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A ONE-DIMENSIONAL SEARCH WITH TRAVELING COST

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Abstract There are $2n+1$ neighboring cells in a straight line. An object is in one of all cells except for the cell which locates at the center of all cells, according to a known probability distribution which is assumed to be symmetric with respect to the cell at the center. A searcher is at the cell which locates at the center of all cells at the beginning of the search, and after he chooses an ordering of the $2n$ labels attached to the $2n$ cells, he examines each cell in that order. An ordering is considered to be optimal when the expected cost of the search is minimized. The cost comprises a traveling cost dependent on the distance from the last cell examined and a fixed examination cost. After basic observations on our model are made the Bellman's Principle of Optimality is applied to it. We have the optimal equation, from which some properties are derived. Approximately optimal search strategies are defined and analyzed. Several discussions are provided.

1. Introduction

Let us imagine a search for a book in a library, for a word in a dictionary, or for a gas station in a street. We will have an experience of changes or reversals of direction.

Gluss [4] considered a model in which there are $n+1$ neighboring cells in a straight line, labeled from 0 to n in that order. An object is in one of them except for cell 0 with a priori probabilities $p_1, \cdots, p_n$. At the beginning of the search the searcher is at cell 0 that is next to cell 1. It is required to determine a strategy that will minimize the expected cost of finding the object. The examination cost $v_i (1 \leq i \leq n)$ is associated with the examination of cell $i (1 \leq i \leq n)$. The only difference between his model and the previous one (See [3]) is that while v_i's are constant in the latter, they vary through time, that is, a traveling cost as well as the fixed examination cost is considered in the former (See [4]). Gluss treated two cases: $p_1 \geq \cdots \geq p_n$ and $p_1 \leq \cdots \leq p_n$. He showed that the former case is trivial, that is, the searcher should examine each cell in the order of $1, 2, \cdots, n$, and in the latter case he found approximately optimal strategies when p_i is proportional to i. These strategies are written by one parameter.

Our model in this paper differs from his model in that at the beginning of the search the searcher is at the cell that locates at the center of all cells. For simplicity a priori proabilities are assumed to be symmetric with respect to the cell at which the searcher is at the beginning. Furthermore only the case is treated that a priori probabilities are monotone non-increasing as the cells become far away from the center.

Both our model and the model by Gluss are special cases of the model developped by Lössner/Wegener [5] (Also see pp. 264–265 of [1]). But it seems to me that they are interested in the case with positive probabilities of overlooking the object. Indeed Theorem 4.1 in [5] is about the existence of a periodic search which is optimal, and about probabilities of overlooking. Theorem 3.3 and Lemma 5.3 in [5] are most interesting. They give critical numbers and conditions in order to check what is the next cell to be examined. But they do not seem to be efficient for our special model since they have been argued in a more general setting. Also see p. 6 of [6]. Nakai [7] is a survey on the search theory.

Given a more concrete model, our purpose is to propose a kind of search strategies and to examine their properties.

In the next section our model is stated in detail and a search strategy is defined. In Section 3 the expected cost under a search strategy is calculated and basic observations on it are made. In Section 4, the Bellman's Principle of Optimality is applied to our model. We have the optimal equation, from which some properties are derived. In Section 5, after the manner of Gluss [4], approximately optimal search strategies are defined and analyzed. Several discussions are provided in Section 6 as concluding remarks.

2. The Model and a Search Plan.

There are $2n+1$ neighboring cells in a straight line, labeled from $-n$ to n in that order. An object is in one of all cells except for cell 0, with a priori probabilities such that

$$p_i > 0, \text{all } i(-n \leq i \leq n \text{ and } i \neq 0), \text{and} \sum_{i=1}^{n}(p_i + p_{-i}) = 1. \tag{2.1}$$

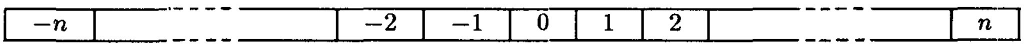

Figure 1

A probability of overlooking the object is equal to zero, when the right cell is searched. Associated with the examination of cell $i(-n \leq i \leq n$ and $i \neq 0)$ is the examination cost that consists of two parts: (i) a traveling cost $d|i-j|(d>0)$ of examining cell i after having examined cell j, and (ii) a fixed examination cost $c \geq 0$. (i) means that the examination cost varies through the search and is a function of which cell was last examined. It is assumed that at the beginning of the search the searcher is at cell 0. Before searching he must determine a search strategy that will minimize the expected cost of finding the object. For simplicity it is also assumed that

