Journal of the Operations Research
Society of Japan
Vol. 34, No. 2, June 1991

ON A PARAMETRIC DIVISOR METHOD FOR THE APPORTIONMENT PROBLEM

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(Received March 19, 1990; Revised September 14, 1990)

Abstract Apportionment problem has been focused upon for more than 200 years by many applied mathematicians and operations researchers. Balinski and Young have done quite extensive works for this problem.

In this paper we propose a parametric divisor method for solving an apportionment problem. Our method is shown to "cover" most traditional apportionment methods. First we introduce the idea of stable regions for the allocation of seats to each political constituency, then show the explicit relation between apportionment methods and corresponding stable regions. Then we look at these apportionment methods from the viewpoints of constrained optimization problem, and we show the corresponding optimization problem for our parametric divisor method.

Finally using Japan's House of Representative data, we show the numerical results for the application of various apportionment methods. We conclude our paper by suggesting appropriate parameter values for our parametric apportionment method.

1. Introduction

In Japan the issue of "weight of one vote" has been controversial since people began to recognize the gap between political constituencies with respect to the weight given to the number of seats per voter. In 1986 our Supreme Court gave a decision responding to the appeal that a gap of more than 2.0 may be unconstitutional. Since then, several similar decisions have been given in various judicial courts. Our ruling Liberal Democratic Party in the Diet has also recognized the importance of this problem. It is considering reform of the election system, which has not been changed since the 1950's. Their plan includes reducing the total number of seats from the present 512, which was established in 1925, to 471 and accepting middle size electoral districting system (the number of seats assigned to each constituency should be between 3 and 5).

Various types of "equity" problems arise in distributing available personnel or resources in "integral parts" to different subdivisions. Typical examples are the allocation of a set of available teachers to classes in order to make timetables, the assignment of a set of individuals to certain jobs, which is the so-called classical assignment problem for operations researchers, and the distribution of seats in a legislature among different political constituencies. Several solution methods have been proposed and established for some problems (*e.g.*, the assignment problem). Others have not yet been "efficiently" solved (*e.g.*, the timetabling problem).

The apportionment problem aims at allocating seats "fairly" among political constituencies when the total number of seats and the distribution of each constituency's population are given. Mathematically, the apportionment problem can be formulated as follows : Given the set of N political constituencies as $S = \{1, 2, \ldots, N\}$, the population of political constituency $i \in S$ as p_i, the total population as P, and the total number of seats as K, the "ideal" number of seats allocated to the constituency i, *i.e.*, the "exact quota" q_i, is given by

$$q_i = \frac{p_i K}{P} \qquad i \in S \tag{1.1}$$

where

$$P = \sum_{i \in S} p_i \tag{1.2}$$

Hence we have

$$\sum_{i \in S} q_i = K \tag{1.3}$$

Then the apportionment problem is to partition a given positive integer K into nonnegative integral parts $\{d_i \mid i \in S\}$ such that

$$\sum_{i \in S} d_i = K \tag{1.4}$$

$$d_i \geq 0. \ integer, \ i \in S \tag{1.5}$$

and such that these parts are "as near as possible" proportional, respectively, to a set of nonnegative integers $\{p_1, p_2, \ldots, p_N\}$, *i.e.*, $\{q_1, q_2, \ldots, q_N\}$. If the exact quotas $\{q_i \mid i \in S\}$ were to be integers for all $i \in S$, then the apportionment would be obtained by setting $d_i = q_i$ for each $i \in S$. But this is an extremely rare case, so usually exact quotas $\{q_i \mid i \in S\}$ all have fractional parts. Therefore, the problem becomes how to round the fractions $\{q_i \mid i \in S\}$ to their "nearby" integral values keeping their sum equal to a given value K.

The apportionment problem may seem to be an easily solved "approximation" problem. However, this is not the case as history shows. The Congress of the United States, for example, has used four different schemes to apportion the seats in the House of Representatives among the various states over the past 200 years, and they have, on many occasions (beginning in 1790), held lengthy debates on this issue. General descriptions of the apportionment problem and its history are given in *e.g.*, [8, 11].

Difficulties of the apportionment problem occur at several points. Firstly, how should we express the measure of "inequity" to be minimized? There may be many definitions representing both global and local "inequities" between various constituencies. These include minimizing the sum of differences between given apportionments and the exact quota of each constituency or minimizing "locally" relative differences of the number of seats allocated per voter. Secondly, the difficulty of the apportionment problem is related to the property which we want our apportionment method to satisfy. For example, we want the apportionment method to have the property that the number of seats given to each constituency is either rounded-up or rounded-down by an exact quota or we may want that a constituency should not be given less representation if the total number of seats increases and the distribution of the population of each constituency remains the same. There are various "natural" requirements for acceptable apportionment methods. Some of these "requirements", however, are inconsistent. As yet, no method has been found to satisfy them simultaneously in the general case. This means that no matter which apportionment method is accepted, it will possess certain "defects". Namely, we may have to decide in advance which properties must be satisfied, and which "defects" are acceptable before we employ our own apportionment method.

Balinski and Young have done extensive work in the area of apportionment problems (see *e.g.*, [1, 2, 3, 4, 5, 6, 7]). Even though they have proposed a new complicated scheme called quota method (see [1, 2]) satisfying certain properties simultaneously, they have been promoting a classical Webster method (see [8]) for its impartialness and simplicity. In this paper we propose a new simple apportionment method, which we call a parametric divisor method. We show that our method is general enough to "cover" most traditional apportionment methods so far employed in several countries. We then propose a range of appropriate parameter values for our apportionment method in order to maintain our method's impartialness and fairness with respect to the population size of each constituency.

In Section 2 we review several representative apportionment methods and introduce the idea of a stable region related to certain locally optimal assignments. In Section 3 we look at those apportionment methods from the viewpoints of constrained optimization problems, then consider what kind of objective function these apportionment methods are trying to minimize. In Section 4 we explain our parametric divisor method with its relation to other methods. In Section 5 we give the results of our numerical experiments using Japan's House of Representative data, and compare these results with the apportionment methods described therein. In the last Section, we conclude our paper by giving certain evaluations obtained from our analysis and numerical experiments.

2. Apportionment methods and the stable region

2.1 Traditional apportionment methods

We explain several very common apportionment methods, some of which are employed or have been employed in some European and American countries. First, we will give a common scheme called the largest fraction method. This scheme is based upon remainders. Next, we will show five divisor methods. These were based upon divisors and go by the name of Huntington methods.

The largest fraction method, which we shall denote by LFM, was first suggested by A. Hamilton at the United States Congress in 1791, and was used by the Congress from 1851 until 1910. The LFM first assigns each constituency $i \in S$ its lower quota $\lfloor q_i \rfloor$, where $\lfloor q \rfloor$ denotes the largest integer less than or equal to q. Then we define the fraction of each constituency t_i as follows.

$$t_i = q_i - \lfloor q_i \rfloor \qquad i \in S \tag{2.1}$$

Sorting the set $\{t_i \mid i \in S\}$ from the largest, arbitrarily for the equal elements, we define the set of suffices of the first $K - \sum_{i \in S} \lfloor q_i \rfloor$ constituencies in the ordering by T. Then the LFM allocates an additional seat to the constituencies belonging to the set T: namely, the whole allocation $\{d_i \mid i \in S\}$ by the LFM is given as follows.

$$d_i = \begin{cases} \lfloor q_i \rfloor + 1 & i \in T \\ \lfloor q_i \rfloor & i \notin T \end{cases} \tag{2.2}$$

Let us define the general divisor method. First, we give a divisor λ in order to compute the quotient of each constituency $i \in S$ with the population p_i as $q_i(\lambda) = \frac{p_i}{\lambda}$. Then, we round the quotients according to values of the number of seats to each constituency. Let us denote the integer value obtained from the quotient $q_i(\lambda) = \frac{p_i}{\lambda}$ by $[q_i(\lambda)]_r = [\frac{p_i}{\lambda}]_r$. Then, in order that these quotients can be an apportionment the following must hold.

$$\sum_{i \in S} [q_i(\lambda)]_r = \sum_{i \in S} [\frac{p_i}{\lambda}]_r = K \tag{2.3}$$

Now we generalize the rounding process by defining the divisor function $v(d)$ as follows. Let $v(d)$ be a monotone increasing function defined for all integers $d \geq 0$ and also satisfy $d \leq v(d) \leq d + 1$. Then, for any positive real number x, there corresponds a unique integer d such that $v(d-1) < x \leq v(d)$. We define the above rounding process by

$$[\frac{p_i}{\lambda}]_r = d_i \qquad i \in S \tag{2.4}$$

where

$$v(d_i - 1) < \frac{p_i}{\lambda} \leq v(d_i) \quad i \in S \tag{2.5}$$

The apportionment method described above is called the divisor method based upon the divisor function $v(d)$. The divisor method can be defined equivalently as follows. From (2.4) and (2.5), the parameter λ has to satisfy

$$\frac{p_i}{v(d_i)} \leq \lambda < \frac{p_i}{v(d_i - 1)} \quad \text{for all } i \in S \tag{2.6}$$

This means that

$$\max_{d_j \geq 0} \frac{p_j}{v(d_j)} \leq \min_{d_i > 0} \frac{p_i}{v(d_i - 1)} \tag{2.7}$$

where we permit dividing by 0 and assume that $\frac{p_i}{0} > \frac{p_j}{0}$ if $p_i > p_j$. Defining the rank function $r(p_i, d_i)$ for $i \in S$ as

$$r(p_i, d_i) = \frac{p_i}{v(d_i)} \qquad i \in S \tag{2.8}$$

then we can write the above relation (2.7) as follows.

$$\max_{d_i \geq 0} r(p_i, d_i) \leq \min_{d_j > 0} r(p_j, d_j - 1) \tag{2.9}$$

We denote the apportionment method based upon the divisor function $v(d)$ by $A(\mathbf{p}, K)$, which expresses a function giving N integral components $d_1, \ldots, d_N$ as an image of a given population distribution vector $\mathbf{p} = (p_1, \ldots, p_N)$ and a total number of seats K. The function $A(\mathbf{p}, K)$ can be written as follows.

$$A(\mathbf{p}, K) = \{\mathbf{d} \mid \sum_{i \in S} d_i = K, \max_{d_j \geq 0} r(p_j, d_j) \leq \min_{d_i > 0} r(p_i, d_i - 1)\} \tag{2.10}$$

where $\mathbf{d}$ denotes the allocation vector given by $\mathbf{d} = (d_1, \ldots, d_N)$.

There exists an alternative way of expressing the general apportionment methods based upon the rank function $v(p_i, d_i)$ recursively. Let d_i^k be the number of seats allocated to the political constituency $i \in S$ given the total number of seats $k \in \{0, 1, \ldots, K\}$. Then an iterative algorithm for the general divisor method can be written as follows.

Algorithm (general divisor method)

Step 1 $\quad d_i^k = 0, \ k \in \{0, 1, \ldots, K\}, \ i \in S. \ k = 0.$

Step 2

$$r(p_t, d_t) = \max_{i \in S} r(p_i, d_i) \tag{2.11}$$

$$\begin{cases} d_t^{k+1} = d_t^k + 1 \\ d_i^{k+1} = d_i^k \qquad i \neq t, \ i \in S \end{cases} \tag{2.12}$$

Step 3 $\quad k = k + 1$. If $k = K$, then stop. Otherwise, go to step 2.

As shown in the above algorithm, for $k = 0$, the allocation must be zero for every constituency. Given that an allocation $\mathbf{d}^k = (d_1^k, \ldots, d_N^k)$ has been determined for a total number of seats k, an allocation for a size $k + 1$ is found by giving one more seat to the constituency i for which the rank function $r(p_i, d_i)$ is a maximum.

Based upon different divisor functions we can define an infinite number of different divisor methods (see *e.g.*, [1, 2, 3, 4, 5, 6, 7, 15]). There are five traditional divisor methods shown in Table 1. The method of greatest divisors, which we denote by GDM, was also called the Jefferson method in Balinski and Young's papers. The method of major fractions, which we denote by MFM, was called the Webster method in their papers. Balinski and Young called the equal proportion method (EPM), the harmonic method (HMM), and the smallest divisor method (SDM) after the names of their advocates, *i.e.*, the Hill method, the Dean method, and the Adams method, respectively.

From the computational aspects, three apportionment methods, GDM, MFM and SDM can be separately described. First, apportionment by the GDM can be obtained as follows.

