, \cdots, m$.
The problem of solving the differential equation (2.4) is reduced to the following initial boundary value problems, for $k-1 < x < k$, $t > 0$,$k = 1, 2, \cdots, m-1$

$$\frac{\partial f_k}{\partial t} = \frac{1}{2} a_k \frac{\partial^2 f_k}{\partial x^2} - b_k \frac{\partial f_k}{\partial x}, \tag{2.14}$$

$$f_k(k-1, t|x_0) = g_{k-1}(t|x_0), \tag{2.15a}$$

$$f_k(k, t|x_0) = g_k(t|x_0), \tag{2.15b}$$

$$f_k(x, 0|x_0) = \delta(x - x_0), \tag{2.15c}$$

and for $m-1 < x < \infty$, $t > 0$,

$$\frac{\partial f_m}{\partial t} = \frac{1}{2} a_m \frac{\partial^2 f_k}{\partial x^2} - b_m \frac{\partial f_m}{\partial x}, \tag{2.16}$$

$$f_m(m-1, t|x_0) = g_{m-1}(t|x_0), \tag{2.17a}$$

$$f_m(x, 0|x_0) = \delta(x - x_0). \tag{2.17b}$$

From the conditions (2.7), we have that $g_0(t|x_0) = 0$ for all $t \geq 0$. In this paper the Laplace transform of a given function $f(t)$ is defined by

$$f^*(s) = \int_0^\infty e^{-st} f(t) dt.$$

**Proposition 1.** Let $\{X(t), t \geq 0\}$ be a elementary return process formulated in this section approximating the number of customers in the $M/G/m$ system. The Laplace transform $f^*(x, s|x_0)$ of the density function $f(x, t|x_0)$ of $X(t)$ given $X(0) = x_0$ is given as follows: For $k-1 < x \leq k, k = 1, 2, \cdots, m-1$

$$\begin{aligned} f_k^*(x, s|x_0) = &\exp(\frac{b_k}{a_k}(x-k)) \frac{\sinh A_k(x-k+1)}{\sinh A_k} g_k^*(s|x_0) \\ &+ \exp(\frac{b_k}{a_k}(x-k+1)) \frac{\sinh A_k(k-x)}{\sinh A_k} g_{k-1}^*(s|x_0) \end{aligned} \tag{2.18}$$

and, for $m-1 < x < \infty$,

$$\begin{aligned} f_m^*(x, s|x_0) = &\exp((\frac{b_m}{a_m} - A_m)(x-m+1)) g_{m-1}^*(s|x_0) \\ &+ \frac{2}{a_m A_m} \exp(\frac{b_m}{a_m}(x - x_0)) \left\{ e^{-A_m(x_0 - m + 1)} \sinh A_m(x-m+1) \right. \\ &\left. - \sinh A_m(x - x_0) U(x - x_0) \right\} 1(x_0 \geq m), \end{aligned} \tag{2.19}$$

and $g_k^*(s|x_0)$ are given by

$$g_1^*(s) = \frac{1}{B_1}(\lambda + s)P^*(s) - \frac{1}{B_1}1(x_0 = 0), \tag{2.20}$$

$$g_2^*(s) = \frac{C_2}{B_2}g_1^*(s) - \frac{1}{B_2}\lambda P^*(s) - \frac{1}{B_2}1(x_0 = 1), \tag{2.21}$$

$$g_k^*(s) = \frac{C_k}{B_k}g_{k-1}^*(s) - \frac{B_{k-1}}{B_k}e^{2\frac{b_{k-1}}{a_{k-1}}}g_{k-2}^*(s) - \frac{1}{B_k}1(x_0 = k-1), \tag{2.22}$$

$$k = 3, 4, \cdots, m-1,$$

$$g_{m-1}^*(s|x_0) = \frac{1}{C_m}B_{m-1}e^{2\frac{b_{m-1}}{a_{m-1}}}g_{m-2}^*(s|x_0) + \frac{1}{C_m}e^{-(\frac{b_m}{a_m}+A_m)(x_0-m+1)}1(x_0 \geq m-1), \tag{2.23}$$

where $1(D)$ is the indicator of $D$ and $U(x) = 1(x \geq 0)$ and

$$A_k = \frac{\sqrt{2a_k s + b_k^2}}{a_k}, \qquad k = 1, 2, \cdots, m, \tag{2.24}$$

$$B_k = \frac{a_k A_k}{2}e^{-\frac{b_k}{a_k}}\frac{1}{\sinh A_k}, \qquad k = 1, 2, \cdots, m-1, \tag{2.25}$$

$$C_k = -\frac{b_{k-1}}{2} + \frac{a_{k-1}A_{k-1}}{2}\frac{\cosh A_{k-1}}{\sinh A_{k-1}} + \frac{b_k}{2} + \frac{a_k A_k}{2}\frac{\cosh A_k}{\sinh A_k}, \tag{2.26}$$

$$k = 2, 3, \cdots, m-1,$$

$$C_m = -\frac{b_{m-1}}{2} + \frac{a_{m-1}A_{m-1}}{2}\frac{\cosh A_{m-1}}{\sinh A_{m-1}} + \frac{b_m}{2} + \frac{a_m A_m}{2}. \tag{2.27}$$

**Proof.** For the derivation of (2.18) and (2.19), see Appendix. Next we will determine $g_k^*(s|x_0)$ in the expression (2.18) and (2.19) in the terms of known parameters. We take the Laplace transform of equation (2.4) with respect to $t$ variable, and then integrate with

respect to $x$ variable. Then we have

$$\frac{1}{2}\frac{\partial}{\partial x}\{a(x)f^*(x,s|x_0)\} - b(x)f^*(x,s|x_0) \tag{2.28}$$

$$=[C_{x,s}f^*]_{x\downarrow 0} + s\int_0^x f^*(y,s|x_0)dy - U(x-x_0) - \lambda P^*(s)U(x-1),$$

where

$$C_{x,s}f^* = \frac{1}{2}\frac{\partial}{\partial x}\{a(x)f^*(x,s|x_0)\} - b(x)f^*(x,s|x_0).$$

Simple calculation from (2.28) gives

$$[C_{x,s}f_2^*]_{x\downarrow 1} = [C_{x,s}f_1^*]_{x\uparrow 1} - \lambda P^*(s) - 1(x_0 = 1), \tag{2.29}$$

$$[C_{x,s}f_k^*]_{x\downarrow k-1} = [C_{x,s}f_{k-1}^*]_{x\uparrow k-1} - 1(x_0 = k-1), \tag{2.30}$$

where $k = 3, 4, \cdots, m$. Taking the Laplace transform of equation (2.5) yields

$$[C_{x,s}f_1^*]_{x\downarrow 0} = (s+\lambda)P^*(s) - P(0). \tag{2.31}$$

From (2.18), (2.19), (2.29), (2.30) and (2.31), we can obtain $g_k^*$'s in terms of $P^*$ as follows. Let us show how to find only $g_1^*$ and $g_2^*$. First, we calculate $C_{x,s}f_1^*$ from (2.18). The left hand side of (2.31) is equal to $B_1 g_1^*(s|x_0)$. Thus (2.20) can be obtained from (2.31). By calculating $C_{x,s}f_1^*$ from (2.18), we obtain that the right hand side of (2.29) is equal to

$$g_1^*(s)\left(-\frac{b_1}{2} + \frac{a_1 A_1}{2}\frac{\cosh A_1}{\sinh A_1}\right) - \lambda P^*(s) - 1(x_0 = 1)$$

and the left hand side of (2.29) is

$$g_2^*(s)B_2 - g_1^*(s)\left(\frac{b_2}{2} + \frac{a_2 A_2}{2}\frac{\cosh A_2}{\sinh A_2}\right).$$

Thus (2.21) is obtained from (2.29). By the same method (2.22) obtained from (2.30). □

We need to express $P^*(s)$ in terms of known parameters. The equations (2.20) - (2.23) tell us that $g_k^*$, $k = 1, 2, \cdots, m-1$ can be represented in terms of only $P^*$. Hence the $f_k^*$'s

and $f_m^*$ in (2.18) and (2.19) can be represented in terms of only $P^*$. Using the conservation of probability

$$P(t) + \int_0^\infty f(x,t|x_0)dx = 1 \tag{2.32}$$

which is equivalent to

$$P^*(s) + \int_0^\infty f^*(x,s|x_0)dx = \frac{1}{s}, \qquad \mathrm{Re}\, s > 0, \tag{2.33}$$

$P*(s)$ can be represented in terms of known parametes.

