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A LAGRANGEAN APPROACH TO THE FACILITY LOCATION PROBLEM WITH CONCAVE COSTS

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Abstract We consider the concave cost capacitated facility location problem, and develop a composite algorithm of lower and upper bounding procedures. Computational results for several instances with up to 100 customers and 25 candidate facility locations are also presented. Our numerical experiments show that the proposed algorithm generates good solutions. The gaps between upper and lower bounds are within 1 percent for all test problems.

1 Introduction

The facility location problem, i.e., finding the optimal location to be established among a given set of candidate sites in order to satisfy known demands specified at a given set of customer locations, has been a popular research area for mathematicians and management scientists. The objective of the standard facility location problem is to minimize the total cost consisting of fixed costs for establishing facilities and linear production and transportation costs. Our model is a variant of the facility location problem called *the capacitated concave cost facility location problem* (CCFLP) in which the costs are represented by nonlinear concave functions. The nonlinear costs usually arise when pull type of inventory policies are used in the operational level.

The assumptions of our model are:

1. The numbers of candidate facilities and customer locations are m and n, respectively.

2. Candidate facility locations are indexed by $1, \cdots, i, \cdots, m$.

3. Customer locations are indexed by $1, \cdots, j, \cdots, n$.

4. The production size in each facility, say i, is limited by a prescribed amount C_i. In the sequel, we call it *the capacity* of the facility.

5. The production and transportation costs are non-decreasing functions of the production and transportation sizes, respectively. We denote the production cost function of facility i by f_i and the transportation cost function between facility i and customer j by g_{ij}.

6. The marginal production and transportation costs are non-increasing functions of the production and transportation sizes, respectively. This leads that functions f_i, $i = 1, \cdots, m$ and g_{ij}, $i = 1, \cdots, m$ and $j = 1, \cdots, n$ are concave. Concavities reflect the economies of scale of the production and transportation costs.

7. The production cost is equal to 0 if no amount is produced, i.e., $f_i(0) = 0$ for $i =$

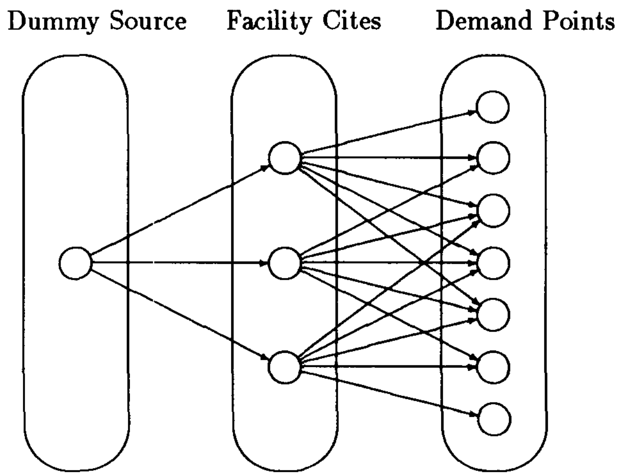

Figure 1: Sample Network of the Concave Cost Facility Location Problem

$1, \cdots, m$. The transportation cost is equal to 0 if no amount is shipped, i.e., $g_{ij}(0) = 0$ for $i = 1, \cdots, m$ and $j = 1, \cdots, n$.

8. The demand of each customer, say j, is a known value and denoted by d_j.

Several variants of the facility location problems have been extensively analyzed by many authors, but very few researches have been done for the CCFLP.

Eilon et al. [3] considered a continuous version of the CCFLP, in which facilities can be located in any place in the plane. Soland [11] developed a branch and bound algorithm for the CCFLP. Recently, Erickson et al. [4] proposed a dynamic programming algorithm for the general concave network flow problems. Of course their algorithm can be used for solving the CCFLP.

We have several motivations of this research. One motivation of this research is the sparsity of the previous research for the CCFLP in contrast to its practical importance. The CCFLP is closely related to the fixed charge facility location problem in which we want to minimize the sum of the fixed costs and linear variable costs. The fixed charge facility location problem and its variants are extensively analyzed in literature, but the linear and fixed charge cost function cannot explain the economy-of-scale that we encounter in several practical situations.

To explain the second motivation, we should analyze the characteristic of the algorithms of the previous works.