(i) Find the maximum $\lambda = \lambda_{GD}$ such that

$$\sum_{i \in S} \lfloor \frac{p_i}{\lambda} \rfloor \geq K \tag{2.13}$$

(ii) If (2.13) holds as equality for $\lambda = \lambda_{GD}$, then the allocation $\mathbf{d}$ is given by

$$d_i = \lfloor \frac{p_i}{\lambda_{GD}} \rfloor \qquad i \in S \tag{2.14}$$

If (2.13) holds as a strict inequality, then let

$$E = \{ i \mid i \in S, \frac{p_i}{\lambda_{GD}} : \text{integer} \} \tag{2.15}$$

Since there exist more than one i such that $\frac{p_i}{\lambda_{GD}}$ is integer valued, $E \neq \phi$. Suppose

$$\sum_{i \in S} \lfloor \frac{p_i}{\lambda_{GD}} \rfloor = K' > K \tag{2.16}$$

then we must decide that $K' - K$ constituencies lose a seat. Hence let D be a subset of E with $\mid D \mid = K' - K$ (we can apply an ad-hoc rule to determine this), then the apportionment can be given as

$$d_i = \begin{cases} \dfrac{p_i}{\lambda_{GD}} & i \notin D,\ i \in E \\ \dfrac{p_i}{\lambda_{GD}} - 1 & i \in D \end{cases} \tag{2.17}$$

In the case of MFM the maximum $\lambda = \lambda_{MF}$ can be obtained as satisfying

$$\sum_{i \in S} \lfloor \frac{p_i}{\lambda_{MF}} + 0.5 \rfloor \geq K \tag{2.18}$$

The remaining parts in the above procedure (ii) are similarly obtained by changing $\frac{p_i}{\lambda_{GD}}$ to $\frac{p_i}{\lambda_{MF}} + 0.5$. Similarly, for the case of SDM the parameter λ_{SD} is obtained as the maximum λ such that

$$\sum_{i \in S} \lfloor \frac{p_i}{\lambda_{SD}} + 1 \rfloor \geq K. \tag{2.19}$$

The remaining parts in the above procedure (ii) are also similarly obtained by changing $\frac{p_i}{\lambda_{GD}}$ to $\frac{p_i}{\lambda_{SD}} + 1$.

There are several properties for each apportionment method to satisfy. If the allocation $\{d_i \mid i \in S\}$ given by the method M satisfies

$$d_i \geq \lfloor q_i \rfloor \qquad i \in S \tag{2.20}$$

then we say that apportionment method M satisfies the lower quota. Suppose the allocation $\{d_i \mid i \in S\}$ satisfies

$$d_i \leq \lceil q_i \rceil \qquad i \in S \tag{2.21}$$

where $\lceil q_i \rceil$ indicates the smallest integer larger than or equal to q_i then method M is said to satisfy the upper quota. If method M satisfies both the lower and the upper quota properties, we say that method M satisfies the quota. Neither divisor method described above satisfies the quota property, while the LFM does satisfy it.

An apportionment method M is said to satisfy the house monotone property if no political constituency $i \in S$ decreases its allocation when the house size increases from k to $k+1$. The violation of this property is often referred to as the "Alabama paradox". The word "Alabama paradox" originates from the fact that when the U.S. Congress was using the LFM in 1881, the state of Alabama was allocated 8 representatives, while they received 7 when the total went to 300 from 299. Therefore, the LFM does not satisfy this property. All other divisor methods satisfy it.

2.2 Local measures of inequity

Now we focus upon the local measures of inequity between pairs of constituencies. Let the population in the constituency $i \in S$ be p_i and the number of seats assigned be d_i. We say that constituency i is favored over j when the number of seats per individual in the constituency i is greater than or equal to that in j; namely, $\frac{d_i}{p_i} \geq \frac{d_j}{p_j}$, (*i.e.*, $\frac{p_i}{d_i} \leq \frac{p_j}{d_j}$). Huntington considered making ratios such as $\frac{d_i}{p_i}$ or $\frac{p_i}{d_i}$ as equal as possible over all constituencies. That these ratios are nearly equal means that, ideally, relative or the absolute differences concerning $\frac{d_i}{p_i}$ or $\frac{p_i}{d_i}$ become zero. Generally, we denote the measure of inequity between two constituencies i and j as $E(p_i, d_i; p_j, d_j)$. Then Huntington's rule says that we should transfer a seat from a more favored constituency i to a less favored constituency j when it brings a smaller measure of inequity. Namely, when $\frac{d_i}{p_i} \geq \frac{d_j}{p_j}$ and

$$E(p_i, d_i; p_j, d_j) > E(p_i, d_i - 1; p_j, d_j + 1) \tag{2.22}$$

we should transfer a seat from i to j. The objective of Huntington's rule is to minimize the measure of inequity between pairs of constituencies. So the "desirable apportionment" is obtained when no switching of seats between constituencies can improve the measure of inequity between any such pair of constituencies. The attainment of this state is referred to a stable assignment of seats.

Huntington's rule was applied to several forms of the measure of inequity $E(p_i, d_i; p_j, d_j)$ as shown in Table 1. For each measure of inequity in Table 1 we can obtain a stable assignment of seats. Moreover, the resulting stable apportionment obtained from each function of $E_{GD}, E_{MF}, E_{EP}, E_{HM}$, and E_{SD} indicating the measure of inequity, corresponds to the solution for the apportionment methods GDM, MFM, EPM, HMM and SDM, respectively. For example, the correspondence between function $E_{EP}(p_i, d_i; p_j, d_j)$ and EPM is shown in the following theorem, which can be proved in a similar way to [8, p101]. [11, p378].

Theorem 2.1 For the pair of constituencies i and j with populations p_i and p_j, apportionments d_i and d_j, respectively, the following holds.

$$E_{EP}(p_i, d_i; p_j, d_j) \leq E_{EP}(p_i, d_i - 1; p_j, d_j + 1) \qquad i, j \in S \tag{2.23}$$

if and only if

$$\frac{p_i}{\sqrt{d_i(d_i+1)}} \leq \frac{p_j}{\sqrt{d_j(d_j-1)}} \qquad i, j \in S \tag{2.24}$$

Table 1. Divisor method, divisor function and measure of inequity

Divisor method	Divisor function $v(d)$	Measure of inequity $E(p_i, d_i; p_j, d_j)$
GDM	$d+1$	$\left\lvert \frac{d_i p_j}{p_i} - d_j \right\rvert$
MFM	$d+0.5$	$\left\lvert \frac{d_i}{p_i} - \frac{d_j}{p_j} \right\rvert$
EPM	$\sqrt{d(d+1)}$	$\frac{\left\lvert \frac{d_i}{p_i} - \frac{d_j}{p_j} \right\rvert}{\min\{\frac{p_i}{d_i}, \frac{p_j}{d_j}\}}$
HMM	$\frac{d(d+1)}{d+0.5}$	$\left\lvert \frac{p_i}{d_i} - \frac{p_j}{d_j} \right\rvert$
SDM	d	$\left\lvert d_i - \frac{p_i d_j}{p_j} \right\rvert$

Using the above theorem, suppose that the relation (2.23) holds for all $i \in S$ and $j \in S$. Then the assignment corresponds to the optimal convergent apportionment. Hence, comparing (2.23) or (2.24) with (2.7) or (2.9), we can conclude that the above case in Theorem 2.1 is equivalent to the case that the divisor function is given as $v(d_i) = \sqrt{d_i(d_i+1)}$. In other words, the pairwise transfering procedure given by the criterion in Theorem 2.1 gives the same apportionment solution as EPM. Similarly, we can prove that the measure of inequity functions $E_{GD}, E_{MF}, E_{EP}, E_{HM}$, and E_{SD} are equivalent to GDM, MFM, EPM, HMM, and SDM, respectively.

From the computational points of view, pairwise comparisons are very inefficient since we probably have to consider each of the $\frac{N(N-1)}{2}$ combinations several times. The above equivalent relations between measures of inequity functions and rank functions indicate that we can apply divisor methods to compute the apportionment method based upon Huntington's criteria.

Huntington examined 64 different measures of inequity including 32 relative and 32 absolute differences (see [10]). All of the relative differences and two of the absolute differences lead to EPM. For example, if we define another relative differences with respect to $\frac{p_i}{d_i}$ and $\frac{p_j}{d_j}$ instead of $\frac{d_i}{p_i}$ and $\frac{d_j}{p_j}$ in E_{EP}, we again obtain the same result as Theorem 2.1. Various absolute differences brought other four methods GDM, MFM, HMM and SDM. There are also some absolute differences for which the measure of inequity function does not work. In [10] an example of the measure of inequity function $E(p_i, d_i; p_j, d_j) = \lvert \frac{d_i}{d_j} - \frac{p_i}{p_j} \rvert$ for which an unstable assignment appears is shown.

Theorem 2.2 Let the measures of inequity be as follows.

$$E_1(p_i, d_i; p_j, d_j) = \frac{\max\{\frac{p_i}{d_i}, \frac{p_j}{d_j}\}}{\min\{\frac{p_i}{d_i}, \frac{p_j}{d_j}\}} \tag{2.25}$$

$$E_2(p_i, d_i; p_j, d_j) = \frac{\max\{\frac{d_i}{p_i}, \frac{d_j}{p_j}\}}{\min\{\frac{d_i}{p_i}, \frac{d_j}{p_j}\}} \tag{2.26}$$

Then the apportionment by the EPM gives a stable solution when the Huntington's rule is applied to the above measures of inequity E_1 and E_2.

Proof By subtracting 1 from each of the functions E_1 and E_2, we obtain the error functions $E_{EP}(p_i, d_i; p_j, d_j)$ in Table 1. From the equivalent correspondence between the function E_{EP} and the method EPM, we can conclude that apportionment by the EPM gives a stable solution to the measure of inequity in (2.25) and (2.26). □

Similarly, we can obtain the following corollary.

Corollary 2.3 Let the measures of inequity be as follows.

$$E_3(p_i, d_i; p_j, d_j) = -\frac{min\{\frac{p_i}{d_i}, \frac{p_j}{d_j}\}}{max\{\frac{p_i}{d_i}, \frac{p_j}{d_j}\}} \tag{2.27}$$

$$E_4(p_i, d_i; p_j, d_j) = -\frac{min\{\frac{d_i}{p_i}, \frac{d_j}{p_j}\}}{max\{\frac{d_i}{p_i}, \frac{d_j}{p_j}\}} \tag{2.28}$$

Then the apportionment by EPM gives a stable solution when Huntington's rule is applied to the above measures of inequity E_3 and E_4.

2.3 The ARPT rule

We consider a new rule, which we call the average ratio pairwise transfer ($ARPT$) rule, for transferring a seat from one constituency to another. Our new rule provides deep insights to traditional apportionment methods and it is also very useful for investigating our new apportionment method proposed later. Our ARPT rule is based upon the measure of inequity $E(p_i, d_i; p_j, d_j)$. Let r be the average number of seats per individual, *i.e.*, $r = \frac{K}{P}$, then the $ARPT$ rule says that, for any two constituencies i and j such that $\frac{d_j}{p_j} \leq r \leq \frac{d_i}{p_i}$, we should make a transfer of one seat from the more favored constituency i to the less favored constituency j if it reduces the measure of inequity, *i.e.*, if $E(p_i, d_i; p_j, d_j) > E(p_i, d_i - 1; p_j, d_j + 1)$. Our transfer rule is different from Huntington's one in that we add a restriction $\frac{d_j}{p_j} \leq r \leq \frac{d_i}{p_i}$, which possibly implies that our stable region is "larger than" Huntington's one. Applying $ARPT$ rule to several types of measures of inequity, we can obtain the following theorems. In the following we always assume constituency i is favored over constituency j, *i.e.*, $\frac{d_i}{p_i} \geq \frac{d_j}{p_j}$.

Theorem 2.4 Let the measure of inequity be

$$E_S(p_i, d_i; p_j, d_j) = |\frac{d_i}{p_i} - r| + |\frac{d_j}{p_j} - r| \tag{2.29}$$

Then applying the $ARPT$ rule to the above measure $E_S(p_i, d_i; p_j, d_j)$, a stable assignment can be obtained as satisfying

$$x_j \geq x_i - \frac{p'_i + p'_j}{2} \tag{2.30}$$

$$\mathrm{sgn}(p_i - p_j) \cdot x_k \geq \mathrm{sgn}(p_i - p_j) \cdot (r + \frac{p'_i - p'_j}{2}) \tag{2.31}$$

where $x_i = \frac{d_i}{p_i}, x_j = \frac{d_j}{p_j}, p'_i = \frac{1}{p_i}, p'_j = \frac{1}{p_j}$, and $k = i$ if $p'_i \geq p'_j; = j$ if $p'_i < p'_j$; and $\mathrm{sgn}(t) = 1$ if $t > 0; = -1$ if $t \leq 0$.

Proof Since constituency i is favored over j, we have $\frac{d_j}{p_j} \leq r \leq \frac{d_i}{p_i}$. So we need to consider the following three cases (i)–(iii). In each case we show the condition that the measure of inequity after a transfer of one seat from i to j is larger than or equal to that before the transfer.

