Journal of the Operations Research
Society of Japan
Vol. 34, No. 2, June 1991

TWO NONCONVEX MINIMIZATION APPROACHES FOR THE PROBLEM OF DETERMINING AN ECONOMIC ORDERING POLICY FOR JOINTLY REPLENISHED ITEMS

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(Received December 7, 1988; Final March 15, 1991)

Abstract This paper describes nonconvex minimization approaches for the problem of determining an economic ordering policy for jointly replenished items. After reducing the object function in the problem to a single-variable nonconvex function, two methods are proposed for solving the problem. One is the Successive Underestimation Method which utilizes effectively the properties of the function $A/t + Bt$. Another is a modification of the Relief Indicator Method. Different from other heuristic methods ever proposed for the problem, the methods developed in this paper can obtain a global approximate solution within any prescribed error bound as we like. Results of computational tests show that both of the proposed methods perform equally well for many test problems.

1. Introduction

In this paper we will develop two methods for the problem of determining economic ordering frequencies of jointly replenished items by applying nonconvex minimization techniques. The problem is encountered when an unpackaged product is packaged into several sizes right after manufacture. These items, each representing a particular type of container, are said to be jointly replenished. The problem has been a matter of considerable interest for researchers over the last 20 years; see Doll and Whybark [1], Goyal [2],[3],[4],[5], Goyal and Belton [6], Shu [12], and Silver [14]. It has a close connection with the problem of determining economic ordering frequencies of items procured from a single supplier. For details of the latter problem refer to Graves [7], Muramatu and Yanai [8], Roundy [10], Starr and Miller [15], and Schwartz [11].

The problem we will treat in this paper is originally investigated by Goyal [3],[4]. It is to determin ordering frequencies minimizing the associated annual cost which consists of:

(a) cost of manufacturing set-ups,
(b) cost of packaging set-ups, and
(c) cost of holding stock for packaged items.

Although a number of methods have been developed for the problem, almost all the approaches are heuristic and only local optimum solutions are obtained. Moreover, good and sensible error bounds are not evaluated. Only one method that guarantees optimality would be that of Goyal[4]. However, his method is rather enumerative. It requires evaluating all the candidate solutions between the lower and the upper bounds of decision variables which will be defined in the following section. Therefore the number of solutions to be evaluated will be enormous as the size of the problem n, which is the number of container types, becomes large.

The purpose of this paper is to develop such methods for the problem as guarantee the error of the solution obtained to be less than a prescribed tolerance $\varepsilon > 0$. In order to accomplish this, we first reduce the original problem to a single-variable nonconvex minimization problem and then apply two methods for the reduced problem. One is a successive underestimation method and the other is the Relief Indicator Method which was developed recently by H.Tuy and one of the authors [16]. The latter was originally a general method for the global minimization of a multivariable constrained nonconvex function. In this paper we will adopt it with some modification to our problem. Numerical experiments show that both of our methods perform well with reference to our purpose.

In both methods, especially in the successive underestimation method, a function of the form $f(t) = A/t + Bt$ plays a central role as an appropriate underestimator for the nonconvex function under consideration. Such a function naturally has applications in other fields, but we will describe the underestimating techniques with reference to obtaining an economic ordering policy for jointly replenished products.

This paper consists of 6 sections. In section 2, we will give the formuation of the original problem and the conversion to a single-variable nonconvex minimization problem. In section 3 the successive underestimation method will be developed. Section 4 is for the application of the relief indicator method. Section 5 is for the numerical experiments. Section 6 contains an extension and conclusions.

2. Problem formulation and its reduction to a single-variable nonconvex minimization

In this section we will introduce a mathematical formulation of the problem. We use the following assumptions and notations.

Assumptions:

(1) rate of demand for each packaged item is constant with time,
(2) no shortages are allowed,
(3) time horizon is infinite, and
(4) rate of packaging is infinite.

Notations:

C : average annual cost,
S : set-up cost of each manufacturing run,
n : number of container types,
t : time interval between manufacturing set-ups (continuous variable),

and for the i-th item

D_i : annual demand,
h_i : holding cost per unit per year,
S_i : set-up cost of packaging,
k_i : ratio of the time interval between packaging set-ups to t (integer variable).

All these variables and constants are positive.

Now the minimization problem of the average annual cost C can be expressed as follows (see Goyal[4]):

$$\text{(2.1)} \qquad minimize \;\; C(t,k) := (S + \sum_{i=1}^{n} S_i/k_i)/t + \sum_{i=1}^{n} h_i D_i k_i t/2$$

$$\text{s.t. } t \geq 0, \; k_i \geq 1 \; \forall \, i \, .$$

We will introduce a single-variable function $F(t)$ defined by

$$\text{(2.2)} \qquad F(t) := \min \{ \, C(t,k) \mid k_i : \text{positive integer}, \; i = 1,2,\ldots,n \, \} .$$

Generally, $F(t)$ is continuous but not convex and it may have many local minima (see Fig.1). So our problem will be reduced to the problem of seeking a global minimum of the

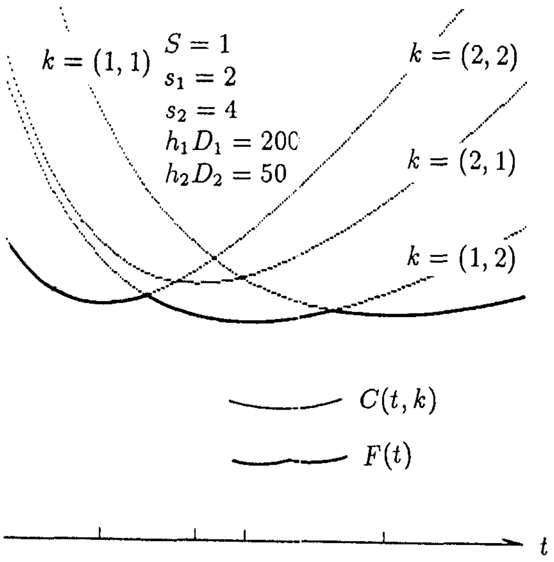

Fig.1. $F(t)$ and local minima for an example of two items.