The branch and bound method of Soland is particularly effective if only small fraction of the arc costs are nonlinear concave function. If many arcs are strictly concave, the algorithm may run in exponential time in the worst case. The remarkable characteristic of Soland's algorithm is that the presence of the capacity constraints does not increase the difficulty of solving the problem. As shown in the numerical experiments in [11], the computational time is decreasing as the capacity constraints become tighter.

In contrast to Soland's algorithm, the dynamic programming algorithm developed by Erickson et al. is effective if many of the arcs are strictly concave. Their algorithm runs in polynomial time if the underling network has a special structure which they call 'k-planar', i.e., planar graph with all demand nodes lying on the boundary of k faces. Unfortunately, the network of the CCFLP does not possess such a special structure (see Figure 1). Further the presence of the capacity of arcs makes the algorithm run in exponential time. Thus we can observe a lack of the algorithm that works well when

1. many arcs are strictly concave,
2. the underlying network does not have a special structure, and
3. there may exist capacity constraints, but not tight.

This fact means that we need a tough algorithm that works relatively well for any network with strictly concave costs and weak capacity constraints.

Another drawback of the previous works is that both of the previous algorithms may run in exponential time. These observations indicate that we need a tough algorithm that runs in low-order polynomial time, and gives relatively good solutions.

Based on the above motivations, we propose a composite algorithm of the lower and upper bounding procedures that uses the Lagrangean dual solution method and produces tight lower and upper bounds.

The remainder of this paper is organized as follows. In section 2 we give two formulations of the CCFLP. In section 3 we describe a lower bounding procedure. In section 4 we describe an upper bounding procedure. Section 5 contains the numerical experiments and the final section gives concluding remarks.

2 Problem Formulations

In this section, we describe two formulations of the CCFLP. The first formulation is derived from a naive generalization of the standard facility location problem. Similar formulation has been given by Soland [11].

(CCFLP : Formulation 1)

$$(1) \qquad Z^* = \min \quad \sum_{i=1}^{m} f_i(\sum_{j=1}^{n} x_{ij}) + \sum_{i=1}^{m}\sum_{j=1}^{n} g_{ij}(x_{ij})$$

subject to

$$(2) \qquad \sum_{i=1}^{m} x_{ij} = d_j \quad j = 1, \cdots, n,$$

$$(3) \qquad \sum_{j=1}^{n} x_{ij} \leq C_i \quad i = 1, \cdots, m,$$

$$(4) \qquad 0 \leq x_{ij} \leq \Gamma_{ij} \quad i = 1, \cdots, m, j = 1, \cdots, n,$$

where

$m =$ the number of candidate facilities,

$n =$ the number of customers,

$x_{ij} =$ the amount shipped from facility i to customer j,

$f_i =$ the concave function that represents the production cost in facility i,

$g_{ij} =$ the concave function that represents the transportation cost from facility i to customer j,

$C_i =$ the upper bound of the amount produced in facility i, i.e., the capacity of facility i,

$d_j =$ the demand of each customer,

$\Gamma_{ij} =$ $\min\{C_i, d_j\}$ that we call the reduced capacity on arc (i,j).

In the formulation above, the objective function (1) minimizes the sum of the production costs and the transportation costs. Constraints (2) represent that each customer's demand must be satisfied. Constraints (3) represent that the total amount shipped from facility i must be less than or equal to the capacity of the facility. Constraints (4) are non-negativity and upper bound constraints of variable x_{ij}.

Next, we describe the second formulation. We introduce new variables y_{ij} for $i = 1,\cdots,m, j = 1,\cdots,n$. Variable y_{ij} represents the amount produced in facility i to satisfy the demand of customer j. It may be seen that these variables are not necessary to formulate the CCFLP, since y_{ij} must be equal to x_{ij} for $i = 1,\cdots,m, j = 1,\cdots,n$. But this formulation is used to derive the tight lower bounds that will be described in the next section. Similar formulation can be found in Magnanti and Wong [6] for the reduction of the facility location problem to the network design problem.