(i) $\frac{d_i - 1}{p_i} \leq r, \quad \frac{d_j+1}{p_j} \geq r$

$$\frac{d_i}{p_i} - r + r - \frac{d_j}{p_j} \leq r - \frac{d_i - 1}{p_i} + \frac{d_j + 1}{p_j} - r$$

$$\frac{d_i - \frac{1}{2}}{p_i} \leq \frac{d_j + \frac{1}{2}}{p_j}$$

$$x_j \geq x_i - \frac{p'_i + p'_j}{2} \tag{2.32}$$

(ii) $\frac{d_i - 1}{p_i} > r, \quad \frac{d_j+1}{p_j} \geq r$

$$\frac{d_i}{p_i} - \frac{d_j}{p_j} \leq \frac{d_i - 1}{p_i} - r + \frac{d_j + 1}{p_j} - r$$

$$x_j \geq r + \frac{p'_i - p'_j}{2} \tag{2.33}$$

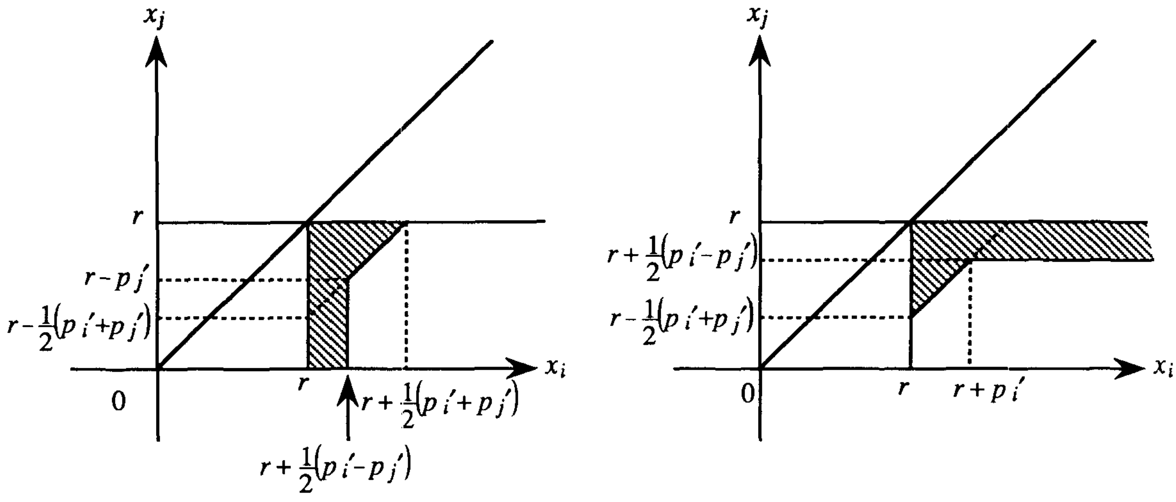
(iii) $\frac{d_i - 1}{p_i} \leq r, \quad \frac{d_j+1}{p_j} < r$

$$\frac{d_i}{p_i} - \frac{d_j}{p_j} \leq r - \frac{d_i - 1}{p_i} + r - \frac{d_j + 1}{p_j}$$

$$x_i \leq r + \frac{p'_i - p'_j}{2} \tag{2.34}$$

From (2.32)–(2.34) we obtain (2.30)–(2.31). □

Illustrating a stable region given by (2.30) and (2.31) on the $x_i - x_j$ plane, shaded areas are obtained as in Fig. 1. Namely the point $(x_i, x_j) = (\frac{d_i}{p_i}, \frac{d_j}{p_j})$ in the shaded area indicates that the corresponding apportionment does not decrease the measure of inequity by transferring a seat from a more favored constituency i to a less favored constituency j. Let us define the allocation of seats $\{d_i \mid i \in S\}$ is stable if the assignment $\{d_i, d_j\}$ is in the stable region for any pair i and j in S when $ARPT$ rule is applied. Then, from the above Theorem 2.4 we can obtain the following corollary.



(a) $p_i' \geq p_j'$ (b) $p_i' < p_j'$

Fig. 1 Stable region for the case $E_S(p_i, d_i; p_j, d_j)$

Corollary 2.5 The allocation of seats obtained from MFM is stable for the application of $ARPT$ rule.

Proof The allocation of seats $\{d_i \mid i \in S\}$ obtained from MFM satisfies

$$\max_i \frac{p_i}{d_i + \frac{1}{2}} \leq \min_j \frac{p_j}{d_j - \frac{1}{2}}$$

Therefore the relation (2.32) can be satisfied for any pair of i and j in S. Hence the allocation given by the MFM is stable for the application of the $ARPT$ rule. □

Applying the $ARPT$ rule to the measure of inequity function defined by maximizing the absolute biases from the average ratios, we obtain the following results, whose proofs are given in the Appendix.

Theorem 2.6 Let the measure of inequity be

$$E_M(p_i, d_i; p_j, d_j) = \max\{|\frac{d_i}{p_i} - r| \, , \, |\frac{d_j}{p_j} - r|\} \tag{2.35}$$

Then applying the ARPT rule to the measure $E_M(p_i, d_i; p_j, d_j)$, a stable allocation can be obtained as satisfying

$$x_j \geq x_i - \max\{p_i', p_j'\} \tag{2.36}$$

$$\text{if } \operatorname{sgn}(p_i' - p_j') \cdot (x_i + x_j) \leq \operatorname{sgn}(p_i' - p_j') \cdot 2r$$

$$\operatorname{sgn}(p_i' - p_j') \cdot x_k \leq \operatorname{sgn}(p_i' - p_j') \cdot r + \frac{1}{2} \max\{p_i', p_j'\} \tag{2.37}$$

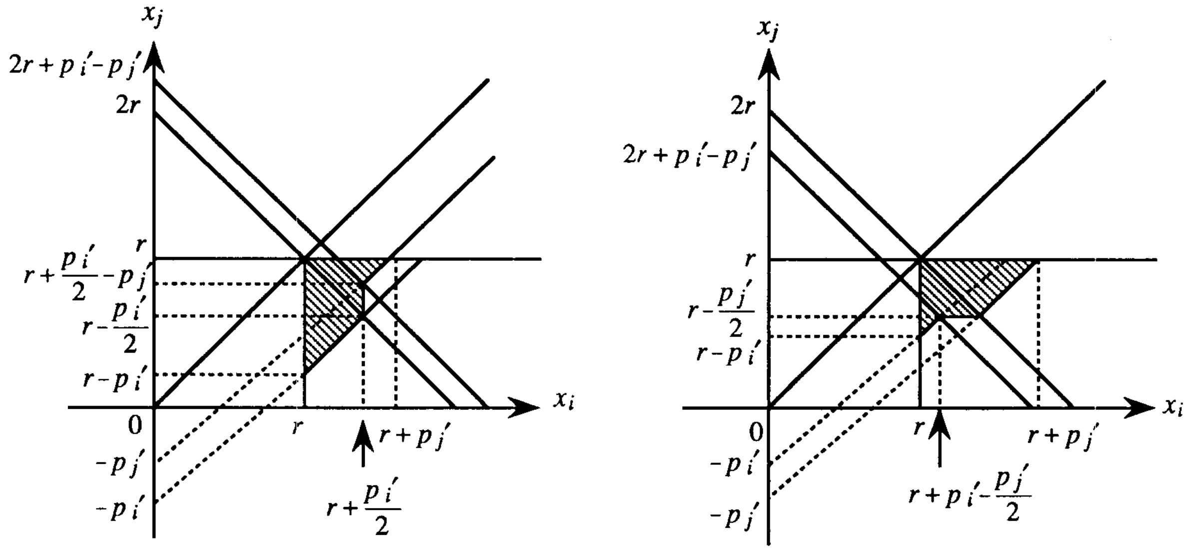
$$\text{if } \operatorname{sgn}(p_i' - p_j') \cdot 2r < \operatorname{sgn}(p_i' - p_j') \cdot (x_i + x_j) \leq \operatorname{sgn}(p_i' - p_j') \cdot (2r + p_i' - p_j')$$

$$x_j \geq x_i - \min\{p_i', p_j'\} \tag{2.38}$$

if $\mathrm{sgn}(p_i' - p_j') \cdot (x_i + x_j) \geq \mathrm{sgn}(p_i' - p_j') \cdot (2r + p_i' - p_j')$ and
either $p_j' \leq p_i' \leq 2p_j'$ or $2p_i' \geq p_j' > p_i'$

where suffix k and function $\mathrm{sgn}(p)$ are defined as in Theorem 2.4.

Illustrating a stable region given by (2.36)-(2.38) on the $x_i - x_j$ plane, shaded areas are obtained as in Fig. 2, while other cases $p_i' > 2p_j'$ and $2p_i' < p_j'$ can be reduced to degenerate ones.



(a) $2p_j' \geq p_i' \geq p_j'$ (b) $2p_i' \geq p_j' > p_i'$

Fig. 2 Stable region for the case $E_M(p_i, d_i; p_j, d_j)$

Corollary 2.7 Suppose that both of the two apportionment methods *GDM* and *SDM* give an identical assignment of seats. Then it is a stable allocation for the application of the *ARPT* rule to the measure of inequity E_M.

Theorem 2.8 Let the measure of inequity be

$$E_P(p_i, d_i; p_j, d_j) = |\frac{d_i}{p_i} - r| \cdot |\frac{d_j}{p_j} - r| \tag{2.39}$$

Then applying the *ARPT* rule to the measure $E_P(p_i, d_i; p_j, d_j)$, a stable allocation can be obtained as satisfying

$$X_j \geq X_i - 1 \quad \text{for } 0 \leq X_i \leq 1,\ -1 \leq X_j \leq 0 \tag{2.40}$$

$$2X_iX_j + X_i - X_j - 1 \geq 0 \quad \text{for } X_i > 1,\ -1 \leq X_j \leq 0 \text{ and } 0 \leq X_i,\ -1 > X_j \tag{2.41}$$

where $X_i = d_i - q_i, X_j = d_j - q_j, q_i = rp_i$ and $q_j = rp_j$.

The shaded areas in Fig. 3 serve to illustrate a stable region given by (2.40) and (2.41) on the $X_i - X_j$ plane.

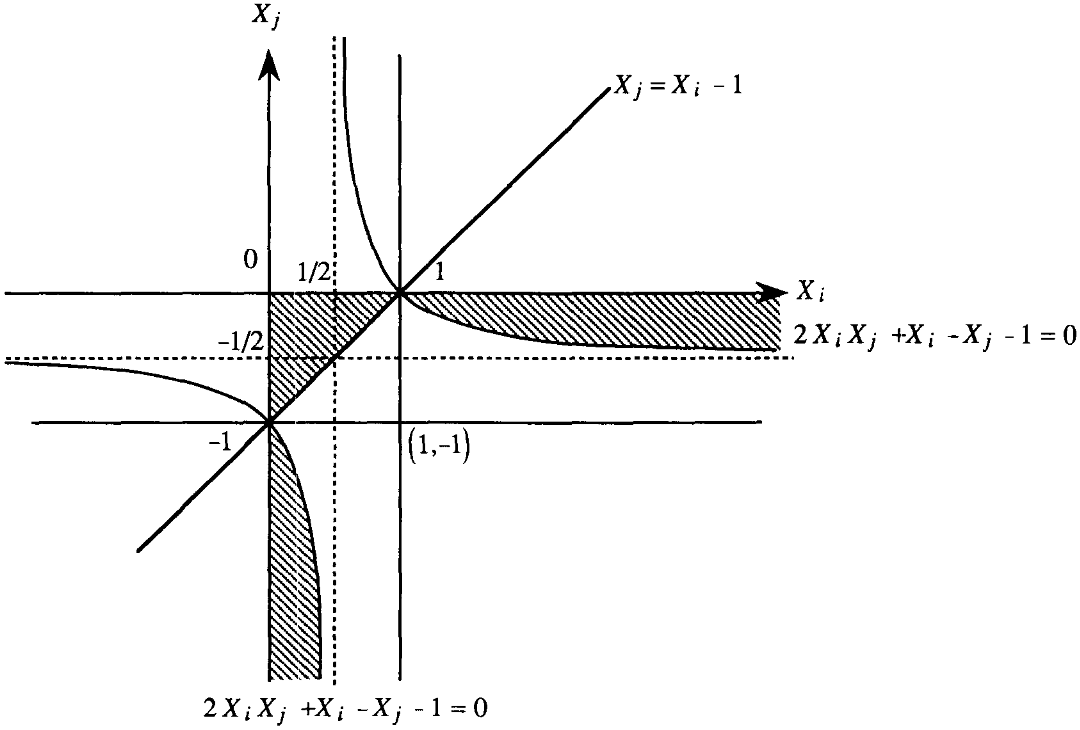


Fig. 3 Stable region for the case $E_M(p_i, d_i; p_j, d_j)$

Corollary 2.9 Suppose $\lambda = \frac{1}{r}$ be a parameter satisfying the MFM condition (2.6) for $v(d_i) = d_i + \frac{1}{2}$. Then the solution of the MFM is a stable assignment for the application of the $ARPT$ rule.

3. Global optimization aspects of apportionment methods

In this section we look at apportionment methods from the viewpoint of constrained optimization problems. In this respect, as far as we know, very few investigations have been done so far except that some preliminary results have been obtained and seen in [8, 11]. The various kinds of constrained optimization problems with respect to the unknown variables $\{d_i \mid i \in S\}$ have the same constraints as follows.

$$\sum_{i \in S} d_i = K \tag{3.1}$$

$$d_i \geq 0, \ integer, \qquad i \in S \tag{3.2}$$

So from now on we abbreviate the above constraints, showing only the objective function for each constrained optimization problem. First, the following theorem shows the constrained optimization problems for which an optimal solution is given by the LFM.

Theorem 3.1 The LFM gives an optimal solution for the following constrained optimization problems.

$$\text{P1} : \min_{\mathbf{d}} \sum_{i \in S} |d_i - q_i| \tag{3.3}$$

$$\text{P2} : \min_{\mathbf{d}} \max_{i \in S} |d_i - q_i| \tag{3.4}$$

$$\text{P3} : \min_{\mathbf{d}} \sum_{i \in S} (d_i - q_i)^2 \tag{3.5}$$

The above theorem can be easily proved, so it is omitted here. Incidentally, the LFM gives an optimal solution to all constrained optimization problems with objective functions with the form of l_p-norm of $|\mathbf{d} - \mathbf{q}|$ (see [9]).

Regarding the GDM and the MFM, we have the following results.