$$p_i = p_{-i}, \text{all } i(1 \leq i \leq n). \tag{2.2}$$

This assumption and (2.1) imply that

$$\sum_{i=1}^{n} p_i = \frac{1}{2}. \tag{2.3}$$

In this paper we treat only the case that

$$p_1 \geq p_2 \geq \cdots \geq p_n. \tag{2.4}$$

This leads to the following definition of a search strategy. A <u>search plan</u> is defined by $s = [\ell_1, \cdots, \ell_m]$ where $\ell_i(1 \leq i \leq m)$ is a positive integer and $1 \leq \ell_1 < \cdots < \ell_{m-1} < \ell_m = n$. s indicates that first he (the searcher) examines each cell from -1 to $-\ell_1$, then goes back to cell 0, then searches each cell from 1 to ℓ_2, then goes back to cell $-\ell_1$, then examine each cell from $-\ell_1 - 1$ to $-\ell_3, \cdots$. Supposing that m is even and he examines from ℓ_{m-2} to ℓ_m, (m is odd and he examines from $-\ell_{m-2}$ to $-\ell_m$) finally he examines from $-\ell_{m-1}$ to $-\ell_m$ (from ℓ_{m-1} to ℓ_m). We denote by S the set of all search plans, and by S_m the set of all search plans that have at most m turnabouts. Then $S = S_n$. $s_1 \equiv [n]$ and $s_n \equiv [1, 2, \cdots, n-1, n]$ are the only elements in S_1 and $S_n \backslash S_{n-1}$ respectively. Hereafter we take into consideration only strategies in S. The first reason for this restriction is to avoid the difficulties in deducing "exactly optimal" search strategies and in enumerating the expected cost of each of the $\frac{(2n)!}{2n}$ possible order of examination (See p. 279 of [4]). The second reason is the paper by Beck [2]. Beck used a similar kind of strategy in case of linear search problem.

Our problem is to find an optimal search plan, that is, a search plan that will minimize the expected cost of finding the object. A search plan in $S_m (1 \le m \le n)$ is called m-optimal if of all search plans in S_m it minimizes the expected cost.

3. Observations.

In this section we dispose of a few cases that are easy to solve. Suppose that c and d are given and that he employs search plan $s = [\ell_1, \cdots, \ell_m]$. The expected cost, written as $E(s, c, d)$, is as follows:

$$\begin{aligned}
E(s,c,d) &= d + c + [1 - p_1][d + c] + \cdots + [1 - q(1, \ell_1 - 1)][d + c] \\
&+ [1 - q(1, \ell_1)][(\ell_1 + 1)d + c] + [1 - q(1, \ell_1) - p_1][d + c] + \cdots \\
&+ [1 - q(1, \ell_1) - q(1, \ell_2 - 1)][d + c] + [1 - q(1, \ell_1) - q(1, \ell_2)][(\ell_1 + \ell_2 + 1)d + c] \\
&+ [1 - q(1, \ell_1) - q(1, \ell_2) - p_{\ell_1+1}][d + c] + \cdots \\
&+ [1 - q(1, \ell_1) - q(1, \ell_2) - q(\ell_1 + 1, \ell_3 - 1)][d + c] + \cdots \\
&+ [1 - q(1, \ell_1) - q(1, \ell_2) - q(\ell_1 + 1, \ell_3) - \cdots - q(\ell_{m-2}, \ell_m)][(\ell_m + \ell_{m-1} + 1)d + c] \\
&+ [1 - q(1, \ell_1) - q(1, \ell_2) - q(\ell_1 + 1, \ell_3) - \cdots - q(\ell_{m-2}, \ell_m) - p_{\ell_{m-1}+1}][d + c] \\
&+ \cdots \\
&+ [1 - q(1, \ell_1) - q(1, \ell_2) - q(\ell_1 + 1, \ell_3) - \cdots - q(\ell_{m-2}, \ell_m) - q(\ell_{m-1} + 1, n - 1)][d + c],
\end{aligned} \tag{3.1}$$