(CCFLP : Formulation 2)

$$(5) \qquad Z^* = \min \quad \sum_{i=1}^{m} f_i\left(\sum_{j=1}^{n} y_{ij}\right) + \sum_{i=1}^{m}\sum_{j=1}^{n} g_{ij}(x_{ij})$$

subject to

$$(6) \qquad \sum_{i=1}^{m} x_{ij} = d_j \quad j = 1,\cdots,n,$$

$$(7) \qquad x_{ij} - y_{ij} = 0 \quad i = 1,\cdots,m, j = 1,\cdots,n,$$

$$(8) \qquad \sum_{j=1}^{n} y_{ij} \le C_i \quad i = 1,\cdots,m,$$

$$(9) \qquad 0 \le x_{ij} \le \Gamma_{ij} \quad i = 1,\cdots,m, j = 1,\cdots,n,$$

$$(10) \qquad 0 \le y_{ij} \le \Gamma_{ij} \quad i = 1,\cdots,m, j = 1,\cdots,n.$$

In the formulation above, the objective function (5) minimizes the sum of the production costs and the shipping costs. Constraints (6) define that each customer's demand must be satisfied. Constraints (7) represent that the amount shipped from facility i to customer j must be equal to the amount produced in facility i for customer j. Constraints (9) and (10) are non-negativity and upper bound constraints of variables x_{ij} and y_{ij} respectively.

3 Lower Bounding Procedure for the CCFLP

In this section, we describe a Lagrangean relaxation formulation of the CCFLP and a solution method of the Lagrangean subproblem for any given Lagrangean multipliers. Then, we describe how to adjust multipliers. We first adjust the initial Lagrangean multipliers by solving the transportation problem, and then use the subgradient optimization method to adjust the multipliers.

3.1 Formulation of the Lagrangean Relaxation Problem

We dualize (6) using multiplier v_j for all $j = 1, \cdots, n$ and (7) using multiplier w_{ij} for all $i = 1, \cdots, m$ and $j = 1, \cdots, n$. Given a vector of Lagrangean multipliers $(v, w) = ([v_i], [w_{ij}])$, we can get the following Lagrangean relaxation problem.

(LRP : Lagrangean Relaxation Problem)

$$(11)\quad Z(v,w) = \min \sum_{i=1}^{m} \{f_i(\sum_{j=1}^{n} y_{ij}) - \sum_{j=1}^{n} w_{ij}y_{ij}\} + \sum_{i=1}^{m}\sum_{j=1}^{n} \{g_{ij}(x_{ij}) + (w_{ij} - v_j)x_{ij}\} + \sum_{j=1}^{n} d_j v_j$$

subject to (8),(9),(10).

The optimal value of the Lagrangean relaxation problem $Z(v, w)$ is a valid lower bound on the optimal value of the CCFLP.

3.2 Solving the Lagrangean Subproblems

The Lagrangean relaxation problem (LRP) can be decomposed into two independent problems; one is to determine variables x_{ij} and another is to determine variables y_{ij}.

The problem for determining variable x_{ij} can be further decomposed into $m \times n$ subproblems each of which determines the optimal shipment size between a facility and a customer. More precisely, for all $i = 1, \cdots, m, j = 1, \cdots, n$ and given Lagrangean multiplier vectors $v = [v_j]$ and $w = [w_{ij}]$, we define $LSX_{ij}(v, w)$ (Lagrangean subproblem corresponding to variable x_{ij}) as follows.

$(LSX_{ij}(v,w)$)

$$(12)\qquad Z^x_{ij}(v,w) = \min \quad g_{ij}(x_{ij}) + (w_{ij} - v_j)x_{ij}$$

subject to (9).

The problem to determine variable y_{ij} can be decomposed into m subproblems each of which gives the optimal production size in a facility. More precisely, for all $i = 1, \cdots, m$ and given Lagrangean multiplier vector $w = [w_{ij}]$, we define $LSY_i(w)$ (Lagrangean subproblem corresponding to variable y_{ij}) as follows.

$(LSY_i(w)$)

$$(13)\qquad Z^y_i(w) = \min \quad f_i(\sum_{j=1}^{n} y_{ij}) - \sum_{j=1}^{n} w_{ij}y_{ij}$$

subject to (8),(10).