Theorem 3.2 The GDM gives an optimal solution for the following constrained optimization problems.

$$\text{P4} : \min_{\mathbf{d}} \max_{i \in S} \frac{d_i}{p_i} \tag{3.6}$$

$$\text{P5} : \max_{\mathbf{d}} \min_{i \in S} \frac{p_i}{d_i} \tag{3.7}$$

Proof If an assignment $\{d_i \mid i \in S\}$ is optimal, then for all d_i, d_j with $i \neq j$ and $d_i > 0$, a transfer from i to j cannot improve the objective criterion. That is, let

$$\frac{d_k}{p_k} = \max_{i \in S} \frac{d_i}{p_i} \tag{3.8}$$

then we have to have the following

$$\frac{d_j + 1}{p_j} \geq \frac{d_k}{p_k} \geq \frac{d_i}{p_i} \quad \text{any } i \in S, \text{ some } j \in S \tag{3.9}$$

Hence the following relation has to be satisfied.

$$\frac{p_i}{d_i + 1} \leq d_j p_j \qquad i \in S, \ j \in S \tag{3.10}$$

The above inequality is equivalent to the following relation.

$$\max_{\substack{i \\ d_i \geq 0}} \frac{p_i}{d_i + 1} \leq \min_{\substack{j \\ d_j > 0}} \frac{p_j}{d_j} \tag{3.11}$$

which is the max-min inequality that characterizes the GDM.

Conversely, if $\{d_i \mid i \in S\}$ is an assignment solution obtained from the GDM, then it satisfies the relation (3.11). Suppose $\{d'_i \mid i \in S\}$ is another assignment different from $\{d_i \mid i \in S\}$, then we define sets of suffices as follows.

$$S^+ = \{i \in S \mid d'_i > d_i\}, \quad S^- = \{j \in S \mid d'_j < d_j\} \tag{3.12}$$

Let

$$\begin{cases} d'_i = d_i + \alpha_i & i \in S^+ \\ d'_j = d_j - \beta_j & j \in S^- \end{cases} \tag{3.13}$$

then positive parameters $\{\alpha_i\}, \{\beta_j\}$ satisfy

$$\sum_{i \in S^+} \alpha_i = \sum_{j \in S^-} \beta_j = \gamma \tag{3.14}$$

Then we need to show

$$\max_i \frac{p_i}{d'_i} = \frac{p_l}{d'_l} \geq \frac{p_k}{d_k} = \max_j \frac{p_j}{d_j} \tag{3.15}$$

Hence there are two cases we need to consider.

$$\begin{cases} \text{Case } 1 : k \notin S^+ \cup S^-, \ l \in S^+ \\ \text{Case } 2 : k \in S^-, \ l \notin S^+ \cup S^- \end{cases}$$

Case 1 : Since $k \notin S^+ \cup S^-$ and $l \in S^+$, we have

$$\frac{p_l}{d'_l} = \frac{p_l}{d_l + \alpha_l} \geq \frac{p_i}{d_i} \quad \text{any } i \notin S^+ \cup S^- \tag{3.16}$$

Hence we obtain

$$\frac{p_k}{d'_k} \leq \frac{p_l}{d_l + \alpha_l} \tag{3.17}$$

which is equivalent to (3.15).

Case 2 : Since $k \in S^-$ and $l \notin S^+ \cup S^-$, we have

$$\frac{p_l}{d'_l} = \frac{p_l}{d_l} \geq \frac{p_i}{d_i - \beta_i} \quad i \in S^- \tag{3.18}$$

Hence we obtain

$$\frac{p_k}{d_k} \leq \frac{p_k}{d_k - \beta_k} \leq \frac{p_l}{d_l} \tag{3.19}$$

which is equivalent to (3.15) again. Thus the criterion P4 is shown to be satisfied by the GDM. The case P5 is equivalent to P4. □

Similarly we can obtain the following results whose proofs are given in the Appendix.

Theorem 3.3 The MFM gives an optimal solution for the following constrained optimization problems.

$$\text{P6} : \min_{\{\mathbf{d}\}} \sum_{i \in S} \left|\frac{d_i}{p_i} - r\right| \tag{3.20}$$

$$\text{P7} : \min_{\{\mathbf{d}\}} \sum_{i \in S} p_i\left(\frac{d_i}{p_i} - r\right)^2 \tag{3.21}$$

Theorem 3.4 The EPM gives an optimal solution for the following constrained optimization problem.

$$\text{P8} : \min_{\{\mathbf{d}\}} \sum_{i \in S} d_i\left(\frac{p_i}{d_i} - s\right)^2 \tag{3.22}$$

where s indicates the average number of individuals per seat, *i.e.*, $s = \frac{1}{r} = \frac{P}{K}$.

Theorem 3.5 The HMM gives an optimal solution for the following constrained optimization problem.

$$P9: \min_{\{d\}} \sum_{i \in S} |\frac{p_i}{d_i} - s| \tag{3.23}$$

Interestingly enough, Theorem 3.3 shows that, when the measure of inequity is given as the bias of the number of seats per individual from its mean, both their absolute sum and weighted squared sum are minimized by the MFM. However, when the measure of inequity is expressed as the difference of the number of individuals per seat from its mean, their weighted squared sum is minimized by the EPM, while the HMM minimizes their absolute sum. We believe that the MFM is more sensitive to the fairness of seat per individual ratio, *i.e.* seat per individual oriented, rather than the EPM and the HMM, which are more individual per seat oriented.

4. Parametric divisor method

Using a parameter t such that $0 \le t \le 1$, the divisor function of the parametric divisor method, which we denote by PDM, can be written as follows.

$$v_{PD}(d, t) = d + t \tag{4.1}$$

Comparing the above function $v_P(d, t)$ with those in Table 1, we find that $t = 0, 1/2$, and 1 correspond to those functions of the SDM, MFM and GDM, respectively.

Now the apportionment method based upon the PDM can be described as follows. Let the parameter for PDM be $\lambda = \lambda_{PD}$, then λ_{PD} can be determined as the maximum λ satisfying

$$\sum_{i \in S} \lfloor \frac{p_i}{\lambda} + 1 - t \rfloor \ge K \tag{4.2}$$

If (4.2) holds as an equality for $\lambda = \lambda_{PD}$, then the allocation $\{d_i \mid i \in S\}$ is given by

$$d_i = \lfloor \frac{p_i}{\lambda} + 1 - t \rfloor \qquad i \in S \tag{4.3}$$

The case that (4.2) holds as an inequality is dealt with similarly as described in (2.15)-(2.17) for the GDM. Also parameter λ_{PD} satisfies

$$\frac{p_i}{d_i + t} \le \lambda_{PD} \le \frac{p_i}{d_i - 1 + t} \qquad i \in S \tag{4.4}$$

Hence we have

$$\max_i \frac{p_i}{d_i + t} \le \min_j \frac{p_j}{d_j - 1 + t} \tag{4.5}$$

In order to look at the parametric method from the viewpoint of applying the $ARPT$ rule, we define the measure of inequity function $E_{SV}(p_i, d_i; p_j, d_j; v)$ using a parameter v as follows.

$$E_{SV}(p_i, d_i; p_j, d_j; v) = |\frac{d_i}{p_i} - r| + v|\frac{d_j}{p_j} - r| \qquad 0 < v \le 1 \tag{4.6}$$

Then we obtain the following theorem.

Theorem 4.1 Applying the $ARPT$ rule to the measure of inequity E_{SV}, a stable region can be obtained as follows. If $p'_i \ge p'_j$, then we have

$$x_j \ge x_i - \frac{vp'_i + p'_j}{1 + v} \quad \text{if } r - p'_j \le x_j \le r \tag{4.7}$$

$$(1-v)x_j \leq -(1+v)x_i + 2r + vp'_i - p'_j$$
$$\text{if} \max\{0, r - \frac{p'_i}{2} - (1+\frac{v}{2})p'_j\} \leq x_j < r - p'_j \tag{4.8}$$

$$x_i \leq r + \frac{p'_i - vp'_j}{2} \quad \text{if } 0 \leq x_j \leq \max\{0, r - \frac{p'_i}{2} - (1+\frac{v}{2})p'_j\} \tag{4.9}$$

where x_i, x_j, p'_i and p'_j are as defined in Theorem 2.4. If $p'_i < p'_j$, then the following has to hold.

$$x_j \geq x_i - \frac{vp'_i + p'_j}{1+v} \quad \text{if } r \leq x_i \leq r + p'_i \tag{4.10}$$

$$(1+v)x_j \geq (1-v)x_i + 2vr + vp'_i - p'_j$$
$$\text{if } r + p'_i < x_i \leq r + (1+\frac{1}{2v})p'_i + \frac{p'_j}{2} \tag{4.11}$$

$$x_j \geq r + \frac{p'_i - vp'_j}{2v} \quad \text{if } (1+\frac{1}{2v})p'_i + \frac{p'_j}{2} < x_i \text{ and } p'_i \leq vp'_j \tag{4.12}$$

If $p'_i > vp'_j$, then only (4.10) and (4.11) have to hold.

Proof Let the constituency i be favored over j, then we have $\frac{d_j}{p_j} \leq r \leq \frac{d_i}{p_i}$. So we need to consider the following cases (i)-(iii). In each case we show the condition that the measure of inequity after a transfer of one seat from i to j does not decrease.

(i) $\frac{d_i-1}{p_i} \leq r, \quad \frac{d_j+1}{p_j} \geq r$

$$\frac{d_i}{p_i} - r + v(r - \frac{d_j}{p_j}) \leq \frac{d_j+1}{p_j} - r + v(r - \frac{d_i-1}{p_i})$$

$$\frac{d_i - \frac{v}{1+v}}{p_i} \leq \frac{d_j + \frac{1}{1+v}}{p_j} \tag{4.13}$$

$$x_j \geq x_i - \frac{vp'_i + p'_j}{1+v} \tag{4.14}$$

(ii) $\frac{d_i-1}{p_i} > r, \quad \frac{d_j+1}{p_j} \geq r$

If $\frac{d_i-1}{p_i} \leq \frac{d_j+1}{p_j}$, then we need to have

$$\frac{d_i}{p_i} - r + v(r - \frac{d_j}{p_j}) \leq \frac{d_j+1}{p_j} - r + v(\frac{d_i-1}{p_i} - r)$$
$$(1+v)x_j \geq (1-v)x_i + 2vr + vp'_i - p'_j \tag{4.15}$$

Otherwise, *i.e.*, $\frac{d_i-1}{p_i} > \frac{d_j+1}{p_j}$, the following needs to be satisfied.

$$\frac{d_i}{p_i} - r + v(r - \frac{d_j}{p_j}) \leq \frac{d_i-1}{p_i} - r + v(\frac{d_j+1}{p_j} - r)$$

$$2vx_j \geq 2vr + p'_i - vp'_j \tag{4.16}$$

(iii) $\frac{d_i-1}{p_i} < r, \quad \frac{d_j+1}{p_j} < r$

If $\frac{d_i-1}{p_i} \leq \frac{d_j+1}{p_j}$, then we need to have

$$\frac{d_i}{p_i} - r + v(r - \frac{d_j}{p_j}) \leq r - \frac{d_j+1}{p_j} + v(r - \frac{d_i-1}{p_i})$$
$$(1-v)x_j \leq -(1+v)x_i + 2r + vp_i' - p_j' \tag{4.17}$$

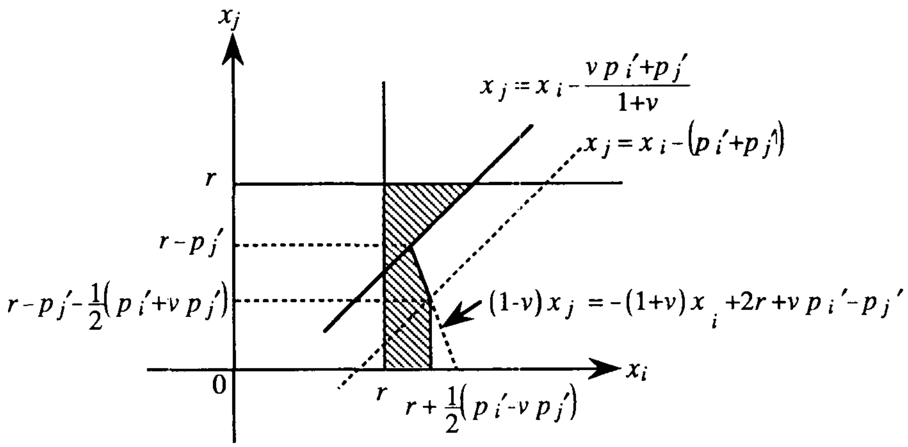
Otherwise, *i.e.*, $\frac{d_i-1}{p_i} > \frac{d_j+1}{p_j}$, the following needs to be satisfied.

$$\frac{d_i}{p_i} - r + v(r - \frac{d_j}{p_j}) \leq r - \frac{d_i-1}{p_i} + v(r - \frac{d_j+1}{p_j})$$

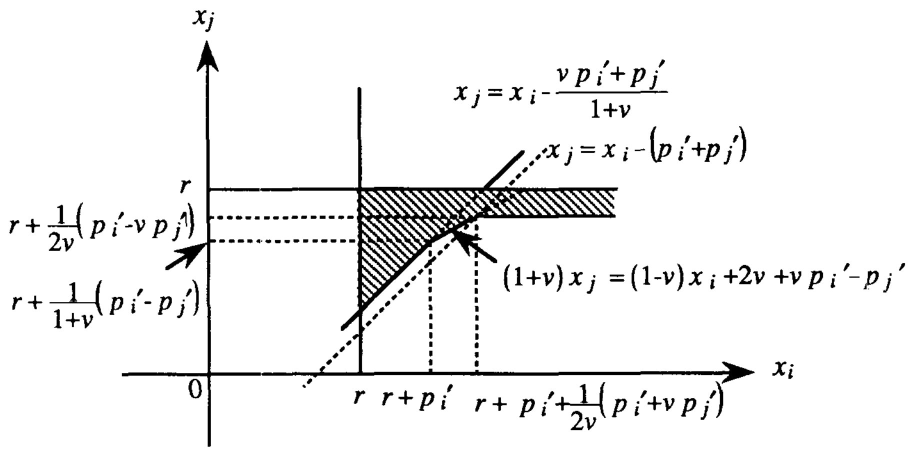
$$2x_i \leq 2r + p_i' - vp_j' \tag{4.18}$$

Summarizing the above results, we obtain the relations given by (4.7)–(4.12). □

The shaded areas in Fig. 4 illustrate stable regions given by (4.7)–(4.12) on the x_i-x_j plane. Note that Theorem 4.1 is a generalization of Theorem 2.4 in the sense that $v = 1$ in (4.6) corresponds to E_S in (2.29).