nonconvex function $F(t)$ over $t \geq 0$. The upper and lower bounds of t and the upper bound of k_i can be estimated respectively as follows (see Appendix):

$$
\begin{aligned}
t_U &= (2(S + \sum_{i=1}^{n} S_i/k_i)/\sum_{i=1}^{n} h_i D_i)^{1/2}, \\
t_L &= S/(F(t_U) - \sum_{i=1}^{n}(2S_i h_i D_i)^{1/2}), \\
k_{U_i} &= [(2S_i/(h_i D_i))^{1/2}/t_L] - 1 \quad for\ i = 1, 2, \ldots, n.
\end{aligned}
$$

3. Successive Underestimation Method

Let us recall our problem:

$$(P) \quad minimize\ F(t), \quad s.t. \quad t_L \leq t \leq t_U ,$$

where t_L and t_U are positive numbers and

$$(3.1) \qquad F(t) = S/t + \sum_{i=1}^{n} \min\{S_i/(k_i/t) + h_i D_i k_i t/2 \mid 1 \leq k_i \leq k_{U_i},\ k_i : \text{integer}\}.$$

In this section we present a successive underestimation method suitably applied to the problem (P). A general scheme of the successive underestimation method is the following. First, we replace the objective function F in the problem (P) by another function F_j which underestimates F on $[t_L, t_U]$, i.e.,

$$(3.2) \qquad F_j(t) \leq F(t) \quad \forall t \in [t_L, t_U],$$

so that it may be easy to handle its minimizer. Instead of solving (P) we solve the relaxed problem:

$$(p_j) \quad minimize\ F_j(t), \quad s.t. \quad t \in [t_L, t_U].$$

Let t_j be a solution of (p_j). From (3.2) it is obvious that if $F_j(t_j) \geq F(t_j)$, then t_j is optimal for the problem (P). Otherwise, i.e., if $F_j(t_j) < F(t_j)$, we construct a new underestimator F_{j+1} of F which is better than F_j, i.e.,

$$\begin{aligned} F_j(t) &\leq F_{j+1}(t) \leq F(t) \quad \forall t \in [t_L, t_U], \\ F_j(t_j) &< F_{j+1}(t_j) = F(t_j), \end{aligned}$$

and enter a new iteration for solving the new relaxed problem:

$$(p_{j+1}) \quad minimize\ F_{j+1}(t), \quad s.t. \quad t \in [t_L, t_U].$$

In the case where F is Lipschitzian on $[t_L, t_U]$ and its Lipschitz constant is effectively estimated, the underestimators F_j $(j = 1, 2, ...)$ will be taken in a class of piecewise linear functions (see Piavskii [9], Shubert [13]).

In our case, an upper bound of the Lipschitz constant of F on $[t_L, t_U]$ may be computed rather easily, but it will be very large (perhaps very far from the actual Lipschitz constant of F on $[t_L, t_U]$). This means that the solution method in Shubert [13] (or in Piavskii [9]) can not be effectively applied to the problem (P). We will provide another class of underestimators, based on which the successive underestimation scheme will be effectively implemented.

For the vector $k = (k_1, k_2, \ldots, k_n)$ of integer variables let us denote

$$\begin{aligned} A(k) &= S + \sum_{i=1}^{n} S_i/k_i, \\ B(k) &= \sum_{i=1}^{n} h_i D_i k_i/2. \end{aligned}$$

From (3.1) we have

$$F(t) = \min_k \{ A(k)/t + B(k)t \}, \tag{3.3}$$

Thus, we see $F(t)$ is a pointwise minimum of a finite family of functions of the form:

$$A/t + Bt. \tag{3.4}$$

Let us describe some basic properties of functions of the form (3.4) which will lead us to a suitable underestimator for the problem (P).

Proposition 3.1 Let $[t_l, t_u]$ be a segment on $t > 0$. If $A > 0$, $B > 0$, and $t_l \leq \sqrt{A/B} \leq t_u$, then the function (3.4) has its minimum value $2\sqrt{AB}$ at $t = \sqrt{A/B}$. Otherwise, either t_l or t_u must be the minimizer of the function (3.4).

Proof: Proposition 3.1 is obvious since the function (3.4) on $t > 0$ is either linear (in case of $A = 0$), strictly concave (in case of $A < 0$), or strictly convex (in case of $A > 0$). □

It is easy to check the following results.

Proposition 3.2 Let (t_l, f_l) and (t_u, f_u) be arbitrary two points in R^2 such that $0 < t_l < t_u$. Then a function of the form (3.4) which involves both of those points is uniquely determined as

$$A(t_l, f_l, t_u, f_u)/t + B(t_l, f_l, t_u, f_u)t,$$

where

$$\begin{aligned} A(t_l, f_l, t_u, f_u) &= (f_u/t_u - f_l/t_l)/(1/(t_u)^2 - 1/(t_l)^2), \\ B(t_l, f_l, t_u, f_u) &= (f_u t_u - f_l t_l)/((t_u)^2 - (t_l)^2). \end{aligned}$$

Corollary 3.1 Let $f_1(t) = A_1/t + B_1t$ and $f_2(t) = A_2/t + B_2t$ be two different functions defined on $t > 0$. They intersect mutually at most at one point over $t > 0$. □

To see that f_1 and f_2 do not contact but intersect mutually, we can check that their derivatives are different at the point where they have the same value.

Now we can introduce an underestimator for the problem (P) by taking advantage of the form $A/t + Bt$.

Proposition 3.3 Let t_l and t_u are two points such that $0 < t_l < t_u$. The function

$$f(t) = A(t_l, F(t_l), t_u, F(t_u))/t + B(t_l, F(t_l), t_u, F(t_u))t \tag{3.5}$$

is an underestimator of F on $[t_l, t_u]$.