Using (12),(13), we can represent the optimal value of the Lagrangean relaxation problem $Z(v, w)$ as follows:

$$(14)\qquad Z(v,w) = \sum_{i=1}^{m} Z^y_i(w) + \sum_{i=1}^{m}\sum_{j=1}^{n} Z^x_{ij}(v,w) + \sum_{j=1}^{n} d_j v_j.$$

We now describe how to solve the subproblem $LSX_{ij}(v, w)$. Since the function $g_{ij}(x_{ij})$ is concave, there exists an optimal solution on the extreme points of the polyhedron defined by the constraints (see [12]). By observing that the constraints of the subproblem

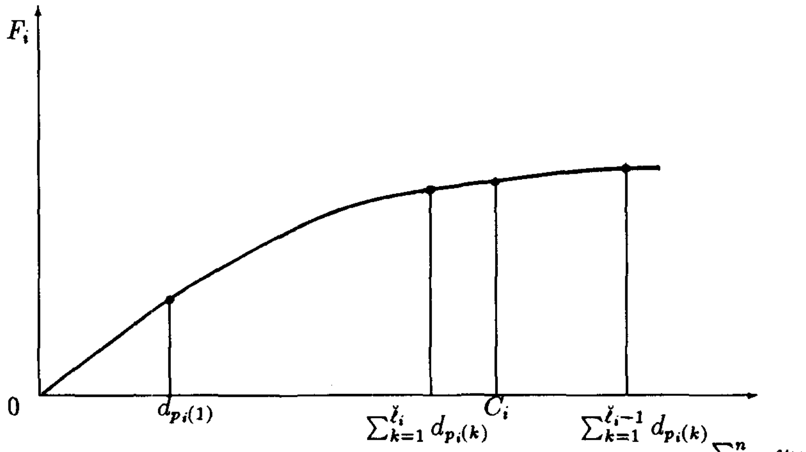

Figure 2: Solving Lagrangean Subproblem $LSY_i(w)$

$LSX_{ij}(v,w)$ are the upper and lower bounds of variable x_{ij} only, the decision variable x_{ij} can be determined as follows.

$$(15) \qquad x_{ij} = \begin{cases} \Gamma_{ij} & \text{if } g_{ij}(\Gamma_{ij}) + (w_{ij} - v_j)\Gamma_{ij} < 0 \\ 0 & \text{otherwise} \end{cases} \quad i = 1, \cdots, m, j = 1, \cdots, n.$$

Next, we describe how to solve $LSY_i(w)$ corresponding to variable y_{ij}. Since the function f_i is also concave, we may restrict our attention to the extreme points only. The extreme points are represented as dots in Figure 2. Using this property, we can get the following procedure for solving $LSY_i(w)$.

Consider a facility $i(\in \{1, 2, \cdots, m\})$ corresponding to $LSY_i(w)$. First, find a permutation p_i that satisfies

$$(16) \qquad w_{ip_i(1)} \geq w_{ip_i(2)} \geq \cdots \geq w_{ip_i(n)}$$

and let

$$(17) \qquad F_{i\ell} = f_i(\sum_{k=1}^{\ell} \Gamma_{ip_i(k)}) - \sum_{k=1}^{\ell} \Gamma_{ip_i(k)} w_{ip_i(k)} \qquad \text{for all} \quad 1 \leq \ell \leq \check{\ell}_i$$

and

$$(18) \qquad F_{i\check{\ell}_i+1} = f_i(C_i) - \sum_{k=1}^{\check{\ell}_i} \Gamma_{ip_i(k)} w_{ip_i(k)} - (C_i - \sum_{k=1}^{\check{\ell}_i} \Gamma_{ip_i(k)}) w_{ip_i(\check{\ell}_i+1)},$$

where $\check{\ell}_i$ is the maximum index ℓ that satisfies

$$(19) \qquad \sum_{k=1}^{\ell} \Gamma_{ip_i(k)} < C_i.$$

Then, find ℓ^* such that

$$(20) \qquad F_{i\ell^*} = \min_{1 \leq \ell \leq \check{\ell}_i + 1} F_{i\ell}.$$

If $F_{i\ell^*} \leq 0$, then the optimal solution is $y_{ij} = 0$ for $j = 1, \cdots, n$, i.e., all variables corresponding to subproblem $LSY_i(w)$ are set to 0. Otherwise, we determine the variable y_{ij} as follows (see Figure 2).
If $\ell^* \neq \check{\ell}_i + 1$,

$$y_{ip_i(k)} = \begin{cases} \Gamma_{ip_i(k)} & k = 1, \cdots, \ell^*, \\ 0 & k = \ell^* + 1, \cdots, n. \end{cases} \tag{21}$$

If $\ell^* = \check{\ell}_i + 1$,

$$y_{ip_i(k)} = \begin{cases} \Gamma_{ip_i(k)} & k = 1, \cdots, \check{\ell}_i, \\ C_i - \sum_{k=1}^{\check{\ell}_i} \Gamma_{ip_i(k)} & k = \check{\ell}_i + 1, \\ 0 & k = \check{\ell}_i + 2, \cdots, n. \end{cases} \tag{22}$$

The computational requirement of the procedure above is very cheap. If we assume that each elementary operation on real numbers requires one unit time and concave functions f_i and g_{ij} are evaluated in one unit time, overall complexity for solving the Lagrangean problem is $O(mn \log n)$, since the subproblem $LSX_{ij}(v, w)$ requires $O(1)$ time for each $i = 1, \cdots, m$ and $j = 1, \cdots, n$ and the subproblem $LSY_i(w)$ requires $O(n \log n)$ time for each $i = 1, \cdots, m$.