## 3. Stationary diffusion approximation for $M/G/m$ system

In this section we obtain the stationary approximation by letting $t \to \infty$ in the transient approximation for the $M/G/m$ system. Let $f_k(x) = \lim_{t\to\infty} f_k(x,t|x_0)$, $g_k = \lim_{t\to\infty} g_k(t|x_0)$, $k = 1,2,\cdots,m-1$, $f_m(x) = \lim_{t\to\infty} f_m(x,t|x_0)$ and $P = \lim_{t\to\infty} P(t)$. To obtain the limiting probability, we use the final value theorem for Laplace transform; $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sf^*(s)$.

**Proposition 2.** Let $\{X(t), t \geq 0\}$ be a elementary return process defined in section 2 approximating the number of customers in $M/G/m$ system. Under the condition $b_m < 0$, that is, $\rho = \frac{\lambda}{m\mu} < 1$, stationsry function $f(x)$ is given as follows: for $0 < x \leq 1$,

$$f_1(x) = \frac{\lambda P}{b_1}(e^{2\frac{b_1}{a_1}x} - 1), \tag{3.1}$$

for $k-1 < x \leq k$, $k = 2,3,\cdots,m-1$,

$$f_k(x) = \frac{\lambda P}{b_1}(e^{\frac{2b_1}{a_1}} - 1)\Big(\prod_{j=2}^{k} e^{\frac{2b_j}{a_j}}\Big)e^{2\frac{b_k}{a_k}(x-k)}, \quad k = 2,3,\cdots,m-1, \tag{3.2}$$

for $m-1 < x < \infty$,

$$f_m(x) = \frac{\lambda P}{b_1}(e^{\frac{2b_1}{a_1}} - 1)\Big(\prod_{j=2}^{m-1} e^{\frac{2b_j}{a_j}}\Big)e^{2\frac{b_m}{a_m}(x-m+1)}. \tag{3.3}$$

When $b_k \neq 0$, $k = 1, 2, \cdots, m-1$,

$$P = \left(1 - \frac{\lambda}{b_1} + \sum_{k=1}^{m-1} \left(\frac{a_k}{2b_k} - \frac{a_{k+1}}{2b_{k+1}}\right) q_k\right)^{-1}. \tag{3.4}$$

When there exists some $i$ with $b_i = 0$ (if it exists it is unique),

$$P = \left(1 - \frac{\lambda}{b_1} + \sum_{\substack{k=1 \\ k \neq i, i-1}}^{m-1} \left(\frac{a_k}{2b_k} - \frac{a_{k+1}}{2b_{k+1}}\right) q_k + q_{i-1}\left(1 + \frac{a_{i-1}}{2b_{i-1}} - \frac{a_i}{2b_{i+1}}\right)\right)^{-1}, \tag{3.5}$$

where

$$q_1 = \frac{\lambda P}{b_1}\left(e^{2\frac{b_1}{a_1}} - 1\right)$$

$$q_k = q_1 \cdot \prod_{j=2}^{k} e^{\frac{2b_j}{a_j}}, \quad k = 2, 3, \cdots, m-1.$$

**Proof.** Since $\lim_{s \to 0} A_k(s) = \frac{|b_k|}{a_k}$, we have from (2.25) that

$$\lim_{s \to 0} B_k(s) = \frac{b_k}{e^{2\frac{b_k}{a_k}} - 1}, \quad k = 1, 2, \cdots, m-1.$$

Thus we have from (2.20) that

$$g_1 = \lim_{s \to 0} s g_1^*(s) = \frac{\lambda}{b_1}\left(e^{2\frac{b_1}{a_1}} - 1\right) P. \tag{3.6}$$

Note from (2.26) that, for $k = 1, 2, \cdots m-1$,

$$\lim_{s \to 0} C_k(s) = b_{k-1} \frac{1}{e^{2\frac{b_{k-1}}{a_{k-1}}} - 1} + b_k \frac{e^{2\frac{b_k}{a_k}}}{e^{2\frac{b_k}{a_k}} - 1}.$$

Thus we have from (2.21) and the above results that

$$g_2 = \frac{b_1}{b_2}\frac{e^{2\frac{b_2}{a_2}}-1}{e^{2\frac{b_1}{a_1}}-1}g_1 - \frac{e^{2\frac{b_2}{a_2}}-1}{b_2}\lambda P + e^{2\frac{b_2}{a_2}}g_1 = e^{2\frac{b_2}{a_2}}g_1, \tag{3.7}$$

where (3.6) has been used in the last equality. We have from (2.22) that, for $k = 2, 3, \cdots, m-1$,

$$\begin{aligned} g_k &= \frac{b_{k-1}}{b_k}\frac{e^{2\frac{b_k}{a_k}}-1}{e^{2\frac{b_{k-1}}{a_{k-1}}}-1}g_{k-1} + e^{2\frac{b_k}{a_k}}g_{k-1} - \frac{b_{k-1}}{b_k}\frac{e^{2\frac{b_k}{a_k}}-1}{e^{2\frac{b_{k-1}}{a_{k-1}}}-1}e^{2\frac{b_{k-1}}{a_{k-1}}}g_{k-2} \\ &= \frac{b_{k-1}}{b_k}\frac{e^{2\frac{b_k}{a_k}}-1}{e^{2\frac{b_{k-1}}{a_{k-1}}}-1}(g_{k-1} - e^{2\frac{b_{k-1}}{a_{k-1}}}g_{k-2}) + e^{2\frac{b_k}{a_k}}g_{k-1}. \end{aligned} \tag{3.8}$$

Since $g_2 = e^{2\frac{b_2}{a_2}}g_1$, (3.8) with $k = 3$ gives $g_3 = e^{2\frac{b_3}{a_3}}g_2$. By induction we obtain

$$g_k = e^{2\frac{b_k}{a_k}}g_{k-1}, \quad k = 3, 4, \cdots, m-1. \tag{3.9}$$

Now we find the equilibrium condition. We have from (2.19) that

$$f_m(x) = g_{m-1}e^{(\frac{b_m}{a_m} - \frac{|b_m|}{a_m})(x-m+1)}, \quad x \geq m-1. \tag{3.10}$$