Proof: By the definition of F (see (3.3)) and Proposition 3.2, one has

$$f(t_l) = F(t_l) = \inf_k \{ A(k)/t_l + B(k)t_l \},$$

$$f(t_u) = F(t_u) = \inf_k \{ A(k)/t_u + B(k)t_u \}.$$

Then, by Corollary 3.1 one has

$$f(t) \leq \inf_k \{ A(k)/t + B(k)t \} = F(t) \quad \forall\, t \in [t_l, t_u]. \quad \square$$

Suppose that we are given an ordered set T which consists of a finite number of points in $[t_L, t_U]$ such that

$$T \supset \{ t_L, t_U \}.$$

Let us define the following function on $[t_L, t_U]$.

$$F_T(t) = A(t_d, F(t_d), t_s, F(t_s))/t + B(t_d, F(t_d), t_s, F(t_s))t \tag{3.6}$$

$$\text{for} \quad t_d \leq t \leq t_s, \quad t_d, t_s \in T, \quad t_d : \text{predecessor of } t_s.$$

Proposition 3.3 immediately gives the following result.

Corollary 3.2 F_T is an underestimator of F on $[t_L, t_U]$. □

By Proposition 3.1 it is easy to handle a minimizer of F_T.

We are now ready to present a successive underestimation method for the problem (P) in each iteration of which an underestimator will be taken in the class of functions of the form (3.6).

Algorithm 3.1
Initialization: Set $j \leftarrow 1$, $T_1 \leftarrow \{t_L, t_U\}$, $F_1(t) \leftarrow F_{T_1}(t)$, and $\varepsilon \leftarrow$ a positive small value.
Iteration j: Solve the relaxed problem:

$$(p_j) \quad \textit{minimize } F_j(t), \quad s.t. \quad t_L \leq t \leq t_U,$$

and let t_j be the solution obtained.
If the stopping criterion

$$F_j(t_j) \geq F(t_j) - \varepsilon \tag{3.7}$$

holds, then stop. Otherwise, set $T_{j+1} \leftarrow T_j \cup \{t_j\}$, $F_{j+1}(t) \leftarrow F_{T_{j+1}}(t)$.
Go to iteration j after setting $j \leftarrow j + 1$. □

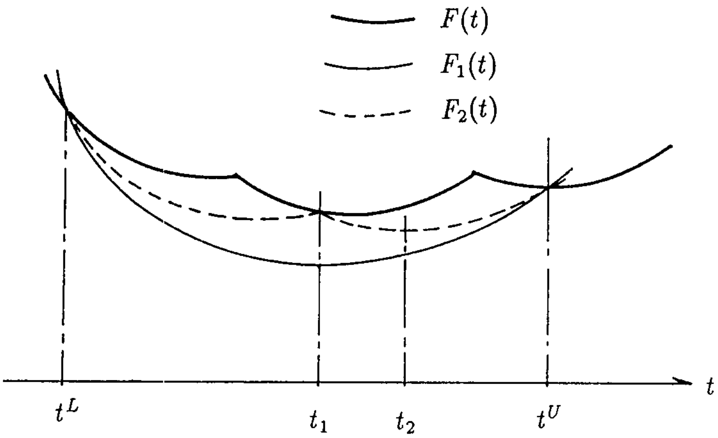

Fig.2. Illustration of Algorithm 2.1.

Fig.2 shows stepwise procedures of the algorithm up to the 2nd iteration.

In the rest of this section we will prove that the Algorithm 3.1 terminates in finite iterations.

Definition 3.1 For a given tolerance $\varepsilon \geq 0$, a point $t^* \in [t_L, t_U]$ is called an $\varepsilon -$ *optimal* solution of the problem (P) if

$$F(t^*) \leq \inf F(t) + \varepsilon \quad \forall t \in [t_L, t_U].$$

It is obvious that $0 -$ *optimal* solution is the exact solution of the problem (P).

Proposition 3.4 F and F_T are Lipschitzians on $[t_L, t_U]$.

Proof: The function (3.4) is Lipschitzian with Lipschitz constant $\max\{A/(t_L)^2, B\}$ over $t \geq t_L > 0$. So, by (3.3), F is a pointwise minimum of a finite family of Lipschitzians of the form (3.4), and by (3.6), F_T is a piecewise Lipschitzian of the form (3.4). Thus, one has Proposition 3.4. □

Theorem 3.1 If $\varepsilon > 0$, Algorithm 3.1 terminates after a finite number of iterations yielding an $\varepsilon -$ *optimal* solution.

Proof: Since F_j is an underestimator of F and t_j is a minimizer of F_j, $F_j(t_j)+\varepsilon \leq \inf F(t)+\varepsilon$ holds. So, if the stopping criterion (3.7) holds for any $\varepsilon > 0$, i.e., $F(t_j) \leq F_j(t_j) + \varepsilon$, then $F(t_j) \leq \inf F(t) + \varepsilon$. Therefore, by Definition 3.1, t_j is an $\varepsilon -$ *optimal* solution. Let us assume that the algorithm is infinite, i.e., it generates an infinite sequence $\{t_j\}$, $j = 1, 2, \cdots$. We are going to prove that

$$F(t_j) - F_j(t_j) \to 0 \quad (j \to \infty). \tag{3.8}$$

Suppose that (3.8) is not satisfied. Since $F(t_j) \geq F_j(t_j)$ for any j, it follows that there is a subsequence $\{t_{j_s}\}$ such that

$$F(t_{j_s}) - F_{j_s}(t_{j_s}) \geq \theta > 0 \quad \forall s. \tag{3.9}$$

Since F and F_j are Lipschitzians on $[t_L, t_U]$, one has

$$\begin{aligned}
|F(t_{j_s}) - F_{j_s}(t_{j_s})| &= |F(t_{j_s}) - F_{j_s}(t_{j_{s-1}}) + F_{j_s}(t_{j_{s-1}}) - F_{j_s}(t_{j_s})| \\
&\leq |F(t_{j_s}) - F_{j_s}(t_{j_{s-1}})| + |F_{j_s}(t_{j_{s-1}}) - F_{j_s}(t_{j_s})| \\
&= |F(t_{j_s}) - F(t_{j_{s-1}})| + |F_{j_s}(t_{j_{s-1}}) - F_{j_s}(t_{j_s})| \\
&\quad (\text{Note that } F_{j_s}(t_{j_r}) = F(t_{j_r}) \ \forall\, r < s) \\
&\leq L_1|t_{j_s} - t_{j_{s-1}}| + L_2|t_{j_{s-1}} - t_{j_s}| \\
&\leq (L_1 + L_2)|t_{j_s} - t_{j_{s-1}}|,
\end{aligned}$$

where L_1 and L_2 are the Lipschitz constants of F and F_j, respectively.
Since $|t_{j_s} - t_{j_{s-1}}| \to 0$ $(s \to \infty)$, this conflicts with (3.9). Thus, we have (3.8). □