3.3 Dual Adjustment Procedure

By duality theory [5], the Lagrangean objective function value $Z(v, w)$ gives a valid lower bound of the optimal solution value Z^* for any real vectors $v = [v_j]$ and $w = [w_{ij}]$.

Therefore, we have to solve the following Lagrangean dual problem to find the best lower bound.

(LDP: Lagrangean Dual Problem)

$$LB = \max_{v,w} \quad Z(v, w). \tag{23}$$

Instead of solving the Lagrangean dual problem optimally, we solve it approximately by adjusting multipliers and using subsequently a subgradient optimization procedure to improve the lower bound. This subsection gives the initial multiplier adjustment procedure and the subgradient optimization procedure for the CCFLP.

3.3.1 Initial Adjustment of Lagrangean Multipliers

Though we may start the subgradient procedure with any Lagrangean multipliers, we set the initial Lagrangean multipliers heuristically for fast and steady convergence.

We use linear and lower approximation of the concave functions to derive initial adjustment procedure. Let $\bar{f}_i$ and $\bar{g}_{ij}$ be the coefficients of the linear-lower approximation of $f_i(\sum_{j=1}^n y_{ij})$ and $g_{ij}(x_{ij})$, respectively, which are defined as follows:

$$\bar{f}_i = \frac{f_i(\Gamma_{ij})}{\Gamma_{ij}} \tag{24}$$

and

$$\bar{g}_{ij} = \frac{g_{ij}(\Gamma_{ij})}{\Gamma_{ij}}. \tag{25}$$

Using the linear and lower approximation above, we can get the following approximation of the Lagrangean relaxation problem.

$$\min \quad \sum_{i=1}^m \sum_{j=1}^n \bar{f}_i y_{ij} + \sum_{i=1}^m \sum_{j=1}^n \bar{g}_{ij} x_{ij} \tag{26}$$

subject to (6),(7), (8),(9),(10).

Since $x_{ij} = y_{ij}$ for $i = 1, \cdots, m$, $j = 1, \cdots, n$, the problem above can be reduced to the following transportation problem :

$$\text{(27)} \qquad \min \quad \sum_{i=1}^{m}\sum_{j=1}^{n}(\bar{f}_i + \bar{g}_{ij})\xi_{ij}$$

subject to

$$\text{(28)} \qquad \sum_{i=1}^{m}\xi_{ij} = d_j \quad j = 1, \cdots, n,$$

$$\text{(29)} \qquad \sum_{j=1}^{n}\xi_{ij} \le C_i \quad i = 1, \cdots, m,$$

$$\text{(30)} \qquad 0 \le \xi_{ij} \le \Gamma_{ij} \quad i = 1, \cdots, m, j = 1, \cdots, n.$$

The initial Lagrangean multipliers $\hat{v}_j$ for $j = 1, \cdots, n$ are set to the optimal dual variables of the corresponding constraints (28) and multipliers $\hat{w}_{ij}$ for $i = 1, \cdots, m$, $j = 1, \cdots, n$ are set as follows :

$$\text{(31)} \qquad \hat{w}_{ij} = \hat{v}_j - \bar{g}_{ij} \quad i = 1, \cdots, m, j = 1, \cdots, n.$$

Since an optimal solution of the transportation problem can be obtained in polynomial time by the primal-dual algorithm, and multipliers $\hat{w}_{ij}$ can be computed in $O(mn)$, the overall complexity is polynomial time order.

For the uncapacitated case, i.e., $C_i \ge \sum_{j=1}^{n} d_j$ for all $i = 1, \cdots, m$, we can determine $\hat{v}_j$ more easily.

$$\text{(32)} \qquad \hat{v}_j = \min_i \quad (\bar{f}_i + \bar{g}_{ij}) \quad j = 1, \cdots, n.$$

In this case, computational complexity can be reduced to $O(mn)$.