Suppose $b_m \geq 0$, that is, $\rho = \lambda/m\mu \geq 1$. Then $f_m(x) = g_{m-1}$ $(> 0)$ for all $x \geq m-1$ and hence $\int_{m-1}^{\infty} f_m(x)dx = \infty$. Thus the conservation of probability

$$P + \sum_{k=1}^{m-1}\int_{k-1}^{k} f_k(x)dx + \int_{m-1}^{\infty} f_m(x)dx = 1 \tag{3.11}$$

holds if and only if $b_m < 0$, which is the equilibrium condition. Now assume $b_m < 0$, that is, $\rho = \lambda/m\mu < 1$. (3.10) becomes

$$f_m(x) = g_{m-1}e^{2\frac{b_m}{a_m}(x-m+1)}, \quad x > m-1 \tag{3.12}$$

Thus (3.3) is obtained from (3.6), (3.9) and (3.12). For $k = 1$, letting $t \to \infty$ in (2.18), we have

$$f_1(x) = \frac{e^{2\frac{b_1}{a_1}x} - 1}{e^{2\frac{b_1}{a_1}} - 1} g_1. \tag{3.13}$$

Substituting (3.6) into (3.13) yields (3.1). To derive $f_k(x)$,$k = 2, 3, \cdots, m-1$, we need to calculate the followings; for $k = 2, 3, \cdots, m-1$,

$$\lim_{s \to 0} \left( \frac{\sinh A_k(x-k+1)}{\sinh A_k} s g_k^*(s) - e^{\frac{b_k}{a_k}} \frac{\sinh A_k(x-k)}{\sinh A_k} s g_{k-1}^*(s) \right)$$

$$= \frac{e^{\frac{b_k}{a_k}(x-k+1)} - e^{-\frac{b_k}{a_k}(x-k+1)}}{e^{\frac{b_k}{a_k}} - e^{-\frac{b_k}{a_k}}} g_k - \frac{e^{\frac{b_k}{a_k}(x-k)} - e^{-\frac{b_k}{a_k}(x-k)}}{e^{\frac{b_k}{a_k}} - e^{-\frac{b_k}{a_k}}} e^{\frac{b_k}{a_k}} g_{k-1}$$

$$= e^{\frac{b_k}{a_k}(x-k)} g_k,$$

where (3.9) has been used in the second equality. Now we use the above result when we let $t$ approach $\infty$ in (2.18). Thus we have

$$f_k(x) = e^{2\frac{b_k}{a_k}(x-k)} g_k \tag{3.14}$$

Thus (3.2)is obtained from (3.6), (3.9) and (3.14). The value of $P$ is determined by the condition (3.11) as follows. Let us assume $b_k \neq 0$, $k = 1, 2, \cdots, m-1$. The case of $b_k = 0$ for some $k = 1, 2, \cdots, m-1$ will be treated later. Simple calculation gives

$$1 = P + \sum_{1}^{m-1} \int_{k-1}^{k} f_k(x)dx + \int_{m-1}^{\infty} f_m(x)dx$$

$$= P \cdot \left( 1 - \frac{\lambda}{b_1} + \sum_{k=1}^{m-1} \left( \frac{a_k}{2b_k} - \frac{a_{k+1}}{2b_{k+1}} \right) q_k \right)$$

and hence we have (3.4). When there exists an $i$ such that $b_i = 0$, then $f_i(x)$ is constant $q_i \cdot P$ for $i - 1 < x \leq i$ and clearly $\int_{i-1}^{i} f_i(x)dx = q_i \cdot P$, by letting $b_i \to 0$ in the right hand side of (3.11), (3.5) is obtained. □

**Remark** The proposition 2 coincides with the Kimura's result [21, p309].

## 4. Numerical results

In order to examine the accuracy of the diffusion approximation, we shall numerically compare the approximate results with the simulation results. The discretization of continuous density function $f(x,t|x_0)$ can be done in several different ways (Chiamsiri and Leonard [6], Gelenbe [12], Halachmi and Franta [14], Kobayashi [22]). We adopt the following one; for $n = 1, 2, \cdots$,

$$P_n(t) = f(n, t|x_0). \tag{4.1}$$

Table 1 (light traffic case $\rho = 0.3$), Table 2 (moderate traffic case $\rho = 0.5$) and Table 3 (heavy traffic case $\rho = 0.7$ ) present the comparison of the transient diffusion approximation results with the simulation results for the $M/M/3$ queueing system. We do the same comparisonfor $M/H_2/3$ bqueueing systemin the case of $\rho = 0.3$, 0.5, 0.7 in the Tables 4 - 6. The hyperexponential density function used in Tables 4 - 6is given by $b(x) = 0.3\mu_1 e^{-\mu_1 x} + 0.7\mu_2 e^{-\mu_2 x}$, where $\mu_1 = 5.0$ and $\mu_2 = \frac{0.7\mu_1}{\rho\mu_1 - 0.3}$ for each traffic intensity $\rho$. In the tables $diff$ denotes the diffusion approximation results with discretization methods (4.1) and $sim$ denotes the simulation results. In case $t = \infty$ we compare the diffusion results with the exact solution. We adopt the Stehfest's method [27] to obtain the numerical inversion of Laplace transform. The approximate numerical inversion $f(t)$ of $f^*(s)$ at time $t$ is given by

$$f(t) = \frac{\ln 2}{t} \sum_{i=1}^{N} V_i f^*(\frac{\ln 2}{t} i)$$

where the coefficient

$$V_i = (-1)^{\frac{N}{2}+i} \sum_{k=[\frac{i+1}{2}]}^{Min(i,\frac{N}{2})} \frac{k^{\frac{N}{2}}(2k)!}{(\frac{N}{2}-k)!k!(k-1)!(i-k)!(2k-i)!}$$

depends only on the constant N. In Tables, $P_n(t)$'s are calculated with double precision arithmetic and $N = 10$. Simulation results in the tables are obtained with thirty thousand times run. The confidence intervals are calculated with the batch means method (Bratley et al. [4]) assuming the confidence level 95 %, which are based on the $t$-statistic applied to the thirty batches of size one thousand. Tables show that the accuracy of diffusion approximation method is high for all traffic cases. It can be seen by comparing the transient probability $P_k(t)$ with the stationary probability $P_k(\infty)$ in the tables that it takes a short period of time to reach the steady-state for the light traffic cases and it takes rather long period of time to reach the steady-state for the heavy traffic cases as we expected.

**TABLE 1**

Comparison of the diffusion results with simulation for $M/M/3$ Queue

($x_0 = 3$, $\lambda = 3.0$, $\rho = 0.3$)