or

$$\begin{aligned}
E(s,c,d) &= \sum_{k=1}^{\ell_1} k(d + c)p_k + \sum_{k=1}^{\ell_2} [(2\ell_1 + k)d + (\ell_1 + k)c]p_k \\
&+ \sum_{k=\ell_1+1}^{\ell_3} [(2\ell_1 + 2\ell_2 + k)d + (\ell_2 + k)c]p_k + \cdots \\
&+ \sum_{k=\ell_{m-1}+1}^{n} [(2\ell_1 + 2\ell_2 + \cdots + 2\ell_m + k)d + (n + k)c]p_k \\
&= 2dA(s) + cB(s) + 2(d + c) \sum_{k=1}^{n} kp_k,
\end{aligned}$$

where $A(s) = \sum_{k=1}^{m} \sum_{j=1}^{k} \ell_j q(\ell_{k-1} + 1, \ell_{k+1})$, $B(s) = \sum_{k=1}^{m} \ell_k q(\ell_{k-1} + 1, \ell_{k+1})$, $q(i,j) = \sum_{k=i}^{j} p_k$, $\ell_0 = 0$ and $\ell_{m+1} = n$. Without loss of generality, we can assume $d = 1$. Then c can be alternatively interpreted as the ratio of fixed examination cost to traveling cost. Thus we have $E(s, c) \equiv E(s, c, 1) = e(s, c) + 2(1 + c) \sum_{k=1}^{n} kp_k$ where

$$e(s,c) = 2A(s) + cB(s). \tag{3.2}$$

It must be noted that $A(s) > 0$ and $B(s) > 0$ for all s in S.

Lemma 1. $s_1 = [n]$ is not optimal if

$$p_n < \frac{1 + c/2}{4n - 2 + nc}. \tag{3.3}$$

Proof: Let $s = [n-1, n]$. From (3.1) and (3.2) we have $e(s,c) = (n-1)(2+c)/2 + (4n - 2 + nc)p_n$ and $e(s_1, c) = n(2+c)/2$. s_1 is not optimal if $e(s_1, c) > e(s,c)$, that is, if condition (3.3) holds. **Q.E.D.**

Lemma 1 asserts that if p_n is appropriately small then he should turn around before examining cell n or cell $-n$. For example, let $p_i = (n+1-i)/(n^2+n)(1 \le i \le n)$ then condition (3.3) holds if $n \ge 3$.

Suppose c is fixed. By $S(c)$ and $S_m(c)(1 \le m \le n)$ denote the set of all optimal search plans and the set of all m-optimal $(1 \le m \le n)$ search plans respectively.

For $s, s' \in S$, let $R(s,s') \equiv 2(A(s') - A(s))/(B(s) - B(s'))$. For $s \in S$, let $L(s) \equiv \max\{R(s,s') : s' \in S, B(s) < B(s')\}$ and $U(s) \equiv \min\{R(s,s') : s' \in S, B(s) > B(s')\}$.

Lemma 2. $s \in S(c)$ if and only if
(i) $L(s) \le c \le U(s)$ and,
(ii) if there is $s' \in S$ such that $B(s) = B(s')$ then $A(s) \le A(s')$.

Proof: $s \in S(c)$ if and only if

$$2A(s) + cB(s) \le 2A(s') + cB(s') \text{ for all } s' \in S,$$

i.e., $$c(B(s) - B(s')) \le R(s,s')(B(s) - B(s')) \text{ for all } s' \in S.$$

Also S is a finite set. **Q.E.D.**

Thus $s \in S$ can be optimal for some $c \ge 0$ if $L(s) \le U(s)$ and Lemma 2 (ii) holds. For $s \in S$, let $I(s) \equiv \{c \ge 0 : s \in S(c)\}$.

Lemma 3. (i) For every $s \in S$, either $I(s) = \phi$ or $I(s) = [L(s), U(s)]$.
(ii) For $s, s' \in S$, one out of the three cases holds: (a) $I(s) \cap I(s') = \phi$, (b) $I(s) \cap I(s')$ is a one-point set, (c) $I(s) = I(s')$.

Proof: (i) From Lemma 2 (i), $I(s) \subset [L(s), U(s)]$. Assume $I(s) \ne \phi$. Assume $[L(s), U(s)] \backslash I(s) \ne \phi$. Then from Lemma 2 (ii), there is $s^* \in S$ such that $B(s) = B(s^*)$ and $A(s) > A(s^*)$. This implies $e(s,c) > e(s^*, c)$ for all $c \ge 0$. This contradicts $I(s) \ne \phi$. Hence $[L(s), U(s)] \backslash I(s) = \phi$.
(ii) Assume $c, c' \in I(s) \cap I(s')$. Then $e(s,c) = e(s',c)$ and $e(s,c') = e(s',c')$. These imply $(c - c')B(s) = (c - c')B(s')$. Hence if $c \ne c', B(s) = B(s')$ and $A(s) = A(s')$. Hence $I(s) = I(s')$. **Q.E.D.**