4. Relief Indicator Method

In this section we present another method to minimize $F(t)$ on $[t_L, t_U]$. The method is a modification of the relief indicator method developed recently by H.Tuy and one of the authors (see [16]) for multivariable, constrained nonconvex minimization problems.

We will start with introducing the basic idea of the relief indicator method which will reduce the problem to a parametric minimization problem of a d.c. function (a function which can be written as a difference of two convex functions).

Choose a point t_j in $[t_L, t_U]$ and let $\theta_j = F(t_j)$. Then, define

(4.1) $$S_j := \{\, t \mid F(t) \leq \theta_j \ , \ t \in [t_L, t_U] \,\},$$

(4.2) $$d_j(t) := \inf_{x \in S_j} |\, t - x| \quad \forall t \in [t_L, t_U] \ ,$$

(4.3) $$g_j(t) := \sup_y \{\, 2yt + d_j^2(y) - y^2 \mid F(y) \geq \theta_j \ y \in [t_L, t_U] \,\} \ ,$$

where $d_j(t)$ denotes the distance from each point $t \in [t_L, t_U]$ to the set S_j. $g_j(t)$ is a convex polygon formed along the quardraric curve t^2 (see Fig.3).

As seen form Fig.3, one has

(4.4) $$g_j(t) > t^2 \iff F(t) > \theta_j$$

(4.5) $$g_j(t) = t^2 \iff F(t) = \theta_j \ ,$$

(4.6) $$g_j(t) < t^2 \iff F(t) < \theta_j \ ,$$

(Proof: see Proposition 3.1 in [16]).

We call $\psi(\theta, t) := g(t) - t^2$ a relief indicator of $F(t)$. From (4.4), (4.5), and (4.6) it follows that if $\min \psi(\theta_j, t) = 0$, then t_j is a minimizer of $F(t)$. This means that the minimization of $F(t)$ can be reduced to the parametric minimization problem:

$$(Q) \quad \textit{find } t_j \textit{ such that } \ \min\{\, \psi(\theta_j, t) \mid t \in [t_L, t_U]\} = 0.$$

We will employ this approach with some modifications.

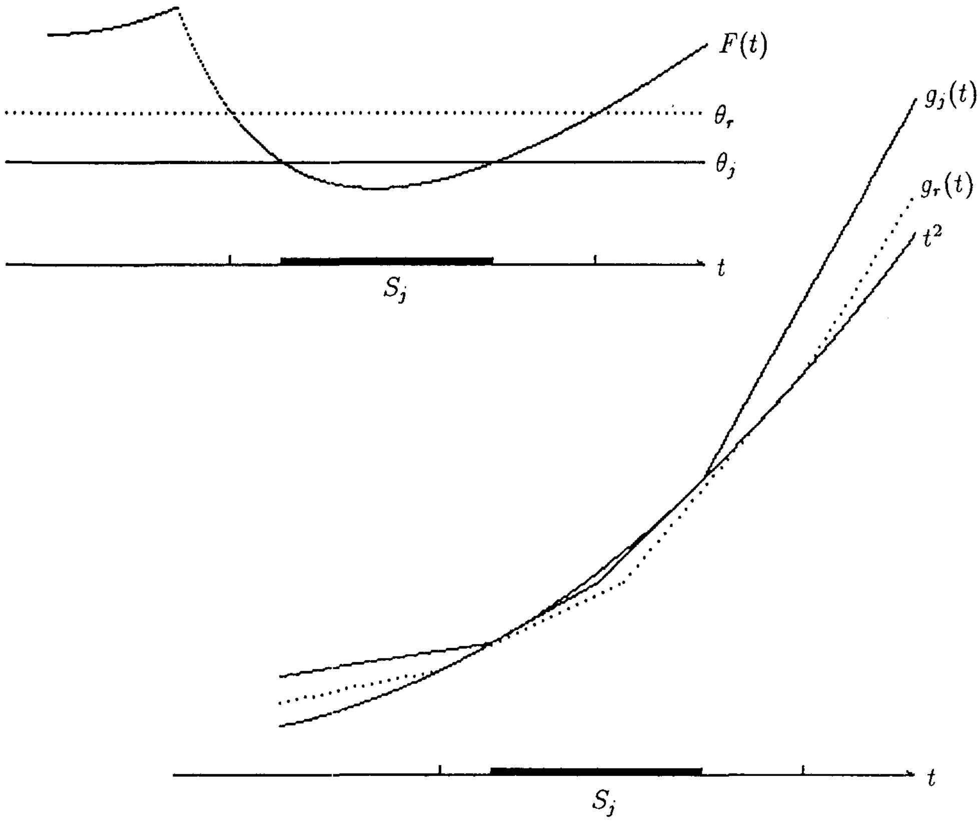

Fig.3. Relations among F, S, θ, g, and t^2.