| t | method | $P_0(t)$ | $P_1(t)$ | $P_2(t)$ | $P_3(t)$ | $P_4(t)$ | $P_5(t)$ | $P_6(t)$ | $P_7(t)$ |
|---|---|---|---|---|---|---|---|---|---|
| 0.1 | diff | 0.0032 | 0.1791 | 0.4176 | 0.3005 | 0.1160 | 0.0211 | 0.0018 | 0.0001 |
| | sim | 0.0160 | 0.1395 | 0.3685 | 0.3679 | 0.0945 | 0.0122 | 0.0014 | 0.0001 |
| | c.i. | 0.0015 | 0.0066 | 0.0071 | 0.0071 | 0.0055 | 0.0016 | 0.0006 | 0.0002 |
| 0.3 | diff | 0.1276 | 0.3559 | 0.2905 | 0.1455 | 0.0659 | 0.0247 | 0.0074 | 0.0017 |
| | sim | 0.1407 | 0.3270 | 0.2971 | 0.1502 | 0.0605 | 0.0192 | 0.0044 | 0.0007 |
| | c.i. | 0.0048 | 0.0079 | 0.0076 | 0.0055 | 0.0049 | 0.0021 | 0.0008 | 0.0004 |
| 0.5 | diff | 0.2494 | 0.3654 | 0.2310 | 0.1011 | 0.0441 | 0.0182 | 0.0067 | 0.0022 |
| | sim | 0.2485 | 0.3610 | 0.2332 | 0.0983 | 0.0389 | 0.0139 | 0.0043 | 0.0015 |
| | c.i. | 0.0066 | 0.0072 | 0.0056 | 0.0045 | 0.0037 | 0.0024 | 0.0012 | 0.0006 |
| 0.7 | diff | 0.3194 | 0.3632 | 0.2017 | 0.0808 | 0.0333 | 0.0136 | 0.0054 | 0.0020 |
| | sim | 0.3180 | 0.3647 | 0.2011 | 0.0728 | 0.0277 | 0.0101 | 0.0037 | 0.0015 |
| | c.i. | 0.0082 | 0.0062 | 0.0070 | 0.0036 | 0.0018 | 0.0019 | 0.0013 | 0.0006 |
| 1.0 | diff | 0.3689 | 0.3618 | 0.1825 | 0.0674 | 0.0257 | 0.0100 | 0.0039 | 0.0015 |
| | sim | 0.3688 | 0.3650 | 0.1763 | 0.0575 | 0.0216 | 0.0070 | 0.0025 | 0.0005 |
| | c.i. | 0.0074 | 0.0076 | 0.0052 | 0.0049 | 0.0022 | 0.0013 | 0.0006 | 0.0004 |
| 3.0 | diff | 0.4034 | 0.3627 | 0.1699 | 0.0579 | 0.0167 | 0.0067 | 0.0023 | 0.0008 |
| | sim | 0.4030 | 0.3631 | 0.1628 | 0.0494 | 0.0147 | 0.0049 | 0.0014 | 0.0004 |
| | c.i. | 0.0070 | 0.0088 | 0.0059 | 0.0033 | 0.0022 | 0.0012 | 0.0006 | 0.0003 |
| 5.0 | diff | 0.4033 | 0.3628 | 0.1699 | 0.0579 | 0.0197 | 0.0067 | 0.0023 | 0.0008 |
| | sim | 0.4049 | 0.3612 | 0.1635 | 0.0505 | 0.0142 | 0.0039 | 0.0013 | 0.0004 |
| | c.i. | 0.0082 | 0.0071 | 0.0062 | 0.0035 | 0.0020 | 0.0011 | 0.0005 | 0.0003 |
| 7.0 | diff | 0.4040 | 0.3631 | 0.1701 | 0.0579 | 0.0197 | 0.0067 | 0.0023 | 0.0008 |
| | sim | 0.4041 | 0.3642 | 0.1633 | 0.0472 | 0.0147 | 0.0048 | 0.0012 | 0.0003 |
| | c.i. | 0.0077 | 0.0072 | 0.0055 | 0.0041 | 0.0020 | 0.0011 | 0.0005 | 0.0003 |
| 10.0 | diff | 0.4034 | 0.3628 | 0.1699 | 0.0579 | 0.0197 | 0.0067 | 0.0023 | 0.0008 |
| | sim | 0.4026 | 0.3606 | 0.1647 | 0.0488 | 0.0157 | 0.0053 | 0.0017 | 0.0004 |
| | c.i. | 0.0080 | 0.0089 | 0.0046 | 0.0035 | 0.0022 | 0.0011 | 0.0006 | 0.0004 |
| 20.0 | diff | 0.4037 | 0.3628 | 0.1699 | 0.0579 | 0.0197 | 0.0067 | 0.0023 | 0.0008 |
| | sim | 0.4092 | 0.3600 | 0.1621 | 0.0471 | 0.0156 | 0.0040 | 0.0012 | 0.0006 |
| | c.i. | 0.0079 | 0.0081 | 0.0048 | 0.0031 | 0.0015 | 0.0007 | 0.0006 | 0.0004 |
| $\infty$ | diff | 0.4034 | 0.3628 | 0.1699 | 0.0579 | 0.0197 | 0.0067 | 0.0023 | 0.0008 |
| | exact | 0.4035 | 0.3631 | 0.1634 | 0.0490 | 0.0147 | 0.0044 | 0.0013 | 0.0004 |

**TABLE 2**

Comparison of the diffusion results with simulation for $M/M/3$ Queue

($x_0 = 3$, $\lambda = 3.0$, $\rho = 0.5$)

| t | method | $P_0(t)$ | $P_1(t)$ | $P_2(t)$ | $P_3(t)$ | $P_4(t)$ | $P_5(t)$ | $P_6(t)$ | $P_7(t)$ |
|---|---|---|---|---|---|---|---|---|---|
| 0.1 | diff | 0.0002 | 0.0668 | 0.3691 | 0.4052 | 0.1647 | 0.0223 | 0.0010 | 0.0000 |
| | sim | 0.0050 | 0.0641 | 0.2923 | 0.4877 | 0.1311 | 0.0180 | 0.0017 | 0.0003 |
| | c.i. | 0.0011 | 0.0032 | 0.0068 | 0.0071 | 0.0062 | 0.0024 | 0.0006 | 0.0002 |
| 0.3 | diff | 0.0359 | 0.2303 | 0.3244 | 0.2326 | 0.1297 | 0.0250 | 0.0146 | 0.0029 |
| | sim | 0.0477 | 0.1950 | 0.3220 | 0.2497 | 0.1262 | 0.0449 | 0.0114 | 0.0023 |
| | c.i. | 0.0032 | 0.0051 | 0.0063 | 0.0067 | 0.0040 | 0.0031 | 0.0015 | 0.0007 |
| 0.5 | diff | 0.0926 | 0.2722 | 0.2872 | 0.1830 | 0.1051 | 0.0512 | 0.0204 | 0.0066 |
| | sim | 0.0967 | 0.2506 | 0.2924 | 0.1851 | 0.1042 | 0.0470 | 0.0174 | 0.0050 |
| | c.i. | 0.0043 | 0.0071 | 0.0074 | 0.0055 | 0.0034 | 0.0031 | 0.0019 | 0.0012 |
| 0.7 | diff | 0.1348 | 0.2879 | 0.2666 | 0.1592 | 0.0909 | 0.0473 | 0.0217 | 0.0087 |
| | sim | 0.1362 | 0.2807 | 0.2653 | 0.1565 | 0.0886 | 0.0430 | 0.0190 | 0.0074 |
| | c.i. | 0.0060 | 0.0056 | 0.0067 | 0.0058 | 0.0035 | 0.0032 | 0.0023 | 0.0016 |
| 1.0 | diff | 0.1772 | 0.2999 | 0.2514 | 0.1415 | 0.0789 | 0.0422 | 0.0212 | 0.0097 |
| | sim | 0.1666 | 0.3018 | 0.2470 | 0.1377 | 0.0760 | 0.0399 | 0.0167 | 0.0090 |
| | c.i. | 0.0049 | 0.0054 | 0.0063 | 0.0058 | 0.0034 | 0.0031 | 0.0019 | 0.0013 |
| 3.0 | diff | 0.2143 | 0.3172 | 0.2391 | 0.1233 | 0.0637 | 0.0330 | 0.0171 | 0.0088 |
| | sim | 0.2095 | 0.3108 | 0.2390 | 0.1185 | 0.0612 | 0.0310 | 0.0150 | 0.0070 |
| | c.i. | 0.0067 | 0.0079 | 0.0068 | 0.0056 | 0.0034 | 0.0031 | 0.0019 | 0.0013 |
| 5.0 | diff | 0.2156 | 0.3182 | 0.2392 | 0.1228 | 0.0631 | 0.0324 | 0.0166 | 0.0086 |
| | sim | 0.2115 | 0.3140 | 0.2400 | 0.1164 | 0.0577 | 0.0290 | 0.0155 | 0.0068 |
| | c.i. | 0.0066 | 0.0091 | 0.0069 | 0.0058 | 0.0041 | 0.0027 | 0.0022 | 0.0014 |
| 7.0 | diff | 0.2160 | 0.3185 | 0.2394 | 0.1229 | 0.0631 | 0.0324 | 0.0166 | 0.0085 |
| | sim | 0.2109 | 0.3134 | 0.2380 | 0.1167 | 0.0592 | 0.0297 | 0.0163 | 0.0078 |
| | c.i. | 0.0060 | 0.0085 | 0.0086 | 0.0060 | 0.0035 | 0.0021 | 0.0019 | 0.0015 |
| 10.0 | diff | 0.2157 | 0.3183 | 0.2392 | 0.1228 | 0.0631 | 0.0324 | 0.0166 | 0.0085 |
| | sim | 0.2074 | 0.3152 | 0.2382 | 0.1162 | 0.0605 | 0.0311 | 0.0160 | 0.0078 |
| | c.i. | 0.0047 | 0.0061 | 0.0069 | 0.0045 | 0.0037 | 0.0030 | 0.0022 | 0.0014 |
| 20.0 | diff | 0.2159 | 0.3183 | 0.2392 | 0.1228 | 0.0631 | 0.0324 | 0.0166 | 0.0085 |
| | sim | 0.2140 | 0.3162 | 0.2348 | 0.1166 | 0.0591 | 0.0294 | 0.0152 | 0.0079 |
| | c.i. | 0.0059 | 0.0094 | 0.0072 | 0.0045 | 0.0040 | 0.0023 | 0.0019 | 0.0015 |
| $\infty$ | diff | 0.2157 | 0.3183 | 0.2392 | 0.1228 | 0.0630 | 0.0324 | 0.0166 | 0.0085 |
| | exact | 0.2105 | 0.3158 | 0.2368 | 0.1184 | 0.0592 | 0.0296 | 0.0148 | 0.0074 |