Lemma 4. (i) Suppose $c_1 < c_2$. Suppose s and s' are in $S(c_1)$ and $S(c_2)$ respectively. Then $A(s) \le A(s')$ and $B(s) \ge B(s')$. If either s is not in $S(c_2)$ or s' is not in $S(c_1)$, then $B(s) > B(s')$ and $A(s) < A(s')$.
(ii) Let S^B be the set of all search plans such that they minimize $B(\cdot)$, i.e., $S^B \equiv \{s \in S : B(s) = \min\{B(s') : s' \in S\}\}$. Let $S^{B,A} \equiv \{s \in S^B : A(s) = \min\{A(s') : s' \in S^B\}\}$. Then there exists c^* such that $S^{B,A} \subset S(c)$ if and only if $c \ge c^*$.
(iii) $s_n \in S^B$. Further there exist $\{p_1, \cdots, p_n\}$ such that s_1 is not in $S(0)$ and $\{p_1, \cdots, p_n\}$ such that s_n is not in $S(c)$ for any $c \ge 0$.

Proof: (i)

$$2A(s) + c_1 B(s) \le 2A(s') + c_1 B(s') < 2A(s') + c_2 B(s')$$

$$\leq 2A(s) + c_2B(s)$$

since s is in $S(c_1), c_1 < c_2, B(s') > 0$ and s' is in $S(c_2)$. From these inequalities we have $(c_2 - c_1)B(s) \geq (c_2 - c_1)B(s')$, which implies $B(s) \geq B(s')$ since $c_2 > c_1$. From the first inequality we have $2A(s') - 2A(s) \geq c_1(B(s) - B(s'))$. This implies $A(s') \geq A(s)$ since $c_1 \geq 0$ and $B(s) \geq B(s')$. Furthermore, if s is not in $S(c_2)$, then the last inequality of the first three inequalities strictly holds. Hence $(c_2 - c_1)B(s) > (c_2 - c_1)B(s')$, from which $B(s) > B(s')$. We also have $c_1(B(s) - B(s')) \leq 2A(s') - 2A(s)$. This implies $A(s') > A(s)$. These are also proved in the case that s' is not in $S(c_1)$.

(ii) By the definition,

$$B(s) < B(s') \text{ for all } s \text{ in } S^B \text{ and all } s' \text{ in } S\backslash S^B. \tag{3.4}$$

Since S is a finite set and $A(\cdot)$ is bounded, $S^{B,A} \subset S(c)$ if c is sufficiently large. Let $c^* = \inf\{c : S^{B,A} \subset S(c)\}$. Since $2A(s) + cB(s)$ is continuous in c and S is a finite set, $c^* = \min\{c : S^{B,A} \subset S(c)\}$. Suppose $c > c^*$. Then for any s in $S^{B,A}$ and any s' in $S\backslash S^{B,A}$,

$$\begin{aligned} 2A(s') + cB(s') &= 2A(s') + c^*B(s') + (c - c^*)B(s') \\ &\geq 2A(s) + c^*B(s) + (c - c^*)B(s) \\ &= 2A(s) + cB(s) \end{aligned}$$

by (3.4) and the fact that s is in $S(c^*)$. Hence s is in $S(c)$.

(iii) By the definition of $B(\cdot)$, for any $s \in S$,

$$\begin{aligned} B(s) - B(s_n) &= \sum_{k=1}^{m} \ell_k q(\ell_{k-1} + 1, \ell_{k+1} - \sum_{k=1}^{n} k(p_k + p_{k+1}) \\ &= \sum_{k=1}^{m} \sum_{j=\ell_{k-1}+1}^{\ell_k} p_j(\ell_{k-1} + \ell_k) - \sum_{j=1}^{n} p_j(2j - 1) \\ &= \sum_{k=1}^{m} \sum_{j=\ell_{k-1}+1}^{\ell_k} p_j[\ell_{k-1} + \ell_k - (2j - 1)] \\ &= \sum_{k=1}^{m} \sum_{j=\ell_{k-1}+1}^{t_k} (p_j - p_{r(j)})[(\ell_k - j) - (j - \ell_{k-1} - 1)] \\ &\geq 0 \end{aligned} \tag{3.5}$$