(a) $p_i' \geq p_j'$



(b) $p_i' < vp_j'$

Fig. 4 Stable region for the case E_{SV}

From Theorem 4.1, the following corollary can be easily obtained.

Corollary 4.2 The assignment obtained from the PDM gives a stable solution for the application of the $ARPT$ rule to the measure of inequity E_{SV}.

Proof The assignment $\{d_i \mid i \in S\}$ given by the PDM satisfies (4.5). Hence let $t = \frac{1}{1+v}$, then we obtain

$$\max_i \frac{p_i}{d_i + \frac{1}{1+v}} \leq \min_j \frac{p_j}{d_j - \frac{v}{1+v}} \tag{4.19}$$

which implies (4.13) for all pairs of i and j in S as given in Theorem 4.1. Thus the solution given by the PDM is a stable allocation of seats for the application of the $ARPT$ rule based upon the measure function (4.6). □

Regarding the measure of inequity with the form

$$E_{SS}(p_i, d_i; p_j, d_j) = \left|\frac{p_i}{d_i} - s\right| + \left|\frac{p_j}{d_j} - s\right| \tag{4.20}$$

we obtain a stable region for the application of the $ARPT$ rule including the state $(p_i, d_i; p_j, d_j)$ satisfying

$$\frac{p_i(d_i + \frac{1}{2})}{d_i(d_i+1)} \leq \frac{p_j(d_j - \frac{1}{2})}{d_j(d_j-1)}$$

The above condition is obtained by the HMM since its allocation satisfies the following relation.

$$\max_i \frac{p_i(d_i + \frac{1}{2})}{d_i(d_i+1)} \leq \min_j \frac{p_j(d_j - \frac{1}{2})}{d_j(d_j-1)} \tag{4.21}$$

Therefore, by defining a parametric divisor function similarly as the measure of inequity with the form (4.6)

$$E_{SW}(p_i, d_i; p_j, d_j; w) = \left|\frac{p_i}{d_i} - s\right| + w\left|\frac{p_j}{d_j} - s\right| \tag{4.22}$$

we obtain a stable region satisfying

$$\frac{p_i(d_i + \frac{w}{1+w})}{d_i(d_i+1)} \le \frac{p_j(d_j - \frac{1}{1+w})}{d_j(d_j-1)}$$

Suppose for a given parameter w such that $0 < w \le 1$, we have

$$\max_i \frac{p_i(d_i + \frac{w}{1+w})}{d_i(d_i+1)} \le \min_j \frac{p_j(d_j - \frac{1}{1+w})}{d_j(d_j-1)} \tag{4.23}$$

Then we determine that a convergent assignment is obtained. Let us call this apportionment method a parametric harmonic mean method ($PHMM$). Following similar procedure as in Theorem 4.1 and Corollary 4.2, we can obtain the following theorem.

Theorem 4.3 The allocation of seats by the $PHMM$ gives a stable assignment for the application of the $ARPT$ rule to the measure of inequity E_{SW}.

From the viewpoint of global optimization, regarding the apportionment method PDM, we can obtain the following theorem.

Theorem 4.4 The PDM gives an optimal solution for the following constrained optimization problem.

$$\mathrm{P10}: \min_{\{\mathbf{d}\}} \sum_{i\in S} p_i\left(\frac{d_i + t - \frac{1}{2}}{p_i} - r\right)^2 \tag{4.24}$$

Proof Criterion of the problem P10 can be written as follows.

$$\min_{\{\mathbf{d}\}} \left\{ \sum_{i\in S} \frac{(d_i+t-\frac{1}{2})^2}{p_i} - 2r\sum_{i\in S}(d_i+t-\frac{1}{2}) + Pr^2 \right\}$$

$$= \min_{\{\mathbf{d}\}} \left\{ \sum_{i\in S} \frac{(d_i+t-\frac{1}{2})^2}{p_i} - 2Nr(t-\frac{1}{2}) - Pr^2 \right\} \tag{4.25}$$

Hence minimizing the criterion (4.24) of P10 is equivalent to minimizing the first term $\sum_{i\in S}\frac{(d_i+t-\frac{1}{2})^2}{p_i}$ in (4.25).

If an assignment $\{d_i \mid i \in S\}$ is optimal, then for all d_i, d_j with $i \ne j$ and $d_i > 0$, a transfer of a seat from any constituency i to j cannot improve the objective criterion, that is,

$$\frac{(d_i - \frac{3}{2} + t)^2}{p_i} + \frac{(d_j + \frac{1}{2} + t)^2}{p_j} \ge \frac{(d_i + t - \frac{1}{2})^2}{p_i} + \frac{(d_j + t - \frac{1}{2})^2}{p_j} \tag{4.26}$$

Hence the following relation has to be satisfied for any $i, j \in S$.

$$\frac{p_j}{d_j + t} \le \frac{p_i}{d_i - 1 + t}$$

Therefore, the above inequality is equivalent to the following relation.

$$\max_{d_j \ge 0} \frac{p_j}{d_j + t} \le \min_{d_i > 0} \frac{p_i}{d_i - 1 + t} \tag{4.27}$$

which is exactly the same criterion as (4.5) for the PDM.

Conversely, if $\{d_i \mid i \in S\}$ is a solution obtained from the GDM, then it satisfies the relation given by (4.27). Suppose $\{d'_i \mid i \in S\}$ be another assignment different from $\{d_i \mid i \in S\}$, then we define sets of suffices S^+ and S^- as in (3.12) and denote $\{d'_i \mid i \in S\}$ as in (3.13). Then we need to show the following relation.

$$\sum_{i\in S} \frac{(d'_i + t - \frac{1}{2})^2}{p_i} \geq \sum_{i\in S} \frac{(d_i + t - \frac{1}{2})^2}{p_i} \tag{4.28}$$

Namely, we have

$$\sum_{i\in S^+} \frac{(d_i + \alpha_i + t - \frac{1}{2})^2}{p_i} + \sum_{j\in S^-} \frac{(d_j - \beta_j + t - \frac{1}{2})^2}{p_j} \geq \sum_{i\in S^+\cup S^-} \frac{(d_i + t - \frac{1}{2})^2}{p_i}$$

$$\sum_{i\in S^+} \frac{\alpha_i(d_i + t - \frac{1}{2} + \frac{\alpha_i}{2})}{p_i} \geq \sum_{j\in S^-} \frac{\beta_j(d_j + t - \frac{1}{2} - \frac{\beta_j}{2})}{p_j} \tag{4.29}$$

From (4.26) we have

$$\frac{d_i + t + \frac{\alpha_i - 1}{2}}{p_i} \geq \frac{d_i + t - 1}{p_i} \geq \frac{d_j + t - 1}{p_j} \geq \frac{d_j + t - \frac{\beta_j + 1}{2}}{p_j} \qquad i \in S^+, j \in S^-$$

Therefore, the inequality (4.29) can be obtained just by adding γ inequalities with the following form.

$$\frac{d_i + t + \frac{\alpha_i - 1}{2}}{p_i} \geq \frac{d_j + t - \frac{\beta_j + 1}{2}}{p_j} \qquad i \in S^+,\ j \in S^- \tag{4.30}$$

Thus the theorem is proved. □

In the next section we investigate a PDM described in (4.3) using Japan's HOR data, then compare this with other traditional apportionment methods.

5. Numerical experiments

Japan's House of Representative (HOR) has 130 political constituencies, each (CNST.) of which has a population (PPL.) and a current allocation (CRT) of representatives as shown in Table 2. Applying six apportionment methods (GDM, MFM, EPM, HMM, SDM and LFM) to Japan's HOR data based upon the 1985 Census, we obtain the results given in Table 2. First we recognize that Japan's current allocation of HOR seats to each constituency does not reflect the "proportionality to the population" and moreover smaller constituencies, which are mostly in rural areas, are favored over larger constituencies, which are mainly in urban areas. The results in Table 2 show that the apportionment methods GDM, MFM, EPM, HMM and SDM are, in this order, relatively more favorable to these constituencies with larger population, and Japan's current allocation of HOR seats is rather close to that of the SDM. The apportionment method LFM always satisfies the quota property since the allocation by the LFM is either rounded up or rounded down of the exact quota, *i.e.*, stays within the quota. We believe that the LFM is the most unbiased method although it violates the house property unfortunately. The result in Table 2 also shows that the method LFM gives similar apportionment to MFM or EPM. In the 1910's and 1920's in the United States there had been very severe controversy over the bias between the MFM and the EPM regarding which method should be more unbiased (see, *e.g.*, [8], ch.6). From our numerical results and historical arguments done so far, we can say that "impartial (unbiased to both larger or smaller constituencies) and appropriate" apportionment methods should be either MFM or EPM, or between or around these methods.

Applying the PDM given in section 4 to our HOR data we obtain the apportionment results as given in Table 3 using the values of a new parameter s, which equals 1-t in (4.3). The results in Table 3 indicate that the PDM with a smaller parameter value s is more favorable to larger constituencies while that with a larger parameter value s is more favorable to smaller constituencies.

Comparing the results of Table 3 with the allocation by the LFM in Table 2, we can easily recognize that if the parameter value s satisfies $s < 0.5$, larger constituencies get more seats and smaller ones have less, while if $s > 0.7$, smaller constituencies obtain more seats and larger ones less. Thus we can conclude that the PDM should be taken into account for the parameter s such that $0.5 \leq s \leq 0.7$ since a parameter s less than 0.5 makes the PDM too favorable to larger constituencies and s larger than 0.7 makes the method too favorable to smaller constituencies.

Let us look at the relation between apportionment results and the parameter s for $0.5 \leq s \leq 0.7$ into more detail. Firstly, we denote the apportionment results obtained from the PDM with parameter s by the row vector $A(s)$ consisting of nine elements such that $A(s) = (i_9, i_8, \dots, i_1)$ where i_k indicates that i_k-th largest constituency is the smallest one such that k seats are assigned. Based upon this notation, we obtain $A(0.5) = (1, 5, 16, 22, 42, 70, 99, 127, 130)$, $A(0.6) = (1, 5, 15, 22, 40, 70, 101, 128, 130)$, $A(0.7) = (1, 4, 12, 21, 39, 70, 106, 129, 130)$. Also defining the apportionment results by the methods MFM, EPM and LFM by $A(MFM), A(EPM)$ and $A(LFM)$, respectively, we obtain $A(MFM) = (1, 5, 16, 22, 42, 70, 99, 127, 130)$, $A(EPM) = (1, 5, 15, 22, 40, 70, 101, 127, 130)$, and $A(LFM) = (1, 5, 15, 22, 41, 70, 101, 127, 130)$, respectively.

From the definition of the vector notation of $A(s)$ and $A(m)$ where $0.5 \leq s \leq 0.7$ and $m \in \{MFM,\ EPM,\ LFM\}$, we can define the difference between two apportionment methods $A(x)$ and $A(y)$ where $x, y \in \{s \mid 0.5 \leq s \leq 0.7\} \cup \{MFM,\ EPM,\ LFM\}$ as follows.

$$\begin{aligned} A(x) - A(y) &= (i_9^x, i_8^x, \dots, i_1^x) - (i_9^y, i_8^y, \dots, i_1^y) \\ &= (i_9^x - i_9^y, i_8^x - i_8^y, \dots, i_1^x - i_1^y) \end{aligned}$$

and the distance between these two methods by

$$|A(x) - A(y)| = \sum_{k=1}^{9} |i_k^x - i_k^y|$$

Obviously, the distance function is symmetric since $|A(x) - A(y)| = |A(y) - A(x)|$ and it gives an even number since $\sum_{k=1}^{9}(i_k^x - i_k^y) = 0$.

From the results of Tables 2 and 3 we have $|A(MFM) - A(EPM)| = 6$, $|A(LFM) - A(EPM)| = 2$, and $|A(LFM) - A(MFM)| = 4$. We know from the definition of distance that $\frac{1}{2}|A(x) - A(y)|$ indicates the number of constituencies such that two apportionment methods $A(x)$ and $A(y)$ give different assignments. Therefore, there exist 6 different assignments between MFM and EPM, 4 different assignments between MFM and LFM, and only 2 between LFM and EPM.