**TABLE 3**

Comparison of the diffusion results with simulation for $M/M/3$ Queue

$(x_0 = 3,\ \lambda = 3.0,\ \rho = 0.7)$

| t | method | $P_0(t)$ | $P_1(t)$ | $P_2(t)$ | $P_3(t)$ | $P_4(t)$ | $P_5(t)$ | $P_6(t)$ | $P_7(t)$ |
|---|---|---|---|---|---|---|---|---|---|
| 0.1 | diff | 0.0000 | 0.0306 | 0.3094 | 0.4650 | 0.1951 | 0.0210 | 0.0005 | 0.0000 |
| | sim | 0.0019 | 0.0358 | 0.2328 | 0.5523 | 0.1526 | 0.0218 | 0.0025 | 0.0003 |
| | c.i. | 0.0007 | 0.0033 | 0.0066 | 0.0073 | 0.0059 | 0.0021 | 0.0008 | 0.0003 |
| 0.3 | diff | 0.0143 | 0.1493 | 0.3028 | 0.2780 | 0.1777 | 0.0739 | 0.0196 | 0.0034 |
| | sim | 0.0195 | 0.1263 | 0.2930 | 0.2973 | 0.1760 | 0.0656 | 0.0175 | 0.0042 |
| | c.i. | 0.0027 | 0.0057 | 0.0069 | 0.0049 | 0.0058 | 0.0038 | 0.0022 | 0.0011 |
| 0.5 | diff | 0.0433 | 0.1884 | 0.2758 | 0.2253 | 0.1534 | 0.0826 | 0.0342 | 0.0109 |
| | sim | 0.0452 | 0.1712 | 0.2768 | 0.2297 | 0.1519 | 0.0806 | 0.0313 | 0.0101 |
| | c.i. | 0.0034 | 0.0073 | 0.0063 | 0.0066 | 0.0055 | 0.0039 | 0.0026 | 0.0014 |
| 0.7 | diff | 0.0672 | 0.2033 | 0.2593 | 0.2000 | 0.1385 | 0.0825 | 0.0412 | 0.0171 |
| | sim | 0.0652 | 0.1922 | 0.2595 | 0.2043 | 0.1375 | 0.0812 | 0.0371 | 0.0152 |
| | c.i. | 0.0038 | 0.0071 | 0.0077 | 0.0051 | 0.0053 | 0.0049 | 0.0028 | 0.0020 |
| 1.0 | diff | 0.0894 | 0.2130 | 0.2458 | 0.1809 | 0.1256 | 0.0798 | 0.0454 | 0.0229 |
| | sim | 0.0810 | 0.2102 | 0.2448 | 0.1822 | 0.1251 | 0.0798 | 0.0424 | 0.0197 |
| | c.i. | 0.0037 | 0.0059 | 0.0070 | 0.0040 | 0.0054 | 0.0050 | 0.0032 | 0.0017 |
| 3.0 | diff | 0.1104 | 0.2166 | 0.2260 | 0.1570 | 0.1079 | 0.0731 | 0.0486 | 0.0316 |
| | sim | 0.1026 | 0.2122 | 0.2229 | 0.1568 | 0.1073 | 0.0714 | 0.0486 | 0.0324 |
| | c.i. | 0.0049 | 0.0064 | 0.0071 | 0.0074 | 0.0042 | 0.0044 | 0.0035 | 0.0021 |
| 5.0 | diff | 0.1075 | 0.2111 | 0.2205 | 0.1537 | 0.1065 | 0.0733 | 0.0500 | 0.0337 |
| | sim | 0.1018 | 0.2097 | 0.2171 | 0.1530 | 0.1034 | 0.0712 | 0.0478 | 0.0333 |
| | c.i. | 0.0046 | 0.0066 | 0.0067 | 0.0049 | 0.0040 | 0.0043 | 0.0033 | 0.0027 |
| 7.0 | diff | 0.1057 | 0.2079 | 0.2176 | 0.1522 | 0.1061 | 0.0736 | 0.0508 | 0.0349 |
| | sim | 0.0987 | 0.2079 | 0.2121 | 0.1508 | 0.1049 | 0.0744 | 0.0479 | 0.0328 |
| | c.i. | 0.0042 | 0.0075 | 0.0064 | 0.0061 | 0.0046 | 0.0041 | 0.0040 | 0.0030 |
| 10.0 | diff | 0.1040 | 0.2050 | 0.2150 | 0.1507 | 0.1055 | 0.0737 | 0.0513 | 0.0356 |
| | sim | 0.0956 | 0.2029 | 0.2127 | 0.1536 | 0.1001 | 0.0721 | 0.0522 | 0.0349 |
| | c.i. | 0.0048 | 0.0056 | 0.0061 | 0.0061 | 0.0048 | 0.0048 | 0.0032 | 0.0027 |
| 20.0 | diff | 0.1028 | 0.2026 | 0.2126 | 0.1494 | 0.1049 | 0.0736 | 0.0517 | 0.0363 |
| | sim | 0.0920 | 0.2032 | 0.2161 | 0.1456 | 0.1027 | 0.0725 | 0.0497 | 0.0348 |
| | c.i. | 0.0036 | 0.0064 | 0.0055 | 0.0050 | 0.0056 | 0.0045 | 0.0040 | 0.0027 |
| $\infty$ | diff | 0.1024 | 0.2021 | 0.2122 | 0.1491 | 0.1048 | 0.0736 | 0.0517 | 0.0363 |
| | exact | 0.0957 | 0.2010 | 0.2110 | 0.1477 | 0.1034 | 0.0724 | 0.0507 | 0.0355 |