To check if t_j is optimal, we solve the subproblem:

$$(q) \quad minimize \ g_j(t) \ - \ t^2 \ \ s.t. \ t \epsilon [t_L, t_U] \ .$$

Unfortunately, S_j and then $g_j(t)$ are usually not available. So, we underestimate $g_j(t)$ by a piecewise linear convex function $h_j(t)$ such that $h_j(t) \leq g_j(t) \ \forall t \epsilon [t_L, t_U]$. We can indeed construct such underestimator $h_j(t)$ depending on the threshold θ_j as we will state later. At first we solve the following subproblem (H) instead of solving the subproblem (q) .

$$(H) \quad minimize \ h_j(t) \ - \ t^2 \ \ s. \ \ t \epsilon [t_L, t_U] \ ,$$

and obtain a soluton of it denoted by t_{j+1}. From (4.5), (4.6), and $h_j(t) \leq g_j(t)$, the following three cases are possible with respect to t_{j+1} (see Fig.4).

Case 1: $h_j(t_{j+1}) = t_{j+1}^2$, i.e., $\theta_j = F(t_{j+1}) = \min F(t)$.
Case 2: $\psi(\theta_j), t_{j+1}) < 0$ and $h_j(t_{j+1}) < t_{j+1}^2$, i.e., $F(t_{j+1}) < \theta_j$.
Case 3: $\psi(\theta_j, t_{j+1}) \geq 0$ and $h_j(t_{j+1}) < t_{j+1}^2$, i.e., $F(t_{j+1}) > \theta_j$.

In Case 1, t_j and t_{j+1} are optimal. In Case 2, t_j is not optimal and we get t_{j+1} which is better than t_j. In Case 3, we find t_j is better than t_{j+1}.

Now we proceed to construct a more accurate underestimator $h_{j+1}(t)$. Using the solution t_{j+1} of the subproblem (H) with $h_j(t)$, we first prepare an improved threshold θ_{j+1} by

$$\theta_{j+1} \ := \ \min\{F(t_{j+1}), \theta_j\} \ ,$$

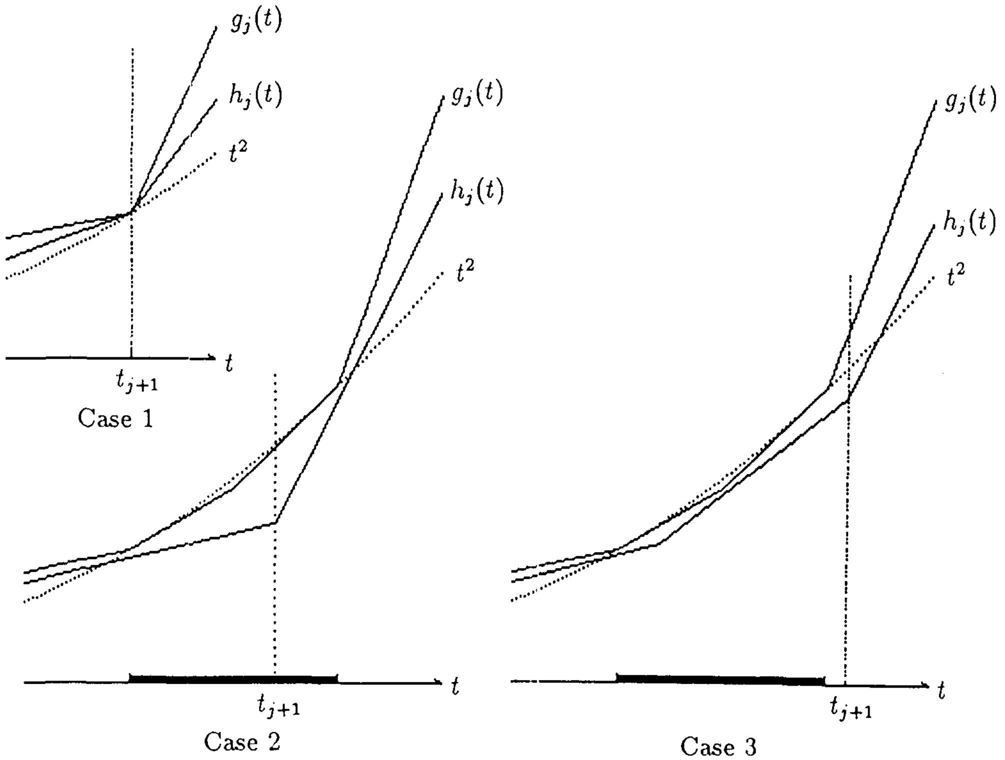

Fig.4. Three cases of solutions of subproblem (H).

and arrange a set of points T_{j+1} such that

$$\{ \ t_L, \ t_U, \ t_{j+1} \ \} \subset T_{j+1} \ .$$

Then construct $h_{j+1}(t)$ through the following routine R1 so that it may satisfy the next conditions:

$$h_{j+1}(t_{j+1}) > h_j(t_{j+1}), \tag{4.7}$$

$$g_{j+1}(t) \geq h_{j+1}(t) \geq h_j(t) \ \forall t \epsilon [t_L, t_U]. \tag{4.8}$$

Routine R1: Construct the underestimator $F_{T_{j+1}}$ of $F(t)$ defined upon T_{j+1} as in the last section (ref. (3.6)). Let u be the largest value of $t(\leq t_{j+1})$ such that $F_{T_{j+1}}(t) = \theta_{j+1}$. If there is no such t, set $u \leftarrow t_L$. Let v be the smallest value of $t(\geq t_{j+1})$ such that $F_{T_{j+1}}(t) = \theta_{j+1}$. If there is no such value, then set $v \leftarrow t_L$ (see Fig.5).
Construct a linear function $\alpha t + \beta$ by setteing $\alpha \leftarrow u + v$, $\beta \leftarrow -uv$ so that it may satisfy the condition :

$$\alpha t + \beta \geq t^2 \ \forall t \in [u, v], \quad \alpha t + \beta < t^2 \ \forall t \notin [u, v], \tag{4.9}$$

then construct the piecewise linear convex function

$$h_{j+1}(t) := max\{ \ h_j(t), \ \alpha t + \beta \ \}. \quad \square \tag{4.10}$$

Thus we obtained a new underestimator $h_{j+1}(t)$ of $g_{j+1}(t)$ depending on the threshold θ_{j+1} $(\leq \theta_j)$. Then we solve again the subproblem (H) with $h_{j+1}(t)$.