Let us look at the apportionment results $A(s)$ by the PDM into more detail. We know that $A(0.5) = A(MFM)$. Our numerical experiments show that $A(EPM) = A(s)$ for the range of $s, 0.543 \leq s \leq 0.6$, *i.e.*, the apportionment method EPM corresponds to the PDM for the approximate parameter value s such that $0.543 \leq s \leq 0.6$. Also $A(HMM) = A(s)$ for the range of $s, 0.64 \leq s \leq 0.645$, *i.e.*, the apportionment method HMM corresponds to the PDM for approximately $0.64 \leq s \leq 0.645$, while $A(LFM) = A(s)$ for the range

Table 2. Political constituency and final apportionments

CNST.	PPL.	*GDM*	*MFM*	*EPM*	*HMM*	*SDM*	*LFM*	CRT
HKID-1	2169716	10	9	9	9	8	9	6
FKOK-1	1939788	9	8	8	8	8	8	5
TKYO-11	1875744	8	8	8	8	7	8	5
KNGW-2	1828593	8	8	8	8	7	8	5
CHBA-1	1790189	8	8	8	7	7	8	5
HYOG-2	1755079	8	7	7	7	7	7	5
OSAK-3	1720428	8	7	7	7	7	7	5
KNGW-4	1711045	8	7	7	7	7	7	4
KYOT-2	1707152	8	7	7	7	7	7	5
CHBA-4	1683125	8	7	7	7	7	7	4
OSAK-5	1637539	7	7	7	7	7	7	4
MIYG-1	1599740	7	7	7	7	6	7	5
TKYO-7	1565417	7	7	7	7	6	7	4
TKYO-10	1556469	7	7	7	7	6	7	5
KNGW-3	1542055	7	7	7	6	6	7	4
SITM-2	1526507	7	7	6	6	6	6	4
OSAK-4	1496406	7	6	6	6	6	6	4
HYOG-1	1410834	6	6	6	6	6	6	5
AITI-2	1381305	6	6	6	6	6	6	4
SZOK-1	1370523	6	6	6	6	6	6	5
SITM-4	1369057	6	6	6	6	6	6	4
NARA-1	1304866	6	6	6	5	5	6	5
KNGW-1	1281881	6	5	5	5	5	5	4
GIFU-1	1263739	6	5	5	5	5	5	5
AITI-4	1236959	5	5	5	5	5	5	4
HRSM-1	1203166	5	5	5	5	5	5	3
OSAK-2	1201348	5	5	5	5	5	5	5
SITM-5	1190106	5	5	5	5	5	5	3
SZOK-2	1183457	5	5	5	5	5	5	5
OKNW-1	1179097	5	5	5	5	5	5	5
OSAK-7	1177473	5	5	5	5	5	5	3
SITM-1	1177247	5	5	5	5	5	5	3
MIEE-1	1172473	5	5	5	5	5	5	5
SIGA-1	1155844	5	5	5	5	5	5	5
HKID-5	1131904	5	5	5	5	5	5	5
TKYO-4	1118220	5	5	5	5	5	5	5
IBRK-1	1107626	5	5	5	5	5	5	4
KMMT-1	1097780	5	5	5	5	5	5	5
TKYO-3	1080470	5	5	5	5	4	5	4
KNGW-5	1068400	5	5	5	5	4	5	3
AITI-6	1058880	5	5	4	4	4	5	4
AITI-1	1057501	5	5	4	4	4	4	4
TKYO-2	1054133	5	4	4	4	4	4	5
TCHG-1	1036612	4	4	4	4	4	4	5
HKID-4	1034861	4	4	4	4	4	4	5
SZOK-3	1020712	4	4	4	4	4	4	4
AITI-3	1010934	4	4	4	4	4	4	3
FKOK-2	996563	4	4	4	4	4	4	5
AOMR-1	987405	4	4	4	4	4	4	4
OKYM-2	976010	4	4	4	4	4	4	5
IBRK-3	967446	4	4	4	4	4	4	5
NGSK-1	964759	4	4	4	4	4	4	5

CNST.	PPL.	*GDM*	*MFM*	*EPM*	*HMM*	*SDM*	*LFM*	CRT
OKYM-1	940896	4	4	4	4	4	4	5
HYOG-3	931342	4	4	4	4	4	4	3
HRSM-3	907690	4	4	4	4	4	4	5
FKOK-4	900537	4	4	4	4	4	4	4
FKOK-3	882371	4	4	4	4	4	4	5
SAGA-1	880013	4	4	4	4	4	4	5
KYOT-1	879422	4	4	4	4	4	4	5
CHBA-3	876419	4	4	4	4	4	4	5
TKYO-9	873135	4	4	4	4	4	4	3
TKYO-5	866342	4	4	4	4	4	4	3
KGSM-1	851854	4	4	4	4	4	4	4
HYOG-4	851743	4	4	4	4	4	4	4
YMGC-2	848828	4	4	4	4	4	4	5
IWTE-1	846892	4	4	4	4	4	4	4
KOTI-1	839784	3	4	4	4	4	4	5
TKSM-1	834889	3	4	4	4	4	4	5
YMNS-1	832832	3	4	4	4	4	4	5
TCHG-2	829454	3	4	4	4	4	4	5
FUKI-1	817633	3	3	3	3	3	3	4
OITA-1	816464	3	3	3	3	3	3	4
TKYO-6	808974	3	3	3	3	3	3	4
CHBA-2	798430	3	3	3	3	3	3	4
SIMN-1	794629	3	3	3	3	3	3	5
ISKW-1	789142	3	3	3	3	3	3	3
FKSM-1	771072	3	3	3	3	3	3	4
NIGT-3	768503	3	3	3	3	3	3	5
HKID-2	767974	3	3	3	3	3	3	4
GIFU-2	764797	3	3	3	3	3	3	4
YMGC-1	752799	3	3	3	3	3	3	4
FKSM-2	747622	3	3	3	3	3	3	5
AKTA-1	746675	3	3	3	3	3	3	4
MYZK-1	745710	3	3	3	3	3	3	3
NIGT-1	743154	3	3	3	3	3	3	3
KMMT-2	739967	3	3	3	3	3	3	5
GNMA-3	725265	3	3	3	3	3	3	4
OSAK-1	724129	3	3	3	3	3	3	3
YMGT-1	715822	3	3	3	3	3	3	4
OSAK-6	710772	3	3	3	3	3	3	3
AITI-5	709593	3	3	3	3	3	3	3
HRSM-2	708344	3	3	3	3	3	3	4
GNMA-1	659408	3	3	3	3	3	3	3
IBRK-2	649933	3	3	3	3	3	3	3
WKYM-1	639756	3	3	3	3	3	3	3
NGSK-2	629209	2	3	3	3	3	3	4
TOYM-1	627226	2	3	3	3	3	3	3
TOTR-1	616024	2	3	3	3	3	3	4
SITM-3	600761	2	3	3	3	3	3	3
IWTE-2	586719	2	2	3	3	3	3	4
NGNO-1	585569	2	2	3	3	3	3	3
TKYO-1	577806	2	2	2	3	3	2	3
MIYG-2	576555	2	2	2	3	3	2	4
HKID-3	574984	2	2	2	2	3	2	3
MIEE-2	574838	2	2	2	2	3	2	4

CNST.	PPL.	*GDM*	*MFM*	*EPM*	*HMM*	*SDM*	*LFM*	CRT
NGNO-3	571726	2	2	2	2	3	2	4
FKSM-3	561610	2	2	2	2	3	2	3
NIGT-2	560065	2	2	2	2	3	2	3
KAGW-1	557122	2	2	2	2	3	2	3
EHIM-2	555415	2	2	2	2	3	2	3
YMGT-2	545840	2	2	2	2	3	2	3
AOMR-2	537043	2	2	2	2	2	2	3
GNMA-2	536586	2	2	2	2	2	2	3
EHIM-1	517401	2	2	2	2	2	2	3
AKTA-2	507357	2	2	2	2	2	2	3
NGNO-4	505719	2	2	2	2	2	2	3
TOYM-2	491143	2	2	2	2	2	2	3
NGNO-2	473913	2	2	2	2	2	2	3
KGSM-2	468450	2	2	2	2	2	2	3
KAGW-2	465447	2	2	2	2	2	2	3
EHIM-3	457167	2	2	2	2	2	2	3
TKYO-8	452653	2	2	2	2	2	2	3
WKYM-2	447450	2	2	2	2	2	2	3
OITA-2	433750	2	2	2	2	2	2	3
MYZK-2	429833	2	2	2	2	2	2	3
NIGT-4	406748	1	2	2	2	2	2	2
ISKW-2	363183	1	2	2	2	2	2	2
KGSM-3	345904	1	1	2	2	2	1	2
HYOG-5	329052	1	1	1	2	2	1	2
KGSM-4	153062	0	1	1	1	1	1	1
Total		512	512	512	512	512	512	512

Table 3. Final apportionments by parametric divisor method

CNST.	PARAMETER										
	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
HKID-1	10	10	10	9	9	9	9	9	9	9	8
FKOK-1	9	9	9	8	8	8	8	8	8	8	8
TKYO-11	8	8	8	8	8	8	8	8	8	7	7
KNGW-2	8	8	8	8	8	8	8	8	7	7	7
CHBA-1	8	8	8	8	8	8	8	7	7	7	7
HYOG-2	8	8	8	8	8	7	7	7	7	7	7
OSAK-3	8	8	8	7	7	7	7	7	7	7	7
KNGW-4	8	8	7	7	7	7	7	7	7	7	7
KYOT-2	8	8	7	7	7	7	7	7	7	7	7
CHBA-4	8	7	7	7	7	7	7	7	7	7	7
OSAK-5	7	7	7	7	7	7	7	7	7	7	7
MIYG-1	7	7	7	7	7	7	7	7	7	6	6
TKYO-7	7	7	7	7	7	7	7	6	6	6	6
TKYO-10	7	7	7	7	7	7	7	6	6	6	6
KNGW-3	7	7	7	7	7	7	7	6	6	6	6
SITM-2	7	7	7	7	7	7	6	6	6	6	6
OSAK-4	7	7	7	6	6	6	6	6	6	6	6
HYOG-1	6	6	6	6	6	6	6	6	6	6	6
AITI-2	6	6	6	6	6	6	6	6	6	6	6
SZOK-1	6	6	6	6	6	6	6	6	6	6	6
SITM-4	6	6	6	6	6	6	6	6	6	6	6
NARA-1	6	6	6	6	6	6	6	5	5	5	5
KNGW-1	6	6	6	6	5	5	5	5	5	5	5
GIFU-1	6	5	5	5	5	5	5	5	5	5	5
AITI-4	5	5	5	5	5	5	5	5	5	5	5
HRSM-1	5	5	5	5	5	5	5	5	5	5	5
OSAK-2	5	5	5	5	5	5	5	5	5	5	5
SITM-5	5	5	5	5	5	5	5	5	5	5	5
SZOK-2	5	5	5	5	5	5	5	5	5	5	5
OKNW-1	5	5	5	5	5	5	5	5	5	5	5
OSAK-7	5	5	5	5	5	5	5	5	5	5	5
SITM-1	5	5	5	5	5	5	5	5	5	5	5
MIEE-1	5	5	5	5	5	5	5	5	5	5	5
SIGA-1	5	5	5	5	5	5	5	5	5	5	5
HKID-5	5	5	5	5	5	5	5	5	5	5	5
TKYO-4	5	5	5	5	5	5	5	5	5	5	5
IBRK-1	5	5	5	5	5	5	5	5	5	5	5
KMMT-1	5	5	5	5	5	5	5	5	5	5	5
TKYO-3	5	5	5	5	5	5	5	5	5	4	4
KNGW-5	5	5	5	5	5	5	5	4	4	4	4
AITI-6	5	5	5	5	5	5	4	4	4	4	4
AITI-1	5	5	5	5	5	5	4	4	4	4	4
TKYO-2	5	4	4	5	4	4	4	4	4	4	4
TCHG-1	4	4	4	4	4	4	4	4	4	4	4
HKID-4	4	4	4	4	4	4	4	4	4	4	4
SZOK-3	4	4	4	4	4	4	4	4	4	4	4
AITI-3	4	4	4	4	4	4	4	4	4	4	4
FKOK-2	4	4	4	4	4	4	4	4	4	4	4