**TABLE 4**

Comparison of the diffusion results with simulation for $M/H_2/3$ Queue

($x_0 = 0$, $\lambda = 3.0$, $\rho = 0.3$)

| t | method | $P_0(t)$ | $P_1(t)$ | $P_2(t)$ | $P_3(t)$ | $P_4(t)$ | $P_5(t)$ | $P_6(t)$ | $P_7(t)$ |
|---|---|---|---|---|---|---|---|---|---|
| 0.1 | diff | 0.7627 | 0.2416 | 0.0356 | 0.0033 | 0.0002 | 0.0000 | 0.0000 | 0.0000 |
| | sim | 0.7778 | 0.1958 | 0.0239 | 0.0022 | 0.0002 | 0.0000 | 0.0000 | 0.0000 |
| | c.i. | 0.0043 | 0.0041 | 0.0025 | 0.0005 | 0.0002 | 0.0000 | 0.0000 | 0.0000 |
| 0.3 | diff | 0.5596 | 0.3241 | 0.1090 | 0.0264 | 0.0055 | 0.0009 | 0.0001 | 0.0000 |
| | sim | 0.5698 | 0.3171 | 0.0930 | 0.0172 | 0.0025 | 0.0004 | 0.0000 | 0.0000 |
| | c.i. | 0.0048 | 0.0051 | 0.0036 | 0.0016 | 0.0005 | 0.0004 | 0.0001 | 0.0000 |
| 0.5 | diff | 0.4822 | 0.3429 | 0.1413 | 0.0428 | 0.0119 | 0.0030 | 0.0007 | 0.0001 |
| | sim | 0.4847 | 0.3500 | 0.1271 | 0.0300 | 0.0069 | 0.0009 | 0.0003 | 0.0000 |
| | c.i. | 0.0061 | 0.0064 | 0.0048 | 0.0022 | 0.0009 | 0.0004 | 0.0002 | 0.0001 |
| 0.7 | diff | 0.4441 | 0.3640 | 0.1556 | 0.0516 | 0.0163 | 0.0049 | 0.0014 | 0.0004 |
| | sim | 0.4462 | 0.3565 | 0.1460 | 0.0395 | 0.0089 | 0.0022 | 0.0004 | 0.0002 |
| | c.i. | 0.0064 | 0.0068 | 0.0043 | 0.0023 | 0.0012 | 0.0007 | 0.0003 | 0.0002 |
| 1.0 | diff | 0.4208 | 0.3482 | 0.1650 | 0.0580 | 0.0199 | 0.0066 | 0.0021 | 0.0007 |
| | sim | 0.4235 | 0.3618 | 0.1536 | 0.0444 | 0.0119 | 0.0039 | 0.0007 | 0.0001 |
| | c.i. | 0.0066 | 0.0084 | 0.0030 | 0.0025 | 0.0011 | 0.0009 | 0.0003 | 0.0001 |
| 3.0 | diff | 0.4046 | 0.3471 | 0.1703 | 0.0625 | 0.0229 | 0.0084 | 0.0031 | 0.0011 |
| | sim | 0.4006 | 0.3649 | 0.1642 | 0.0479 | 0.0161 | 0.0043 | 0.0012 | 0.0005 |
| | c.i. | 0.0073 | 0.0062 | 0.0033 | 0.0031 | 0.0016 | 0.0007 | 0.0004 | 0.0003 |
| 5.0 | diff | 0.4035 | 0.3459 | 0.1699 | 0.0623 | 0.0228 | 0.0084 | 0.0031 | 0.0011 |
| | sim | 0.4061 | 0.3647 | 0.1595 | 0.0481 | 0.0151 | 0.0044 | 0.0015 | 0.0004 |
| | c.i. | 0.0059 | 0.0051 | 0.0043 | 0.0019 | 0.0018 | 0.0009 | 0.0005 | 0.0003 |
| 7.0 | diff | 0.4064 | 0.3486 | 0.1710 | 0.0627 | 0.0230 | 0.0084 | 0.0031 | 0.0011 |
| | sim | 0.4047 | 0.3628 | 0.1623 | 0.0481 | 0.0157 | 0.0045 | 0.0010 | 0.0007 |
| | c.i. | 0.0064 | 0.0065 | 0.0046 | 0.0026 | 0.0016 | 0.0007 | 0.0004 | 0.0003 |
| 10.0 | diff | 0.4024 | 0.3450 | 0.1694 | 0.0621 | 0.0228 | 0.0084 | 0.0031 | 0.0011 |
| | sim | 0.4033 | 0.3646 | 0.1631 | 0.0488 | 0.0141 | 0.0045 | 0.0014 | 0.0003 |
| | c.i. | 0.0082 | 0.0064 | 0.0049 | 0.0035 | 0.0013 | 0.0008 | 0.0004 | 0.0002 |
| 20.0 | diff | 0.4057 | 0.3482 | 0.1708 | 0.0626 | 0.0230 | 0.0084 | 0.0031 | 0.0011 |
| | sim | 0.4045 | 0.3652 | 0.1610 | 0.0478 | 0.1151 | 0.0045 | 0.0014 | 0.0005 |
| | c.i. | 0.0060 | 0.0050 | 0.0040 | 0.0024 | 0.0019 | 0.0006 | 0.0004 | 0.0002 |

**TABLE 5**

Comparison of the diffusion results with simulation for $M/H_2/3$ Queue

($x_0 = 0$, $\lambda = 3.0$, $\rho = 0.5$)