since $j \leq t_k$ and $p_j \geq p_{r(j)}$ for all j and all k, where $t_k \equiv \lfloor \frac{\ell_{k-1}+\ell_k+1}{2} \rfloor$ is the greatest integer less than or equal to $\frac{\ell_{k-1}+\ell_k+1}{2}$, and $r(j) \equiv \ell_k + \ell_{k-1} + 1 - j$. From Lemma 1, if $p_n < 1/(4n-2)$ then s_1 is not in $S(0)$. Next assume $p_1 = p_2 = \cdots = p_{n-1}$ and $0 < p_n < 1/(4n-2)$. Let $s = [2, 3, \cdots, n-1, n]$. Then by the definition of $B(\cdot), B(s) = 2q(1,3) + 3q(3,4) + 4q(4,5) + \cdots + (n-1)q(n-1,n) + nq(n,n)$ and $B(s_n) = q(1,2) + 2q(2,3) + 3q(3,4) + 4q(4,5) + \cdots + (n-1)q(n-1,n) + nq(n,n)$. Hence if $n \geq 3$ then $p_1 = p_2$ and $B(s) = B(s_n)$. Further $A(s_n) = A(s) + q(3,4) + q(4.5) + \cdots + q(n-1,n) + q(n,n) + q(1,2) + 3q(2,3) - 2q(1,3) > A(s)$ if $n \geq 3$. Hence $s_n \notin S^{B,A}$. **Q.E.D.**

The values of c^* are given in Table 1 below when n is small and $p_i = (n+1-i)/(n^2+n)(1 \leq i \leq n)$. $S^B = S^{B,A} = \{s_n\}$. If $c \geq c^*$ then s_n is optimal, i.e., $s_n \in S(c)$.

Table 1.

n	5	10	15	20	25	30
c^*	40	322	1078	2520	4862	8358

From Lemma 4 we see that the problem remains to be solved when c is appropriately small. Suppose $c = 0$. Intuitively s_1 seems to be optimal. But, in general, a priori probabilities affect the choice of strategy. Lemma 4 (iii) asserts this fact. For example let $n = 10, c = 0$ and $p_i = (n+1-i)/(n^2+n)(1 \le i \le n)$. $s = [7, 10]$ is optimal and $s_1 = [10]$ not. p_8, p_9 and p_{10} have relatively small values (See Table 2).

Thus far we have seen that s_n minimizes $B(\cdot)$ but in general neither s_1 nor s_n minimizes $A(\cdot)$. From Lemmas 2, 3, and 4, we have a theorem.

Theorem 5. There are numbers $c_0, c_1, c_2, \cdots, c_k$ and search plans $s^{(0)}, s^{(1)}, \cdots, s^{(k)}$ such that

$c_0 = 0 < c_1 < c_2 < \cdots < c_k < c_{k+1} = +\infty$, and
$s^{(i)} \in S(c)$ if and only if $c \in [c_i, c_{i+1}]$ for $i = 0, 1, \cdots, k$, and
$A(s^{(0)}) < A(s^{(1)}) < \cdots < A(s^{(k)})$, and
$B(s^{(0)}) > B(s^{(1)}) > \cdots > B(s^{(k)})$, and
$e(s^{(0)}, c_0) < e(s^{(1)}, c_1) < \cdots < e(s^{(k)}, c_k)$.

Proof: Since S is a finite set, there is at least one $s \in S$ such that $I(s)$ has an interior point. Starting at $c = c_0 = 0$, we can find finite sequences of numbers and search plans as above. Here $c_i = L(s^{(i)}) = U(s^{(i-1)})$ for $i = 1, \cdots, k$. Also $s^{(k)} = s_n$ and c_k is equal to c^* in Lemma 4 (ii), $c_i \ne c_{i+1}$ for $i = 0, \cdots, k$ and Lemma 4 (i) imply $A(s^{(i)}) < A(s^{(i+1)})$ and $B(s^{(i)}) > B(s^{(i+1)})$ for $i = 0, \cdots, k-1$. $e(s^{(i)}, c_i) < e(s^{(i+1)}, c_{i+1})$ since $e(s, c)$ is increasing in c for any $s \in S$. **Q.E.D**

Remark 6. For each s in $S, g(s) \equiv (A(s), B(s))$ can be interpreted as a point in R^2_+. Let $G(S) \equiv \{g(s) : s \in S\}$ and $\overline{G}(S)$ be the convex hull of $G(S)$. s is called efficient if $g(s)$ is an efficient point of $\overline{G}(S)$, that is, there is no $x = (x_1, x_2)$ in $\overline{G}(S)$ such that either $x_1 < A(s)$ and $x_2 \le B(s)$ or $x_1 \le A(s)$ and $x_2 < B(s)$. Then we see easily that s is in $S(c)$ for some c if and only if s is efficient. Moreover from Theorem 5, as c increases $g(s)(s \in S(c))$ moves in the south-east direction. Since s_n is in $S(c)$ if c is sufficiently large (Lemma 4 (ii)), an algorithm may be considered such that it departs from s_n and traces the efficient search plans as c decreases. This algorithm, however, is not pursued in this note.