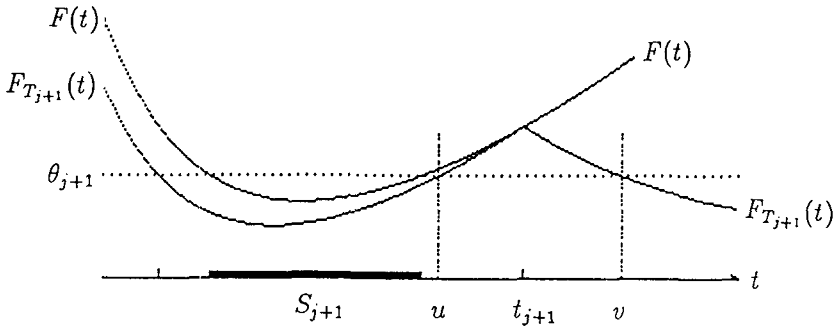

Fig.5. Determination of u and v used F_T.

To see that $h_{j+1}(t)$ is a underestimator of $g_{j+1}(t)$ better than $h_j(t)$, it is sufficient to prove that $h_{j+1}(t)$ satisfies the conditions (4.7) and (4.8). The proof is as follows:

Condition $t_{j+1}^2 > h_j(t_{j+1})$ always holds in Cases 2 and 3. Since we set $u \leq t_{j+1} \leq v$ in the routine R1, (4.9) gives $\alpha t_{j+1} + \beta \geq t_{j+1}^2$. Moreover,(4.10) implies $h_{j+1}(t_{j+1}) \geq \alpha t_{j+1} + \beta$. Combining those facts, one has

$$h_{j+1}(t_{j+1}) \geq \alpha t_{j+1} + \beta \geq t_{j+1}^2 > h_j(t_{j+1}).$$

Thus, condition (4.7) holds. Next, by the definition (4.2) one has

$$d_{j+1}(\alpha/2) = \inf_{x \in S_{j+1}} \mid \alpha/2 - x \mid \geq \inf_{x \notin [u,v]} \mid \alpha/2 - x \mid = (v - u)/2.$$

Hence, from $F(\alpha/2) \geq \theta_{j+1}$ and (4.3), we have

$$\begin{aligned} \alpha t + \beta &= 2(\alpha/2)t + ((v - u)/2)^2 - (\alpha/2)^2 \\ &\leq 2(\alpha/2)t + d_{j+1}^2(\alpha/2) - (\alpha/2)^2 \\ &\leq \sup_y\{ 2yt + d_{j+1}^2(y) - y^2 \mid F(y) \geq \theta_{j+1},\ y \epsilon [t_L, t_U] \} \\ &= g_{j+1}(t) \quad \forall t \,\epsilon [t_L, t_U]. \end{aligned} \tag{4.11}$$

On the other hand, $h_j(t)$ is the underestimator of $g_j(t)$ depending on the threshold θ_j. Hence, $g_j(t) \geq h_j(t)$ holds. By (4.3), $\theta_{j+1} \leq \theta_j$ gives $g_{j+1}(t) \geq g_j(t)$. Hence, from (4.10) and (4.11) one has

$$g_{j+1}(t) \geq \max\{\alpha t + \beta, g_j(t)\} \geq \max\{\alpha t + \beta, h_j(t)\} = h_{j+1}(t) \geq h_j(t).$$

Thus, (4.8) holds. □

To effectively improve the threshold θ_j we can use Goyal's method for finding a stationary point (see [5]) as follows.

Routine R2: Let $\theta_j := F(t_j)$ be the current threshold. We first minimize $C(t_j, k)$(see (2.1)) with respect to variable k, obtaining k'. Then minimize $C(t, k')$ with respect t, obtaining t'. if $t_j = t'$, then t_j is a stationary point. We use θ_j as the threshold. If $t_j \neq t'$, then we have $F(t') < \theta_j$ and repeat this procedure by setting $t_j \leftarrow t'$. □

Before describing our algorithm, we will introduce some concept about an approximate solution. From the definition of $g_j(t)$, one has

$$g_j(t) - t^2 = -\inf_y\{ |t - y|^2 \,|F(y) > F(t_j),\ y\epsilon[t_L,t_U] \} \quad \forall t \mid F(t) \leq F(t_j)$$

(see Proposition 3.1 in [16]). Then, if

$$\inf\{ g_j(t) - t^2 \mid t\epsilon[t_L,t_U] \} \geq \inf\{h_j(t) - t^2 \mid t\epsilon[t_L,t_U] \} = h_j(t_j) - t_j^2 \geq -\varepsilon^2,$$

there exists a point $t'\epsilon[t_L, t_U]$ such that

$$F(t') \geq F(t_j) \quad \text{and} \mid t' - t \mid \leq \varepsilon \,\forall t\epsilon[t_L,t_U]. \tag{4.12}$$

We call t_j an $\varepsilon-$ *approximate* solution if (4.12) holds for every $t\epsilon[t_L, t_U]$. It is obvious that a $0-$ *approximate* solution is exactly optimal. Thus we have

Theorem 4.1 If $h_j(t_j) - t_j^2 \geq -\varepsilon^2$, then t_j is an $\varepsilon-$ *approximate* solution . □

Now we are led to the following algorithm.