| t | method | $P_0(t)$ | $P_1(t)$ | $P_2(t)$ | $P_3(t)$ | $P_4(t)$ | $P_5(t)$ | $P_6(t)$ | $P_7(t)$ |
|---|---|---|---|---|---|---|---|---|---|
| 0.1 | diff | 0.7537 | 0.2579 | 0.0405 | 0.0032 | 0.0001 | 0.0000 | 0.0000 | 0.0000 |
| | sim | 0.7691 | 0.2019 | 0.0263 | 0.0026 | 0.0002 | 0.0000 | 0.0000 | 0.0000 |
| | c.i. | 0.0040 | 0.0039 | 0.0016 | 0.0006 | 0.0001 | 0.0000 | 0.0000 | 0.0000 |
| 0.3 | diff | 0.5104 | 0.3350 | 0.1373 | 0.0390 | 0.0087 | 0.0015 | 0.0002 | 0.0000 |
| | sim | 0.5273 | 0.3321 | 0.1127 | 0.0233 | 0.0038 | 0.0007 | 0.0001 | 0.0000 |
| | c.i. | 0.0052 | 0.0066 | 0.0043 | 0.0015 | 0.0006 | 0.0004 | 0.0001 | 0.0000 |
| 0.5 | diff | 0.3997 | 0.3408 | 0.1844 | 0.0711 | 0.0238 | 0.0069 | 0.0017 | 0.0003 |
| | sim | 0.4123 | 0.3590 | 0.1642 | 0.0486 | 0.0125 | 0.0029 | 0.0004 | 0.0001 |
| | c.i. | 0.0052 | 0.0555 | 0.0045 | 0.0029 | 0.0010 | 0.0006 | 0.0003 | 0.0001 |
| 0.7 | diff | 0.3371 | 0.3315 | 0.2060 | 0.0917 | 0.0370 | 0.0134 | 0.0044 | 0.0013 |
| | sim | 0.3486 | 0.3625 | 0.1934 | 0.0662 | 0.0213 | 0.0061 | 0.0015 | 0.0003 |
| | c.i. | 0.0046 | 0.0060 | 0.0041 | 0.0033 | 0.0016 | 0.0007 | 0.0004 | 0.0002 |
| 1.0 | diff | 0.2883 | 0.3192 | 0.2196 | 0.1089 | 0.0506 | 0.0219 | 0.0089 | 0.0033 |
| | sim | 0.2960 | 0.3511 | 0.2199 | 0.0847 | 0.0323 | 0.0106 | 0.0040 | 0.0009 |
| | c.i. | 0.0055 | 0.0057 | 0.0049 | 0.0041 | 0.0019 | 0.0015 | 0.0005 | 0.0004 |
| 3.0 | diff | 0.2257 | 0.2867 | 0.2228 | 0.1266 | 0.0713 | 0.0397 | 0.0218 | 0.0118 |
| | sim | 0.2162 | 0.3218 | 0.2415 | 0.1102 | 0.0564 | 0.0269 | 0.0139 | 0.0070 |
| | c.i. | 0.0048 | 0.0059 | 0.0048 | 0.0043 | 0.0018 | 0.0017 | 0.0014 | 0.0012 |
| 5.0 | diff | 0.2199 | 0.2813 | 0.2205 | 0.1126 | 0.0726 | 0.0415 | 0.0236 | 0.0134 |
| | sim | 0.2166 | 0.3150 | 0.2357 | 0.1141 | 0.0577 | 0.0288 | 0.0145 | 0.0086 |
| | c.i. | 0.0064 | 0.0056 | 0.0058 | 0.0041 | 0.0017 | 0.0022 | 0.0014 | 0.0015 |
| 7.0 | diff | 0.2205 | 0.2824 | 0.2213 | 0.1272 | 0.0731 | 0.0420 | 0.0241 | 0.0138 |
| | sim | 0.2109 | 0.3192 | 0.2292 | 0.1166 | 0.0593 | 0.0305 | 0.0154 | 0.0085 |
| | c.i. | 0.0048 | 0.0066 | 0.0048 | 0.0041 | 0.0024 | 0.0019 | 0.0014 | 0.0015 |
| 10.0 | diff | 0.2182 | 0.2797 | 0.2196 | 0.1264 | 0.0727 | 0.0418 | 0.0241 | 0.0138 |
| | sim | 0.2113 | 0.3142 | 0.2355 | 0.1127 | 0.0595 | 0.0305 | 0.0162 | 0.0093 |
| | c.i. | 0.0035 | 0.0047 | 0.0038 | 0.0036 | 0.0032 | 0.0021 | 0.0022 | 0.0011 |
| 20.0 | diff | 0.2197 | 0.2817 | 0.2209 | 0.1270 | 0.0731 | 0.0420 | 0.0242 | 0.0139 |
| | sim | 0.2122 | 0.3122 | 0.2358 | 0.1143 | 0.0580 | 0.0307 | 0.0165 | 0.0088 |
| | c.i. | 0.0046 | 0.0044 | 0.0050 | 0.0048 | 0.0032 | 0.0017 | 0.0014 | 0.0011 |

**TABLE 6**

Comparison of the diffusion results with simulation for $M/H_2/3$ Queue

($x_0 = 0$, $\lambda = 3.0$, $\rho = 0.7$)

| t | method | $P_0(t)$ | $P_1(t)$ | $P_2(t)$ | $P_3(t)$ | $P_4(t)$ | $P_5(t)$ | $P_6(t)$ | $P_7(t)$ |
|---|---|---|---|---|---|---|---|---|---|
| 0.1 | diff | 0.7499 | 0.2690 | 0.0416 | 0.0028 | 0.0001 | 0.0000 | 0.0000 | 0.0000 |
| | sim | 0.7654 | 0.2054 | 0.0262 | 0.0027 | 0.0002 | 0.0000 | 0.0000 | 0.0000 |
| | c.i. | 0.0045 | 0.0044 | 0.0019 | 0.0007 | 0.0002 | 0.0000 | 0.0000 | 0.0000 |
| 0.3 | diff | 0.4878 | 0.3399 | 0.1515 | 0.0450 | 0.0099 | 0.0016 | 0.0002 | 0.0000 |
| | sim | 0.5071 | 0.3421 | 0.1186 | 0.0257 | 0.0057 | 0.0008 | 0.0001 | 0.0000 |
| | c.i. | 0.0047 | 0.0065 | 0.0039 | 0.0021 | 0.0013 | 0.0004 | 0.0001 | 0.0000 |
| 0.5 | diff | 0.3616 | 0.3340 | 0.2046 | 0.0874 | 0.0311 | 0.0092 | 0.0022 | 0.0004 |
| | sim | 0.3769 | 0.3677 | 0.1773 | 0.0579 | 0.0156 | 0.0036 | 0.0008 | 0.0003 |
| | c.i. | 0.0056 | 0.0078 | 0.0048 | 0.0033 | 0.0014 | 0.0010 | 0.0003 | 0.0002 |
| 0.7 | diff | 0.2874 | 0.3130 | 0.2267 | 0.1154 | 0.0514 | 0.0200 | 0.0068 | 0.0020 |
| | sim | 0.2990 | 0.3632 | 0.2146 | 0.0818 | 0.0293 | 0.0090 | 0.0024 | 0.0006 |
| | c.i. | 0.0040 | 0.0052 | 0.0049 | 0.0027 | 0.0016 | 0.0014 | 0.0007 | 0.0004 |
| 1.0 | diff | 0.2262 | 0.2862 | 0.2366 | 0.1384 | 0.0736 | 0.0356 | 0.0156 | 0.0062 |
| | sim | 0.2359 | 0.3370 | 0.2408 | 0.1118 | 0.0465 | 0.0182 | 0.0065 | 0.0023 |
| | c.i. | 0.0055 | 0.0076 | 0.0054 | 0.0042 | 0.0028 | 0.0014 | 0.0011 | 0.0005 |
| 3.0 | diff | 0.1327 | 0.2103 | 0.2113 | 0.1525 | 0.1073 | 0.0735 | 0.0489 | 0.0315 |
| | sim | 0.1223 | 0.2496 | 0.2404 | 0.1508 | 0.0913 | 0.0610 | 0.0363 | 0.0206 |
| | c.i. | 0.0030 | 0.0057 | 0.0042 | 0.0038 | 0.0037 | 0.0024 | 0.0018 | 0.0013 |
| 5.0 | diff | 0.1177 | 0.1908 | 0.1964 | 0.1459 | 0.1072 | 0.0778 | 0.0557 | 0.0393 |
| | sim | 0.1073 | 0.2261 | 0.2309 | 0.1383 | 0.0969 | 0.0664 | 0.0455 | 0.0322 |
| | c.i. | 0.0035 | 0.0043 | 0.0034 | 0.0049 | 0.0031 | 0.0029 | 0.0024 | 0.0021 |
| 7.0 | diff | 0.1132 | 0.1847 | 0.1908 | 0.1428 | 0.1063 | 0.0786 | 0.0577 | 0.0420 |
| | sim | 0.1039 | 0.2160 | 0.2183 | 0.1440 | 0.0936 | 0.0662 | 0.0502 | 0.0337 |
| | c.i. | 0.0034 | 0.0044 | 0.0038 | 0.0043 | 0.0025 | 0.0030 | 0.0020 | 0.0022 |
| 10.0 | diff | 0.1083 | 0.1777 | 0.1847 | 0.1390 | 0.1044 | 0.0781 | 0.0583 | 0.0433 |
| | sim | 0.0994 | 0.2060 | 0.2125 | 0.1374 | 0.0958 | 0.0730 | 0.0485 | 0.0371 |
| | c.i. | 0.0033 | 0.0058 | 0.0048 | 0.0048 | 0.0038 | 0.0033 | 0.0027 | 0.0025 |
| 20.0 | diff | 0.1060 | 0.1743 | 0.1813 | 0.1368 | 0.1032 | 0.0778 | 0.0586 | 0.0442 |
| | sim | 0.0970 | 0.2034 | 0.2079 | 0.1336 | 0.0938 | 0.0687 | 0.0503 | 0.0369 |
| | c.i. | 0.0027 | 0.0046 | 0.0039 | 0.0034 | 0.0036 | 0.0036 | 0.0031 | 0.0026 |