Example 7. Let $n = 5$ and $p_i = (n+1-i)/(n^2+n)$ for $i = 1, \cdots, 5$. There are $2^5 = 16$ search plans. By calculating directly $A(s)$ and $B(s)$ for each s, we see that $c_1 = 4, c_2 = 20, c_3 = 40, s^{(0)} = [3,5], s^{(1)} = [2,4,5], s^{(2)} = [1,3,4,5]$, and $s^{(3)} = [1,2,3,4,5]$. $e(s^{(0)}, c) = 4.6 + 2c, e(s^{(1)}, c) = 5 + 1.9c, e(s^{(2)}, c) = 5.666 \cdots + 1.8666 \cdots \times c$, and $e(s^{(3)}, c) = 7 + 1.8333 \cdots \times c$.

Comparing the expected costs directly we have the next necessary condition for a search plan to be optimal. In other words we are considering difference inequalities.

Theorem 8. If $s = [\ell_1, \cdots, \ell_m]$ is optimal then $p_{i(k)} > p_{i(k)+1}$ for all $k = 1, \cdots, m-1$ where $i(k) = \ell_k$.

Proof: By $s_k(\ell)(1 \le k \le m-1)$ we mean a search strategy such that only ℓ_k is replaced by $\ell(\ell_{k-1} \le \ell \le \ell_{k+1})$ and the others are the same as those of s. If either $\ell = \ell_{k+1}$ or $\ell = \ell_{k-1}$ then $s_k(\ell)$ is not in S. But even if not, the expected cost for $s_k(\ell)$ can be calculated. Observing the symmetry of a priori probabilities, that is, assumption (2.2), we see that $s_k(\ell)$ is redundant in this case. For example, suppose that he examined cells 1 and 2 in this order, then went back to cell 0, then examined cells -1 and -2 in this order, and he is now at cell -2 . The object has not been detected. Then clearly he should not turn around but examine cell -3. Hence if $s_k(\ell)$ is redundant there should exist s' in S such that $e(s', c) \le e(s_k(\ell), c)$. We finally have

$$e(s,c) \le e(s_k(\ell_k+1), c) \text{ and } e(s,c) \le e(s_k(\ell_k-1), c) \tag{3.6}$$

for all $k(1 \le k \le m-1)$ if s is optimal. (3.6) becomes $p_{i(k)+1} \le Q/L$ and $p_{i(k)} \ge (2p_{i(k)} + 2p_{i(k)+1} + Q)/L$ where $Q \equiv 2q(\ell_k+2, n) + 2q(\ell_{k-1}+1, n) + cq(\ell_{k-1}+1, \ell_{k+1})$ and $L \equiv 2(\ell_k + \ell_{k+1}) + c(\ell_{k+1} - \ell_{k-1})$. By (1) and the definition of search plan, Q and L are positive. This with (2.1) implies $p_{i(k)} > Q/L \ge p_{i(k)+1}$. **Q.E.D.**

Corollary 9. (i) Assume $p_1 = \cdots = p_n$. Then $s_1 \equiv [n] \in S(c)$ for all c. Furthermore $S^B = S$ and $S^{B,A} = \{s_1\}$.
(ii) Assume $p_1 > p_2 > \cdots > p_n > 0$. Then $S^B = S^{B,A} = \{s_n\}$.

Proof: (i) By theorem 8 and the definitions of $B(\cdot)$ and $A(\cdot)$.
(ii)By (3.5), $B(s_n) = B(s)$ and $p_1 > p_2 > \cdots > p_n$ imply $s = s_n$. **Q.E.D.**

4. An Approach by Dynamic Programming.

The purpose of this section is to develop a method of finding optimal search plans. First we have the recursive relation, applying Bellman's Principle of Optimality ([3]). Second we examine the case of $c = 0$, applying this relation, since this case is not clear intuitively.