Algoriths 4.1
Initialization: Set $h_1(t) :\equiv 0$, $T_1 \leftarrow \{t_L,t_U\}$, $\theta_1 \leftarrow \min\{F(t_L), F(t_U)\}$, $j \leftarrow 1$, and $\varepsilon \leftarrow$ a positive small value.
Iteration j:
j.a.: Solve the subproblem

$$(H) \quad minimize\ h_i(t) - t^2, \quad s.t.\ t\epsilon[t_L,t_U]$$

obtaining an optimal solution t_{j+1} of it.
If

$$h_j(t_{j+1}) - (t_{j+1})^2 \geq -\varepsilon^2 ,$$

then stop. t_{j+1} is an $\varepsilon-$ *approximate* solution.
Otherwise go to step j.b.
j.b: If $F(t_{j+1}) < \theta_j$, then improve $F(t_{j+1})$ by routine R2.
Set $\theta_{j+1} \leftarrow \min\{F(t_{j+1}),\ \theta_j\}$, $T_{j+1} \leftarrow T_j \cup \{t_{j+1}\}$, and $F_{j+1} \leftarrow F_{T_{j+1}}$.
Construct $h_{j+1}(t)$ by routine R1.
Set $j \leftarrow j + 1$. Then, go to step j.a. □

By the analogous arguments as in Theorem 6.2 in [16], one has the following convergence property of our approach.

Theorem 4.2 If $\varepsilon > 0$, then Algorithm 4.1 terminates after a finite number of iterations yielding an $\varepsilon-$ *approximate* solution. □

5. Numerical experiements

In this section we report the computational results of G4: Goyal's method [4], G5: Goyal's method [5], SUM: the Successive Underestimation Method, and RIM: the Relief Indicator Method on over 240 test examples. G5 is a method for finding a stationary point with a heuristic initial point and the rest are methods for finding a global optimal solution. Input data were generated randomly. The computational results are given in Tables 1,2,3, and 4. Notations in the tables are as follows:
n: number of items
m: number of test problems
R: average of ratios of t_U to t_L
N1(%): percentage of test examples in which G5 gives a global optimal solution
E1(%): average of relative errors by G5

T1(s): average of computational times by G5
N2: average of numbers of local minima to be examined in G4
N3: maximum of numbers of local minima to be examined in G4
T2(s): average of computational times by G4
N4: average number of iterations by SUM
N5: maximum numbers of iterations by SUM
$\varepsilon 1$: parameter which gives the tolerance $\varepsilon := \varepsilon 1 \cdot F(t_j)/100$ in the stopping criterion of SUM
T3(s): average of computational times by SUM
N6: average number of iterations by RIM
N7: maximum number of iterations by RIM
E3(%): average of relative errors by RIM
E4(%): maximum relative error by RIM
$\varepsilon 2$: parameter which gives the tolerance $\varepsilon^2 := \varepsilon 2(t_U - t_L)/2$ in the stopping criterion of RIM

The programs were coded in BASIC(compiler) and ran on a PC(16bit/12MHz). The results in Table 1 and those in Table 2 correspond to two different random distributions of data.

Observations: Although G5 is very good in many examples, sometimes an error by this method is rather large ($\approx 7\%$). Furthermore, even if we obtain a very good solution by this method we do not know how good it is. On the other hand, SUM and RIM can control their errors within any tolerance as we like. Additionally, from the computational results we see that they are not so expensive even when the number of items becomes large. The computational times of G4 are much larger than those of SUM and RIM when the number of items becomes large. For the cases of the same number of items, the computational time of G4, compared with SUM and RIM, increases quickly as the ratio R:= t_U/t_L increases.

6. Conclusions

In this paper the problem of determining an economic ordering policy for jointly replenished items has been reduced to the problem of minimizing the single-variable nonconvex function $F(t)$ over $t > 0$. We proposed two methods: the Successive Underestimation Method in Section 3 and the Relief Indicator Method in Section 4. Different from the conventional approaches of seeking a local optimal policy, the proposed methods can obtain a global approximate solution within any prescribed error bound as we like.

As was observed in Section 5 both of the proposed methods performed equally well for many test problems. To evaluate the performance we have numerically tested our methods and Goyal's methods [4],[5]. The computational time of Goyal's method [4] for a global optimal solution was much larger than those of the other tree methods and increased quickly as the number of items n became large and as the ratio R:= t_U/t_L became large even for a small n. Goyal's method [5] which is supposed to be the best heuristic method ever developed was the fastest but sometimes gave a large relative error over 7% to the solution. Moreover it cannot provide an error bound for a local optimum solution obtained. Our methods are slower than Goyal's method [5], but this will be justified by more accurate results obtained by our methods in most practical situations.

In the course of development of the methods described in Sections 3 and 4 we have realized that the approaches employed there can be used in a class of problems, for example the problem:

$$minimize\ (c_1x)\cdot(c_2x) \quad s.t. \quad Ax = b,\ x \geq 0.$$

We will merely indicate such an approach and leave the detailed development to another paper.

Table 1. Computational results of four methods for the first rundom distribution of data.

			Goyal [5]				Goyal [4]			SUM ($\varepsilon 1 = 0.01$)			RIM					
n	m	R	N1 (%)	E1 (%)	E2 (%)	T1 (s)	N2	N3	T2 (s)	N4	N5	T3 (s)	N6	N7	E3 (%)	E4 (%)	$\varepsilon 2$	T4 (s)
5	20	4.0	60	0.54	4.06	0.2	15.8	96	0.5	6.0	10	1.0	5.3	11	1E-05	4E-04	0.01	0.7
10	20	3.5	50	0.52	5.92	0.3	27.6	132	1.1	6.8	15	1.2	5.7	13	0.004	0.06	0.01	1.5
15	20	8.5	50	0.42	5.87	1.0	121.7	722	3.9	11.0	25	3.0	8.6	16	5E-05	9E-04	0.01	2.8
20	20	14.5	25	1.19	7.51	1.3	269.5	2160	9.5	15.3	40	5.2	9.3	13	0.002	0.03	0.01	4.4
25	20	25.6	15	0.84	4.34	2.1	648.5	3624	25.3	26.5	47	11.1	16.5	24	0.009	0.06	0.001	8.7
30	20	34.7	0	0.97	2.30	2.3	1039.9	3885	44.0	34.7	60	17.1	16.5	19	0.01	0.05	0.001	11.3

Table 2. Computational results of four methods for the second rundom distribution of data.