**Appendix. Derivation of the transient solution of diffusion equation**

The problem can be formulated as follows. Find the solution of the equation

$$\frac{\partial f}{\partial t} = \frac{a}{2}\frac{\partial^2 f}{\partial x^2} - b\frac{\partial f}{\partial x} + h \tag{A.1}$$

in region $l_1 < x < l_2$, $0 \leq t$ subject to the conditions

$$\begin{aligned} f(x,0) &= \delta(x - x_0) \\ f(l_1,t) &= g_1(t) \\ f(l_2,t) &= g_2(t) \quad \text{for} \quad t > 0 \\ h(x,t) &= \lambda p(t)\delta(x - c). \end{aligned} \tag{A.2}$$

Let $y = x - l_1$ and $F(y,t) = f(y + l_1, t)$, $0 \leq y \leq l_2 - l_1$ and $t \geq 0$. Then $(A.1)$ and $(A.2)$ become as follows.

$$\frac{\partial F}{\partial t} = \frac{a}{2}\frac{\partial^2 F}{\partial y^2} - b\frac{\partial F}{\partial y} + H(y,t) \quad 0 < y < l_2 - l_1, \quad t \geq 0. \tag{A.1$'$}$$

$$\begin{aligned} F(y,0) &= \delta(y + l_1 - x_0) \\ F(0,t) &= g_1(t) \\ F(l_2 - l_1, t) &= g_2(t) \\ H(y,t) &= \lambda p(t)\delta(y + l_1 - c). \end{aligned} \tag{A.2$'$}$$

A standard method of solving differential equations is to make a change of variable transforms the given equation to an equation whose solution is known. By letting

$$\begin{aligned} W(y,t) &= F(y,t)\exp(-\frac{b}{a}y + \frac{b^2}{2a}t), \\ H_1(y,t) &= H(y,t)\exp(-\frac{b}{a}y + \frac{b^2}{2a}t), \end{aligned} \tag{A.3}$$

we have the canonical heat equation with nonhomogeneous boundary conditons

$$\frac{\partial W}{\partial t} = \frac{a}{2}\frac{\partial^2 W}{\partial y^2} + H_1, \quad 0 < y < l_2 - l_1, \quad t > 0 \tag{A.4}$$

and

$$W(y,0) = \delta(y + l_1 - x_0)e^{-\frac{b}{a}y}, \quad 0 < y < l_2 - l_1$$

$$(A.5) \qquad W(0,t) = g_1(t)e^{\frac{b^2}{2a}t}, \quad t \geq 0$$

$$W(l_2 - l_1, t) = g_2(t)\exp(-\frac{b}{a}(l_2 - l_1) + \frac{b^2}{2a}t), \quad t \geq 0.$$

Let $W^*(y,s)$, $g_1^*(s)$, $g_2^*(s)$ and $H_1^*(y,s)$ be the Laplace transforms with respect to $t$ variable of $W$, $g_1$, $g_2$ and $H_1$, respectively. Then the Laplace transform $W^*(y,s)$ of the solution $W(y,t)$ of $(A.4)$ under the condition $(A.5)$ is given by (Carrier and Carl [5])

$$\begin{aligned}
&W^*(y,s) \\
&= \frac{\sinh(\sqrt{\frac{2s}{a}}y)}{\sinh(\sqrt{\frac{2s}{a}}(l_2 - l_1))}\Bigg(g_2^*(s - \frac{b^2}{2a})e^{-\frac{b}{a}(l_2 - l - 1)} - g_1^*(s - \frac{b^2}{2a})\cosh(\sqrt{\frac{2s}{a}}(l_2 - l_1) \\
&(A.6) \qquad + \sqrt{\frac{2}{as}}\int_0^{l_2 - l_1}\left\{\delta(\xi + l_1 - x_0)e^{-\frac{b}{a}\xi} + H_1^*(\xi,s)\right\}\sinh(\sqrt{\frac{2s}{a}}(l_2 - l_1 - \xi))d\xi\Bigg) \\
&+ g_1^*(s - \frac{b^2}{2a})\cosh(\sqrt{\frac{2s}{a}}y) \\
&- \sqrt{\frac{2}{as}}\int_0^y\left\{\delta(\xi + l_1 - x_0)e^{-\frac{b}{a}\xi} + H_1^*(\xi,s)\right\}\sinh(\sqrt{\frac{2s}{a}}(y - \xi))d\xi.
\end{aligned}$$

We have from $(A.3)$ and $(A.6)$ that

$$\begin{aligned}
F^*(y,s) &= \exp(\frac{b}{a}y)W^*(y, s + \frac{b^2}{2a}) \\
&= \exp(\frac{b}{a}(y - (l_2 - l_1)))\frac{\sinh Ay}{\sinh A(l_2 - l_1)}g_2^*(s) \\
(A.7) \qquad &- \exp(\frac{b}{a}y)\frac{\sinh A(y - (l_2 - l_1))}{\sinh A(l_2 - l_1)}g_1^*(s) \\
&+ \frac{2}{aA}\exp(\frac{b}{a}(y - (x_0 - l_1)))\Bigg(\frac{\sinh Ay}{\sinh A(l_2 - l_1)}\sinh A(l_2 - x_0)
\end{aligned}$$

$$- \sinh A(y-(x_0-l_1))U(y-(x_0-l_1)))\, 1_{l_1<x_0<l_2}$$

$$+ \frac{2}{aA}\exp(\frac{b}{a}(y-(c-l_1)))\lambda p^*(s)\left(\frac{\sinh A(l_2-c)}{\sinh A(l_2-l_1)}\sinh Ay\right.$$

$$- \sinh A(y-(c-l_1))U(y-(c-l_1)))\, 1_{l_1<c<l_2},$$

where $A = \sqrt{2as+b^2}/a$. Hence the Laplace transform $f^*(x,s)$ of the solution $f(x,t)$ of $(A.1)$ is
$(A.8)$

$$f^*(x,s) = \exp(\frac{b}{a}(x-l_2))\frac{\sinh A(x-l_1)}{\sinh A(l_2-l_1)}g_2^*(s) + \exp(\frac{b}{a}(x-l_1))\frac{\sinh A(l_2-x)}{\sinh A(l_2-l_1)}g_1^*(s)$$

$$+ \frac{2}{aA}\exp(\frac{b}{a}(x-x_0))\left(\frac{\sinh A(x-l_1)}{\sinh A(l_2-l_1)}\sinh A(l_2-x_0)\right.$$

$$- \sinh A(x-x_0)U(x-x_0))\, 1_{l_1<x_0<l_2}$$

$$+ \frac{2}{aA}\exp(\frac{b}{a}(x-c))\left(\frac{\sinh A(l_2-c)}{\sinh A(l_2-l_1)}\sinh A(x-l_1)\right.$$

$$- \sinh A(x-c)U(x-c))\, \lambda p^*(s) 1_{l_1<c<l_2}.$$

(2.18) is obtained by applying (A.8) to (2.14) with $l_1 = k-1$ and $l_2 = k$, (2.19) is obtained by first applying (A.8) to (2.16) with $l_1 = m-1$, $l_2 = N$ and $g_n(t|x_0) = 0$ and then letting $N \to \infty$.

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B. D. Choi and Y. W. Shin
Department of Mathematics
Korea Advanced Institute of Science and Technology
P.O. Box 150, Cheongryang,
Seoul 130-650, Korea