Let $p_j^{x-1}(x \le |j| \le n)$ be the conditional probability that the object is in cell j, given that all cells labeled from $-(x-1)$ to $x-1$ were examined, the object was not detected, and he is at cell $x-1$ now. That is, $p_j^{x-1} = p_j/[1 - 2q(1, x-1)]$. This probability depends on only x. Observing this, we define by $W(x)(1 \le x \le n)$ the expected cost of examining $2(n-x+1)$ cells under an optimal search plan, given the same condition as in the definition of p_j^{x-1}. We let

$$W(n+1) = 0. \tag{4.1}$$

By Bellman's Principle of Optimality, we have

$$\begin{aligned} W(x) = \min_{x \le \ell \le n} \{ &(1+c) \sum_{j=x}^{\ell} (j - x + 1) p_j^{x-1} \\ &+ \sum_{j=x}^{\ell} (j + 2\ell - x + 1) p_j^{x-1} + c \sum_{j=x}^{\ell} (j + \ell - 2x + 2) p_j^{x-1} \\ &+ [3\ell - x + 1 + 2(\ell - x + 1)c + W(\ell+1)] \times 2q^{x-1}(\ell+1, n) \} \end{aligned} \tag{4.2}$$

where $q^{x-1}(i,j) = \sum_{k=i}^{j} p_k^{x-1}$. In the right hand side ℓ is the control variable. After examining each cell from x to ℓ, he goes back to cell $-x$, then examines each cell from $-x$ to $-\ell$, and follows an optimal search plan there after.

For simplicity of calculation let $V(x) = 2q(x,n)W(x)+(2x-2)q(x,n)+c(3x-3)q(x,n)+2(1+c)\sum_{j=1}^{x-1} jp_j$. Then (4.2) becomes

$$V(x) = \min_{x\le \ell\le n}\{(2\ell + c(1-x))q(\ell+1,n) + (2+c)\ell q(x,n) + V(\ell+1)\} \tag{4.3}$$

and

$$V(n+1) = 2(1+c)\sum_{j=1}^{n} jp_j. \tag{4.4}$$

Thus we can find recursively $\ell_m = n, \ell_{m-1}, \cdots, \ell_2, \ell_1$, which give an optimal search plan. When n is small, (4.3) and (4.4) are very useful for numerical computation. Let $p_i = (n+1-i)/(n^2+n)(1 \le i \le n)$. Table 2 below indicates optimal search plans when $n = 10$.

Table 2.

optimal search plan	Range of c
[7,10]	$0 - 0.4$
[6,10]	$0.4 - 28/15 = 1.87$
[5,9,10]	$1.87 - 44/9 = 4.89$
[4,8,10]	$4.89 - 34/3 = 11.33$
[3,7,9,10]	$11.33 - 52/3 = 17.33$
[3,6,9,10]	$17.33 - 25$
[3,6,8,10]	$25 - 30$
[1,4,7,9,10]	$30 - 31$
[2,5,7,9,10]	$31 - 178/3 = 59.33$
[2,4,6,8,9,10]	$59.33 - 106$
[1,3,5,7,8,9,10]	$106 - 214$
[1,2,4,6,7,8,9,10]	$214 - 286$
[1,2,3,5,6,7,8,9,10]	$286 - 322$
[1,2,3,4,5,6,7,8,9,10]	$322 - +\infty$

Define $U(x) \equiv V(x+1) - V(n+1)$ for $x = 0, \cdots, n-1$. Also define $f(x,y) \equiv (2y - cx)q(y+1,n) + (2+c)yq(x+1,n)$ for $0 \le x < y \le n$. Then (4.3) and (4.4) become

$$U(x) = \min_{x+1\le y\le n}\{f(x,y) + U(y)\}, \text{and} \tag{4.5}$$

$$U(n) = 0. \tag{4.6}$$

Example 7 (Continued). Suppose $c = 0$. $f(x,y) = y\{(x-6)(x-5) + (y-6)(y-5)\}/30$. Let $a_y \equiv 30U(y)$. Then $a_5 = 0, a_4 = 10, a_3 = \min\{32 + a_4, 30 + a_5\}, a_2 = \min\{54 + a_3, 56 + a_4, 60 + a_5\}, a_1 = \min\{64 + a_2, 78 + a_3, 88 + a_4, 100 + a_5\}$, and $a_0 = \min\{50 + a_1, 84 + a_2, 108 + a_3, 128 + a_4, 150 + a_5\}$. From these we see [3, 5] and [4, 5] are optimal search plans. $a_0 = 138$. Hence $U(0) = 138/30 = 4.6 = e(s^{(0)}, 0)$.