			Goyal [5]				Goyal [4]			SUM ($\varepsilon 1 = 0.01$)			RIM					
n	m	R	N1 (%)	E1 (%)	E2 (%)	T1 (s)	N2	N3	T2 (s)	N4	N5	T3 (s)	N6	N7	E3 (%)	E4 (%)	$\varepsilon 2$	T4 (s)
5	20	37.5	75	0.08	1.13	0.3	198.8	991	4.1	17.8	23	2.5	16.3	20	0.004	0.07	0.002	3.2
10	20	32.5	40	0.33	1.77	0.6	278.4	1063	7.2	20.6	29	4.6	12.7	16	0.01	0.08	0.002	3.5
15	20	40.3	50	0.16	1.01	0.8	577.1	3281	17.0	22.9	33	6.5	12.3	16	0.01	0.08	0.002	4.6
20	20	39.8	20	0.23	0.99	1.1	583.5	1353	19.7	24.1	31	8.4	14.7	16	0.009	0.09	0.002	6.5
25	20	40.8	20	0.29	1.43	1.6	727.7	1328	28.0	26.0	35	10.3	14.9	19	0.01	0.09	0.002	8.0
30	20	73.4	20	0.40	1.45	2.6	1882.2	5012	79.3	34.8	44	16.9	15.4	23	0.008	0.04	0.002	10.2

Table 3. Relation between computational times of Goyal's method[4], SUM and RIM, and R = t_U/t_L in case n = 5.

		Goyal [4]		SUM			RIM			
m	R	N2	T2(s)	N4	$\varepsilon 1$	T3(s)	N6	E3(%)	$\varepsilon 2$	T4(s)
20	4.0	15.8	0.5	6.0	0.01	1.0	5.3	1E-05	0.01	0.7
20	12.8	53.1	1.2	14.4	0.01	2.0	13.3	0.006	0.002	2.5
20	37.5	198.8	4.1	17.8	0.01	2.5	16.3	0.004	0.002	3.2
20	49.9	299.4	6.3	18.9	0.01	2.6	18.1	0.004	0.002	4.0
20	89.3	727.1	14.9	19.3	0.01	2.8	18.1	0.008	0.002	4.1

Table 4. Relation between computational times of Goyal's method[4], SUM and RIM, and R = t_U/t_L in case n = 20.

		Goyal [4]		SUM			RIM			
m	R	N2	T2(s)	N4	$\varepsilon 1$	T3(s)	N6	E3(%)	$\varepsilon 2$	T4(s)
20	14.5	269.5	9.5	15.3	0.01	5.2	9.3	0.002	0.01	4.4
20	25.5	355.2	12.1	23.2	0.01	8.1	15.1	0.01	0.002	6.3
20	39.8	583.5	19.7	24.1	0.01	8.4	14.7	0.009	0.002	6.5
20	51.2	767.0	26.0	24.0	0.01	8.4	14.8	0.008	0.002	6.8
20	71.8	1114.2	37.7	24.7	0.01	8.6	15.0	0.02	0.001	7.1

Acknowlegement

The authors are grateful to the referees for many suggestions to improve the paper.

Appendix

Determination of t_U: From Proposition 3.1 and (2.1),

$$t(k) \ := \ \sqrt{2(S+\sum_{i=1}^{n} S_i/k_i)/\sum_{i=1}^{n} h_iD_ik_i}$$

is the minimizer of $C(t,k)$ with any fixed $k := (k_1, k_2, \ldots, k_n)$. Noting that $t(k)$ is a decreasing function of k and $k_i \geq 1$ for all $i \leq n$, one has

$$t_U \ := \ \sqrt{2(S+\sum_{i=1}^{n} S_i)/\sum_{i=1}^{n} h_iD_i}$$

as an upper bound of t for the problem of minimizing $C(t,k)$.

Determination of t_L: From Proposition 3.1, the function $S_i/k_it \ + \ h_iD_ik_it/2$ on $t > 0$ has its minimum value $\sqrt{2S_ih_iD_i}$. Then, one has

$$\begin{aligned} C(t,k) \ &= \ S/t \ + \ \sum_{i=1}^{n}\{S_i/k_it \ + \ h_iD_ik_it/2\} \\ &\geq \ S/t \ + \ \sum_{i=1}^{n}\sqrt{2S_ih_iD_i} \ . \end{aligned}$$

Let t' be the root of the equation:

$$S/t \;+\; 2\sum_{i=1}^{n}\sqrt{S_ih_iD_i} \;=\; F(t_U)\;.$$

For $t \leq t'$ one has

$$\begin{aligned} F(t) \;=\; \inf_k C(t,k) \quad &\geq \quad S/t \;+\; \sum_{i=1}^{n}\sqrt{2S_ih_iD_i} \\ &\geq \quad S/t' \;+\; \sum_{i=1}^{n}\sqrt{2S_ih_iD_i} \;=\; F(t_U). \end{aligned}$$

Thus, we obtain

$$t_L \;:=\; t' \;=\; S/(F(t_U) - \sum_{i=1}^{n}\sqrt{2S_ih_iD_i})$$

as a lower bound of t for the problem of minimizing $C(t,k)$.
Note that $F(t_L) \geq F(t_U)$ holds.

Determination of k_{U_i}: From Proposition 3.1, $k_i(t) := \sqrt{2S_i/h_iD_it^2}$ is the minimizer of the function $S_i/tk_i + h_iD_itk_i/2$ with any fixed $t > 0$. Letting $k(t) := (k_1(t), k_2(t), \ldots, k_n(t))$, one has

$$C(t,k) \;=\; S/t \;+\; \sum_{i=1}^{n}\{S_i/k_it \;+\; h_iD_ik_it/2\} \;\geq C(t,k(t)).$$

Then, noting that k_i is a positive integer and $k_i(t)$ is a decreasing function of t for all $i \leq n$, we obtain the following k_{U_i} as an upper bound of k_i for the problem of minimizing $C(t,k)$.

$$k_{U_i} \;=\; [\sqrt{2S_i/h_iD_i(t_U)^2}\;] \;+\; 1,$$

where $[x]$ denotes the integer part of x.

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