The validity of the initial lower bound is shown in the following theorem.

Theorem 1 *By setting the Lagrangean multipliers equal to $\hat{v}$ and $\hat{w}$ above, we can get the initial lower bound*

$$\text{(33)} \qquad LB = \sum_{j=1}^{n} d_j \hat{v}_j.$$

Furthermore, the initial primal solution of the Lagrangean subproblem is $x_{ij} = 0$ and $y_{ij} = 0$ for $i = 1, \cdots, m$, $j = 1, \cdots, n$.

Proof: Consider the linear and lower approximation of the LRP:

$$\text{(34)} \qquad \min \quad \sum_{i=1}^{m}\sum_{j=1}^{n}(\bar{f}_i - \hat{w}_{ij})y_{ij} + \sum_{i=1}^{m}\sum_{j=1}^{n}(\bar{g}_{ij} + \hat{w}_{ij} - \hat{v}_j)x_{ij} + \sum_{j=1}^{n} d_j \ v_j$$

subject to (8),(9),(10).

Since $\hat{v}_j \le \bar{f}_i + \bar{g}_{ij}$, we get $\hat{w}_{ij} \le \bar{f}_i$ for $i = 1, \cdots, m$, $j = 1 \cdots, n$. From the definition of $\hat{w}$, we get $\bar{g}_{ij} \ge \hat{v}_j - \hat{w}_{ij}$ for $i = 1, \cdots, m, j = 1, \cdots, n$. Therefore, we can say that the optimal solution of the LRP is $x_{ij} = 0$ and $y_{ij} = 0$ for $i = 1, \cdots, m$, $j = 1, \cdots, n$ and this gives the initial lower bound $\sum_{j=1}^{n} d_j \hat{v}_j$. ∎

3.3.2 Subgradient Optimization

We apply a subgradient optimization procedure to increase the value of the Lagrangean lower bound. Let $\hat{x}_{ij}, \hat{y}_{ij}$ be the solution derived from the Lagrangean relaxation problem and let

$$\lambda_j = \sum_{i=1}^{m} \hat{x}_{ij} - d_j \quad j = 1, \cdots, n \tag{35}$$

and

$$\mu_{ij} = \hat{x}_{ij} - \hat{y}_{ij} \quad i = 1, \cdots, m, j = 1, \cdots, n. \tag{36}$$

The vector $(\lambda, \mu) = ([\lambda_j], [\mu_{ij}])$ defines a subgradient for the Lagrangean function $Z(v, w)$ and specifies a candidate direction for changing the current values. The multipliers are adjusted as

$$v_j := v_j + \theta\lambda_j \quad j = 1, \cdots, n \tag{37}$$

and

$$w_{ij} := w_{ij} + \theta\mu_{ij} \quad i = 1, \cdots, m, j = 1, \cdots, n, \tag{38}$$

where θ is the step-size given by

$$\theta = s\frac{\hat{Z} - Z(v, w)}{\sum_{i=1}^{m}(\lambda_i)^2 + \sum_{i=1}^{m}\sum_{j=1}^{n}(\mu_{ij})^2}. \tag{39}$$

$\hat{Z}$ is an upper bound on the solution value and s is a prespecified step-size in the range $0 < s < 2$.

4 Upper Bounding Procedure for the CCFLP

In this section, we give a simple approximate algorithm for the CCFLP. After determining the open facility using the information of the Lagrangean solution described in the previous section, we use the transportation problem as a subroutine to get an approximate solution. If the transportation problem is infeasible, we add the facilities to be opened using the greedy criteria. This criteria also uses the information of the Lagrangean solution. This procedure should be periodically applied during the subgradient optimization procedure. In the numerical experiments in section 5, we apply the upper bounding procedure once every 5 iterations.

(Upper Bounding Procedure)

Step 1 Let the optimal solution of the Lagrangean subproblem be $\hat{x}_{ij}$ and $\hat{y}_{ij}$ for $i = 1, \cdots, m, j = 1, \cdots, n$. Calculate the reduced fixed charge $\hat{f}_i$ for each facility $i = 1, \cdots m$ as follows :

$$\hat{f}_i = f_i(\sum_{j=1}^{n} \hat{y}_{ij}) - \sum_{j=1}^{n} w_{ij}\hat{x}_{ij}. \tag{40}$$

Step 2 Determine the set of opened facilities I as follows:

$$I = \{i \mid \hat{f}_i < 0, i = 1, \cdots, m\}. \tag{41}$$

Step 3 If $I \neq \emptyset$, then goto Step 5.