CNST.	PARAMETER										
	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
AOMR-1	4	4	4	4	4	4	4	4	4	4	4
OKYM-2	4	4	4	4	4	4	4	4	4	4	4
IBRK-3	4	4	4	4	4	4	4	4	4	4	4
NGSK-1	4	4	4	4	4	4	4	4	4	4	4
OKYM-1	4	4	4	4	4	4	4	4	4	4	4
HYOG-3	4	4	4	4	4	4	4	4	4	4	4
HRSM-3	4	4	4	4	4	4	4	4	4	4	4
FKOK-4	4	4	4	4	4	4	4	4	4	4	4
FKOK-3	4	4	4	4	4	4	4	4	4	4	4
SAGA-1	4	4	4	4	4	4	4	4	4	4	4
KYOT-1	4	4	4	4	4	4	4	4	4	4	4
CHBA-3	4	4	4	4	4	4	4	4	4	4	4
TKYO-9	4	4	4	4	4	4	4	4	4	4	4
TKYO-5	4	4	4	4	4	4	4	4	4	4	4
KGSM-1	4	4	4	4	4	4	4	4	4	4	4
HYOG-4	4	4	4	4	4	4	4	4	4	4	4
YMGC-2	4	4	4	4	4	4	4	4	4	4	4
IWTE-1	4	4	4	4	4	4	4	4	4	4	4
KOTI-1	3	4	4	4	4	4	4	4	4	4	4
TKSM-1	3	3	3	4	4	4	4	4	4	4	4
YMNS-1	3	3	3	4	4	4	4	4	4	4	4
TCHG-2	3	3	3	4	4	4	4	4	4	4	4
FUKI-1	3	3	3	3	3	3	3	3	4	3	3
OITA-1	3	3	3	3	3	3	3	3	3	3	3
TKYO-6	3	3	3	3	3	3	3	3	3	3	3
CHBA-2	3	3	3	3	3	3	3	3	3	3	3
SIMN-1	3	3	3	3	3	3	3	3	3	3	3
ISKW-1	3	3	3	3	3	3	3	3	3	3	3
FKSM-1	3	3	3	3	3	3	3	3	3	3	3
NIGT-3	3	3	3	3	3	3	3	3	3	3	3
HKID-2	3	3	3	3	3	3	3	3	3	3	3
GIFU-2	3	3	3	3	3	3	3	3	3	3	3
YMGC-1	3	3	3	3	3	3	3	3	3	3	3
FKSM-2	3	3	3	3	3	3	3	3	3	3	3
AKTA-1	3	3	3	3	3	3	3	3	3	3	3
MYZK-1	3	3	3	3	3	3	3	3	3	3	3
NIGT-1	3	3	3	3	3	3	3	3	3	3	3
KMMT-2	3	3	3	3	3	3	3	3	3	3	3
GNMA-3	3	3	3	3	3	3	3	3	3	3	3
OSAK-1	3	3	3	3	3	3	3	3	3	3	3
YMGT-1	3	3	3	3	3	3	3	3	3	3	3
OSAK-6	3	3	3	3	3	3	3	3	3	3	3
AITI-5	3	3	3	3	3	3	3	3	3	3	3
HRSM-2	3	3	3	3	3	3	3	3	3	3	3
GNMA-1	3	3	3	3	3	3	3	3	3	3	3
IBRK-2	3	3	3	3	3	3	3	3	3	3	3
WKYM-1	3	3	3	3	3	3	3	3	3	3	3
NGSK-2	2	3	3	3	3	3	3	3	3	3	3

CNST.	PARAMETER										
	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
TOYM-1	2	3	3	3	3	3	3	3	3	3	3
TOTR-1	2	2	3	3	3	3	3	3	3	3	3
SITM-3	2	2	2	2	3	3	3	3	3	3	3
IWTE-2	2	2	2	2	2	2	3	3	3	3	3
NGNO-1	2	2	2	2	2	2	3	3	3	3	3
TKYO-1	2	2	2	2	2	2	2	3	3	3	3
MIYG-2	2	2	2	2	2	2	2	3	3	3	3
HKID-3	2	2	2	2	2	2	2	3	3	3	3
MIEE-2	2	2	2	2	2	2	2	3	3	3	3
NGNO-3	2	2	2	2	2	2	2	3	3	3	3
FKSM-3	2	2	2	2	2	2	2	2	2	3	3
NIGT-2	2	2	2	2	2	2	2	2	2	3	3
KAGW-1	2	2	2	2	2	2	2	2	2	3	3
EHIM-2	2	2	2	2	2	2	2	2	2	3	3
YMGT-2	2	2	2	2	2	2	2	2	2	2	3
AOMR-2	2	2	2	2	2	2	2	2	2	2	2
GNMA-2	2	2	2	2	2	2	2	2	2	2	2
EHIM-1	2	2	2	2	2	2	2	2	2	2	2
AKTA-2	2	2	2	2	2	2	2	2	2	2	2
NGNO-4	2	2	2	2	2	2	2	2	2	2	2
TOYM-2	2	2	2	2	2	2	2	2	2	2	2
NGNO-2	2	2	2	2	2	2	2	2	2	2	2
KGSM-2	2	2	2	2	2	2	2	2	2	2	2
KAGW-2	2	2	2	2	2	2	2	2	2	2	2
EHIM-3	2	2	2	2	2	2	2	2	2	2	2
TKYO-8	2	2	2	2	2	2	2	2	2	2	2
WKYM-2	2	2	2	2	2	2	2	2	2	2	2
OITA-2	2	2	2	2	2	2	2	2	2	2	2
MYZK-2	2	2	2	2	2	2	2	2	2	2	2
NIGT-4	1	1	2	2	2	2	2	2	2	2	2
ISKW-2	1	1	1	1	1	2	2	2	2	2	2
KGSM-3	1	1	1	1	1	1	2	2	2	2	2
HYOG-5	1	1	1	1	1	1	1	2	2	2	2
KGSM-4	0	0	0	0	1	1	1	1	1	1	1
Total	512	512	512	512	512	512	512	512	512	512	512

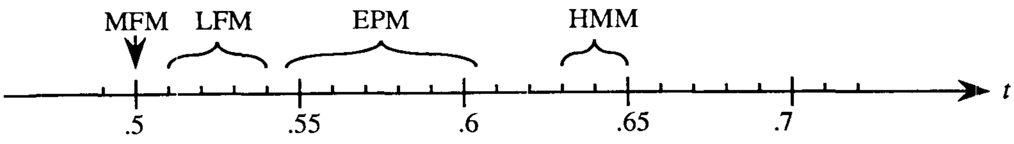


Fig. 5

of $s, 0.51 \leq s \leq 0.54$, *i.e.*, the method LFM is equivalent to the PDM for approximately $0.51 \leq s \leq 0.54$.

From these results we can conclude that the correspondence between PDM and other traditional apportionment methods MFM, EPM, HMM, and LFM can be illustrated as in Fig. 5.

6. Summary and conclusion

In this paper we proposed apportionment method PDM based upon the parameter t given in (4.3). As mentioned in sections 4 and 5, the parametric method PDM covers all six traditional methods GDM, MFM, EPM, HMM, SDM, and LFM by changing the parameter t from 0.0 to 1.0. The PDM satisfies the house monotone property for any t such that $0 \leq t \leq 1$ as it belongs to a divisor method. It does not guarantee the quota property as do other apportionment methods (with the exception of the LFM). From the results of our numerical experiments as illustrated in Fig. 5 we can conclude that the apportionment method LFM is located between MFM and EPM from the viewpoint of biasedness to the population size of the constituency. As history shows (see *e.g.* [8, 9]), there was a harsh controversy in the U.S. Congress in 1950's over whether the MFM or the EPM should be accepted. Although Balinski and Young [8] says that the MFM is the only unbiased divisor method, we believe that generally the MFM is still more favorable to larger constituencies since most numerical examples violate the quota property (see *e.g.* [8, 12, 13, 14]).

In conclusion, we believe that the method LFM, which of course satisfies the quota property, gives a most reasonable assignment of seats to the constituency although it does not satisfy the house monotone property. We would like to strongly recommend the PDM with the parameter value $0.52 \leq s \leq 0.54$ since it gives almost the same assignment as the LFM as shown in section 5, and importantly, it satisfies the house monotone property. Interestingly enough, the method PDM provides a solution to the local and global optimization problems given in Corollary 4.2 and Theorem 4.4.

Presently we are investigating other properties of population monotonicity, constituency and so on (see *e.g.* [5, 8, 11]) for the PDM to see if this method can be made to more closely satisfy these properties.

Acknowledgements

The author would like to express his sincere thanks to Professor M. L. Balinski of l'Ecole de Polytechnique in France, who has guided the author to the apportionment problem and has given him a lot of invaluable suggestions and comments. He is also very grateful to Professors L. E. Trotter and J. A. Muckstadt of Cornell University in the U. S. A., who have given him the chances to conduct the research on the apportionment problem at their University and also have assisted him with extensive contributions and helpful comments on this paper. Valuable and helpful comments provided by the anonymous referees are also gratefully acknowledged.

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Appendix

Proof of Theorem 2.6 Let constituency i be favored over j, then we have $\frac{d_j}{p_j} \le r \le \frac{d_i}{p_i}$. So we need to consider the following two cases (i)–(ii). In each case we show the condition that the measure of inequity after a transfer of one seat from i to j is larger than or equal to that before the transfer.

(i) $\frac{d_i}{p_i} - r \ge r - \frac{d_j}{p_j}$

Suppose $\frac{d_i-1}{p_i} \le r$ and $\frac{d_j+1}{p_j} \ge r$. If $\frac{d_j+1}{p_j} - r \ge r - \frac{d_i-1}{p_i}$, we need to have

$$\frac{d_i}{p_i} \le \frac{d_j+1}{p_j} \tag{A.1}$$

Otherwise, *i.e.*, $\frac{d_i}{p_i} + \frac{d_j}{p_j} < 2r + \frac{1}{p_i} - \frac{1}{p_j}$, we need to have

$$\frac{d_i}{p_i} \le r + \frac{1}{2p_i} \tag{A.2}$$

Suppose $\frac{d_i-1}{p_i} > r$ and $\frac{d_j+1}{p_j} \ge r$. If $\frac{d_j+1}{p_j} \ge \frac{d_i-1}{p_i}$, the condition can be reduced to (A.1). Otherwise, a transfer has to be made since $r < \frac{d_i-1}{p_i} < \frac{d_i}{p_i}$.

Suppose $\frac{d_i-1}{p_i} \le r$ and $\frac{d_j+1}{p_j} < r$. If $\frac{d_i-1}{p_i} \le \frac{d_j+1}{p_j}$, the condition can be reduced to (A.2). Otherwise, a transfer has to be made since $r > \frac{d_j+1}{p_j} > \frac{d_j}{p_j}$.

(ii) $\frac{d_i}{p_i} - r < r - \frac{d_j}{p_j}$

Suppose $\frac{d_i-1}{p_i} \le r$ and $\frac{d_j+1}{p_j} \ge r$. If $r - \frac{d_i-1}{p_i} \ge \frac{d_j+1}{p_j} - r$, we need to have

$$\frac{d_i - 1}{p_i} \le \frac{d_j}{p_j} \tag{A.3}$$

Otherwise, *i.e.*, $\frac{d_i}{p_i} + \frac{d_j}{p_j} > 2r + \frac{1}{p_i} - \frac{1}{p_j}$, we need to have

$$\frac{d_j}{p_j} \ge r - \frac{1}{2p_j} \tag{A.4}$$

Suppose $\frac{d_i-1}{p_i} \le r$ and $\frac{d_j+1}{p_j} < r$. If $\frac{d_i-1}{p_i} \le \frac{d_j+1}{p_j}$, the condition can be reduced to (A.3). Otherwise, a transfer has to be made since $\frac{d_j}{p_j} < \frac{d_j+1}{p_j} < r$.

Suppose $\frac{d_i-1}{p_i} > r$ and $\frac{d_j+1}{p_j} \ge r$. If $\frac{d_i-1}{p_i} \le \frac{d_j+1}{p_j}$, the condition can be reduced to (A.4). Otherwise, a transfer has to be made since $r < \frac{d_i-1}{p_i} < \frac{d_i}{p_i}$.

Summarizing the above two cases, if $2p'_j \ge p'_i \ge p'_j$, then the following has to hold

$$x_j \ge x_i - p'_i \qquad \text{if } x_i + x_j \le 2r \tag{A.5}$$

$$x_i \le r + \frac{p'_i}{2} \qquad \text{if } 2r \le x_i + x_j \le 2r + p'_i - p'_j \tag{A.6}$$

$$x_j \ge x_i - p'_j \qquad \text{if } x_i + x_j \ge 2r + p'_i - p'_j \tag{A.7}$$

and if $p'_i > 2p'_j$, only (A.5) and (A.6) have to hold. If $p'_i < p'_j \le 2p'_i$, then we must have

$$x_j \ge x_i - p'_j \qquad \text{if } x_i + x_j \ge 2r \tag{A.8}$$

$$x_i \ge r - \frac{p'_j}{2} \qquad \text{if } 2r > x_i + x_j \ge 2r + p'_i - p'_j \tag{A.9}$$

$$x_j \ge x_i - p'_i \qquad \text{if } x_i + x_j \le 2r + p'_i - p'_j \tag{A.10}$$

and if $2p'_i < p'_j$, only (A.8) and (A.9) have to hold.

All relations above have to be considered within the region of $x_j \le r$ and $r \le x_i$. Thus the relation (A.5)–(A.10) can be written as (2.36)–(2.38). □

Proof of Corollary 2.7 Let the identical assignment of seats by GDM and SDM be $\{d_i \mid i \in S\}$. Then it should satisfy the following max-min conditions.

$$\max_i \frac{p_i}{d_i + 1} \le \min_j \frac{p_j}{d_j}$$

$$\max_i \frac{p_i}{d_i} \le \min_j \frac{p_j}{d_j - 1}$$

Therefore the conditions of both (A.1) and (A.3) can be satisfied for all pairs i and j in S. Thus the allocation gives a stable solution. □

Proof of Theorem 2.8 Since we have $\frac{d_j}{p_j} \le r \le \frac{d_i}{p_i}$, we consider the following three cases (i)–(iii) in order that a transfer of one seat from a more favored constituency i to a less favored constituency j should not decrease the measure of inequity.