Step 4 Determine i^* (cheapest facility) as follows:

$$f_{i^*} = \min_{i \in \{1,\cdots,m\}-I} \hat{f}_i \tag{42}$$

and add i^* to I.

Step 5 Solve the following transportation problem:

$$\min \sum_{i \in I} \sum_{j=1}^{n} g_{ij}(\xi_{ij}) \tag{43}$$

subject to

$$\sum_{i \in I} \xi_{ij} = d_j \quad j = 1, \cdots, n, \tag{44}$$

$$\sum_{j=1}^{n} \xi_{ij} \leq C_i \quad i \in I, \tag{45}$$

$$0 \leq \xi_{ij} \leq \Gamma_{ij} \quad i \in I, j = 1, \cdots, n. \tag{46}$$

If the solution is infeasible, then goto Step 4. Otherwise we get a feasible solution by setting $x_{ij} = \xi^*_{ij}$ for all $i \in I$ and $j = 1, \cdots, n$, where ξ^*_{ij} is an optimal solution of the transportation problem above.

Most time consuming part of the upper bounding procedure above is Step 5. Since the transportation problem can be solved in polynomial time and Step 5 is used at most m times, the upper bounding procedure described above runs in polynomial time order.

For uncapacitated case, we can replace Step 5 in the algorithm by the following simple procedure:

Step 5' The demands of customer j are shipped by facility $\hat{i}_j$ where

$$\hat{i}_j = \min_{i \in I} \quad g_{ij}(d_j) \quad j = 1, \cdots, n. \tag{47}$$

For uncapacitated case, only $O(mn)$ steps are required.

5 Numerical Experiments

We coded our procedures in BASIC and ran them on a PC9801 VM2 (NEC) computer. Test problems have been randomly generated so that all nodes are distributed randomly on 100×100 grids. We set the data of the random instances so that both the branch and bound algorithm proposed of Soland [11] and the dynamic programming algorithm of Erickson et al. [4] do not work well. Demands are identical at all customers; we set $d = 1$. The capacity of each facility is set to infinity, i.e., all the test problems are uncapacitated.

The production costs are generated as follows:

$$f_i(\sum_{j=1}^{n} y_{ij}) = F \times (\sum_{j=1}^{n} y_{ij})^{\alpha}. \tag{48}$$

We set F=1000 and α=0.5 in our experiments.

Table 1: Summary Statics of the CCFLP

Problem Size	No. of Iter.	Gap	CPU Time
(5, 20)	36	.34	2:07
(10, 40)	26	.96	5:37
(15, 60)	53	.76	25:36
(20, 80)	56	.33	48:29
(25, 100)	96	.98	2:09:50

Legend: Problem Size = (No. of Facilities, No. of Customers), No. of Iter. = No. of subgradient iterations, Gap = $\frac{\text{Best upper bound} - \text{Best lower bound}}{\text{Best lower bound}} \times 100$, CPU time = (Hours: Minute : Seconds) on PC9801 VM2 (10 MHz), Basic Compiler. The computational time is about 3600 times longer than IBM 3090 VS-FORTRAN when measured by addition only.

The transportation costs are generated as follows:

$$g_{ij}(x_{ij}) = \text{(Euclidean distance between nodes } i \text{ and } j) \times (x_{ij})^{\beta} \tag{49}$$

We set $\beta = 0.5$ in all instances.

Table 1 gives the results with various problem sizes.

Our algorithm performs well for randomly generated test problems with up to 100 customers and 25 candidate facilities. For all the computational runs, the gaps between the lower and upper bounds are within 1 percent.

6 Concluding Remarks

We developed a composite algorithm of lower and upper bounding procedures. We reported on computational results for several instances with up to 100 customers and 25 candidate facility location sites. Our numerical experiments showed that our algorithm generates good solutions within reasonable computational time. For all problems, the iterations of our algorithm are less than 100, and one iteration requires $O(mn \log n)$ time. Since the growth of the number of iterations is relatively slow, we can predict that the computational time does not grow so rapidly as the problem size increases. This indicates that our composite algorithm of upper and lower bounding procedures can be used for large instances that we encounter the practical situations.

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