(i) $\frac{d_i-1}{p_i} \le r, \frac{d_j+1}{p_j} \ge r$

$$(\frac{d_i}{p_i} - r)(r - \frac{d_j}{p_j}) \le (r - \frac{d_i-1}{p_i})(\frac{d_j+1}{p_j} - r)$$

The above relation can be written as follows.

$$d_j - q_j \ge d_i - q_i - 1$$

Hence using $X_i \ge 0$ and $X_j \le 0$, we obtain

$$X_j \ge X_i - 1 \qquad \text{for } 0 \le X_i \le 1, -1 \le X_j \le 0 \tag{A.11}$$

(ii) $\frac{d_i-1}{p_i} > r, \frac{d_j+1}{p_j} \ge r$

$$(\frac{d_i}{p_i} - r)(r - \frac{d_j}{p_j}) \le (\frac{d_i-1}{p_i} - r)(\frac{d_j+1}{p_j} - r)$$

$$(d_i - q_i)(q_j - d_j) \le (d_i - q_i - 1)(d_j - q_j + 1)$$

Hence rewriting the above using X_i and X_j, we obtain

$$-X_iX_j \le (X_i - 1)(X_j + 1) \qquad \text{for } X_i > 1, -1 \le X_j \le 0 \tag{A.12}$$

(iii) $\frac{d_i-1}{p_i} \le r, \frac{d_j+1}{p_j} < r$

$$(\frac{d_i}{p_i} - r)(r - \frac{d_j}{p_j}) \le (r - \frac{d_i-1}{p_i})(r + \frac{d_j+1}{p_j})$$

$$(d_i - q_i)(q_j - d_j) \le (q_i - d_i + 1)(q_j - d_j - 1)$$

Hence rewriting the above using X_i and X_j, we obtain

$$-X_iX_j \le (1 - X_i)(-1 - X_j) \qquad \text{for } 0 \le X_i, -1 > X_j \tag{A.13}$$

Thus the relation (2.40)–(2.41) is obtained. □

Proof of Corollary 2.9 The parameter $\frac{1}{r}$ satisfies the MFM condition for the case $v(d_i) = d_i + \frac{1}{2}$. Hence

$$\frac{p_j}{d_j + \frac{1}{2}} \le \frac{1}{r} \le \frac{p_i}{d_i - \frac{1}{2}} \qquad \text{for all } i, j \in S \tag{A.14}$$

and we obtain the following relation.

$$X_i = d_i - q_i \le \frac{1}{2} \tag{A.15}$$

$$X_j = d_j - q_j \ge -\frac{1}{2} \tag{A.16}$$

Considering $X_i \ge 0$ and $X_j \le 0$ for applying the $ARPT$ rule, the region given by (A.15) and (A.16) is a stable region. □

Proof of Theorem 3.3 We show that the MFM solves the problem P6. If an assignment $\{d_i \mid i \in S\}$ is optimal, then for all d_i, d_j with $i \neq j$ and $d_i > 0$, a transfer of a seat from constituency i to j cannot improve the objective criterion, that is,

$$|\frac{d_i - 1}{p_i} - r| + |\frac{d_j + 1}{p_j} - r| \geq |\frac{d_i}{p_i} - r| + |\frac{d_j}{p_j} - r| \tag{A.17}$$

Suppose that the constituency i is favored over j, the inequality (A.17) can be written as follows.

$$r - \frac{d_i - 1}{p_i} + \frac{d_j + 1}{p_j} - r \geq \frac{d_i}{p_i} - r + r - \frac{d_j}{p_j}$$

$$\frac{p_i}{d_i - \frac{1}{2}} \geq \frac{p_j}{d_j + \frac{1}{2}} \tag{A.18}$$

The above inequality indicates that an optimal assignment $\{d_i \mid i \in S\}$ satisfies

$$\min_{d_i > 0} \frac{p_i}{d_i - \frac{1}{2}} \geq \max_{d_i \geq 0} \frac{p_j}{d_j + \frac{1}{2}} \tag{A.19}$$

which is the max-min inequality characterizing the MFM.

Conversely, suppose the assignment $\{d_i \mid i \in S\}$ satisfies the MFM max-min inequality (A.19). Let $\{d'_i \mid i \in S\}$ be another assignment different from $\{d_i \mid i \in S\}$, then we define sets of suffices S^+ and S^- as in (3.12) and denote $\{d'_i \mid i \in S\}$ as in (3.13). Then we need to show the following relation.

$$\sum_{i \in S} |\frac{d'_i}{p_i} - r| \geq \sum_{i \in S} |\frac{d_i}{p_i} - r| \tag{A.20}$$

From (A.19) we have

$$\frac{d_j - \beta_j}{p_j} \leq \frac{d_j - \frac{1}{2}}{p_j} \leq \frac{d_i + \frac{1}{2}}{p_i} \leq \frac{d_i + \alpha_i}{p_i} \qquad i \in S^+, j \in S^- \tag{A.21}$$

Using the above relation and also using

$$\max_i \frac{d_i - \frac{1}{2}}{p_i} \leq \min_j \frac{d_j + \frac{1}{2}}{p_j}$$

we obtain the following.

$$\sum_{i \in S} |\frac{d'_i}{p_i} - r| - \sum_{i \in S} |\frac{d_i}{p_i} - r|$$

$$= \sum_{i \in S^+} |\frac{d_i + \alpha_i}{p_i} - r| + \sum_{j \in S^-} |\frac{d_j - \beta_j}{p_j} - r| - \sum_{i \notin S^+ \cup S^-} |\frac{d_i}{p_i} - r|$$

$$= \sum_{i \in S^+} \frac{\alpha_i}{p_i} + \sum_{j \in S^-} \frac{\beta_j}{p_j} \geq 0$$

Thus the relation (A.20) can be obtained. That the MFM gives an optimal solution to the problem P6 can be proved in a similar way. □

Proof of Theorem 3.4 If an assignment $\{d_i \mid i \in S\}$ is optimal, then for all d_i, d_j with $i \neq j$ and $d_i > 0$, a transfer of a seat from any constituency i to j cannot improve the objective criterion, that is,

$$\frac{p_i^2}{d_i - 1} + \frac{p_j^2}{d_j + 1} \geq \frac{p_i^2}{d_i} + \frac{p_j^2}{d_j} \tag{A.22}$$

Namely,

$$\frac{p_i^2}{d_i(d_i - 1)} \geq \frac{p_j^2}{d_j(d_j + 1)}$$

$$\frac{p_i}{\sqrt{d_i(d_i - 1)}} \geq \frac{p_j}{\sqrt{d_j(d_j + 1)}} \tag{A.23}$$

Thus we obtain the following.

$$\max_{d_i \geq 0} \frac{p_i}{\sqrt{d_i(d_i + 1)}} \leq \min_{d_j > 0} \frac{p_j}{\sqrt{d_j(d_j - 1)}} \tag{A.24}$$

Conversely, suppose the assignment $\{d_i \mid i \in S\}$ satisfies the EPM max-min inequality (A.24). Let $\{d_i' \mid i \in S\}$ be another assignment different from $\{d_i \mid i \in S\}$. Defining sets of suffices S^+ and S^- as in (3.12) and denote $\{d_i' \mid i \in S\}$ as in (3.13), we need to show the following relation.

$$\sum_{i \in S} \frac{p_i^2}{d_i'} \geq \sum_{i \in S} \frac{p_i^2}{d_i} \tag{A.25}$$

Namely, we need to have

$$\sum_{i \in S^+} \frac{p_i^2}{d_i + \alpha_i} + \sum_{i \in S^-} \frac{p_j^2}{d_j - \beta_j} \geq \sum_{i \in S^+ \cup S^-} \frac{p_i^2}{d_i}$$

$$\sum_{i \in S^+} \frac{\alpha_i p_i^2}{d_i(d_i + \alpha_i)} \leq \sum_{j \in S^-} \frac{\beta_j p_j^2}{d_j(d_j - \beta_j)} \tag{A.26}$$

From (A.24) we have

$$\frac{p_j^2}{d_i(d_i + \alpha_i)} \leq \frac{p_i^2}{d_i(d_i + 1)} \leq \frac{p_j^2}{d_j(d_j - 1)} \leq \frac{p_j^2}{d_j(d_j - \beta_j)}$$

Thus the inequality (A.26) is obtained just by adding γ inequalities with the following form.

$$\frac{p_i^2}{d_i(d_i + \alpha_i)} \leq \frac{p_j^2}{d_j(d_j - \beta_j)} \qquad i \in S^+, j \in S^- \tag{A.27}$$

Thus the proof is complete. □

Proof of Theorem 3.5 If an assignment $\{d_i \mid i \in S\}$ is optimal, then for all d_i, d_j with $i \neq j$ and $d_i > 0$, a transfer of a seat from any constituency i to j cannot improve the objective criterion, that is,

$$\left|\frac{p_i}{d_i - 1} - s\right| + \left|\frac{p_j}{d_j + 1} - s\right| \geq \left|\frac{p_i}{d_i} - s\right| + \left|\frac{p_j}{d_j} - s\right|$$

Without loss of generality, we can assume that $\frac{p_j}{d_j} \geq s \geq \frac{p_i}{d_i}$, *i.e.*, constituency i is favored over j. Then we have

$$\frac{p_i}{d_i - 1} - s + s - \frac{p_j}{d_j + 1} \geq s - \frac{p_i}{d_i} + \frac{p_j}{d_j} - s$$

$$\frac{(d_i - \frac{1}{2})p_i}{d_i(d_i - 1)} \geq \frac{(d_j + \frac{1}{2})p_j}{d_j(d_j + 1)} \tag{A.28}$$

The above inequality implies that for any $i, j \in S$

$$\max_i \frac{(d_i + \frac{1}{2})p_i}{d_i(d_i + 1)} \leq \min_j \frac{(d_j - \frac{1}{2})p_j}{d_j(d_j - 1)} \tag{A.29}$$

which is the max-min inequality that characterizes the HMM method.

Conversely, suppose that $\{d_i \mid i \in S\}$ satisfies the EPM max-min inequality (A.29). Let $\{d'_i \mid i \in S\}$ be another assignment different from $\{d_i \mid i \in S\}$. Defining sets of suffices S^+ and S^- as in (3.12) and denote $\{d'_i \mid i \in S\}$ as in (3.13), we need to show the following relation.

$$\sum_{i \in S} \left|\frac{p'_i}{d_i} - s\right| \geq \sum_{i \in S} \left|\frac{p_i}{d_i} - s\right| \tag{A.30}$$

Namely, we need to show

$$\sum_{i \in S^+} \left|\frac{p_i}{d_i + \alpha_i} - s\right| + \sum_{i \in S^-} \left|\frac{p_j}{d_j - \beta_j} - s\right| \geq \sum_{i \in S^+ \cup S^-} \left|\frac{p_i}{d_i} - s\right|$$

$$\sum_{i \in S^+} \left|\frac{p_i}{d_i + \alpha_i} - s\right| + \sum_{i \in S^+} \left|\frac{p_i}{d_i} - s\right|$$

$$\leq \sum_{i \in S^-} \left|\frac{p_j}{d_j - \beta_j} - s\right| - \sum_{i \in S^-} \left|\frac{p_i}{d_i} - s\right| \tag{A.31}$$

Without loss of generality, we can assume that

$$\frac{p_i}{d_i + \alpha_i} \leq \frac{p_i}{d_i} \leq s \leq \frac{p_j}{d_j} \leq \frac{p_j}{d_j - \beta_j} \qquad i \in S^+, j \in S^- \tag{A.32}$$

Hence the above inequality (A.31) can be written as follows.

$$\sum_{i \in S^+} \left\{\frac{p_i}{d_i} - s - s - \frac{p_i}{d_i + \alpha_i}\right\} \leq \sum_{j \in S^-} \left\{\frac{p_j}{d_j - \beta_j} - s - s - \frac{p_j}{d_j}\right\} \tag{A.33}$$

Namely, we have

$$\sum_{i \in S^+} \frac{(d_i + \frac{\alpha_i}{2})p_i}{d_i(d_i + \alpha_i)} + s|S^-| \leq \sum_{j \in S^-} \frac{(d_j - \frac{\beta_j}{2})p_j}{d_j(d_j - \beta_j)} + s|S^+| \tag{A.34}$$

From the max-min inequality (A.29) for the HMM, we have for $\alpha_i \geq 1$ and $\beta_j \geq 1$

$$\frac{(d_i + \frac{\alpha_i}{2})p_i}{d_i(d_i + \alpha_i)} \leq \frac{(d_i + \frac{1}{2})p_i}{d_i(d_i + 1)} \leq \frac{(d_j - \frac{1}{2})p_j}{d_j(d_j - 1)} \leq \frac{(d_j - \frac{\beta_j}{2})p_j}{d_j(d_j - \beta_j)}$$

Hence we also have the following relation.

$$\frac{(d_i + \frac{\alpha_i}{2})p_i}{d_i(d_i + \alpha_i)} \leq s \leq \frac{(d_j - \frac{\beta_j}{2})p_j}{d_j(d_j - \beta_j)} \qquad i \in S^+, j \in S^- \tag{A.35}$$

Therefore, by adding $|S^+|$ inequalities from the left and $|S^-|$ inequalities from the right, we obtain the relation (A.34). □

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