The following theorem states a property of $f(\cdot)$ or $U(\cdot)$.

Theorem 10. Suppose the optimality equation is given as (4.5) with (4.6), without giving $f(\cdot)$ explicitly. Assume $f_{x,z}(y) \equiv f(x,y) - f(z,y)$ is strictly increasing in y, where $x < z \leq y \leq n$. Then there are $y^{(0)}, y^{(1)}, \cdots, y^{(k)}$ such that

(i) $0 < y^{(k)} < y^{(k-1)} < \cdots < y^{(0)} < n$, and

(ii) for $j = 0, \cdots, k+1$, for all x with $y^{(j)} \leq x < y^{(j-1)}$,

$$U(x) = f(x,y) + U(y) \tag{4.7}$$

for some $y \equiv y(x)$ with $y^{(j-1)} \leq y < y^{(j-2)}$, where $y^{(k+1)} = 0, y^{(-1)} = n$ and $y^{(-2)} = +\infty$. Moreover $x < x'$ implies $y(x) \leq y(x')$.

Proof: From (4.5) and (4.6), $y(n-1) = n$. Suppose $y(x) = n$ for some $x \leq n-1$. Assume $y(w) = n$ for $w = z+1, \cdots, n-1$, where $x \leq z$. Let's check if $y(z) = n$. For w with $z+1 \leq w \leq n-1$,

$$\begin{aligned} f(z,w) + U(w) - f(z,n) &\geq f(z,w) + f(x,n) - f(x,w) - f(z,n) \\ &= f_{x,z}(n) - f_{x,z}(w) > 0. \end{aligned}$$

Hence $y(z) = n$. Thus there exists $y^{(0)}$ such that $0 \leq y^{(0)} \leq n-1, y(x) = n$ if and only if $y^{(0)} \leq x$. $y^{(0)} = 0$ means $k+1 = 0$.

Assume $y^{(0)} > 0$. $U(y^{(0)} - 1) = f(y^{(0)} - 1, w) + f(w,n)$ by the definition of $y^{(0)}$ and since $y^{(0)} \leq w \leq n-1$. This implies $y(y^{(0)} - 1) = w$. Suppose $y^{(0)} \leq y(x)$ for some $x \leq y^{(0)} - 1$. We show $y(z) \geq y^{(0)}$ for all z such that $x \leq z \leq y^{(0)} - 1$. Assume $y^{(0)} \leq y(w)$ for $w = z+1, \cdots, y^{(0)} - 1$, where $z \geq x$. Let's check if $y^{(0)} \leq y(z)$. Let

$$f_1(t) \equiv f(z,t) + U(t) - \min\{f(z,u) + f(u,n) : y^{(0)} \leq u < n\}.$$

If $t \geq y^{(0)}$ then $f_1(t) \geq 0$ by the definition of $f_1(t)$. If $z+1 \leq t < y^{(0)}$, then

$$\begin{aligned} f_1(t) &\geq f(z,t) + [f(x,y(x)) + f(y(x),n) - f(x,t)] - \min\{f(z,u) + f(u,n) : y^{(0)} \leq u < n\} \\ &\geq f(z,t) + [f(x,y(x)) + f(y(x),n) - f(x,t)] - [f(z,y(x)) + f(y(x),n)] \\ &= f_{x,z}(y(x)) - f_{x,z}(t) > 0. \end{aligned}$$

Hence we have $U(z) = f(z,y(z)) + f(y(z),n)$ and $y^{(0)} \leq y(z)$.

Moreover, by the definition of $y(z)$,

$$\begin{aligned} 0 &\leq f(z,y(x)) + f(y(x),n) - [f(z,y(z)) + f(y(z),n)] \\ &\leq f(z,y(x)) + [f(x,y(z)) + f(y(z),n) - f(x,y(x))] - [f(z,y(z)) + f(y(z),n)] \\ &= f_{x,z}(y(z)) - f_{x,z}(y(x)). \end{aligned}$$

Hence we must have $y(z) \geq y(x)$ since $f_{x,z}$ is strictly increasing.

Thus there is $y^{(1)}$ such that $0 \leq y^{(1)} < y(0) < n$, and $y^{(0)} \leq y(x) < n$ if and only if $y^{(1)} \leq x < y^{(0)}$.

Assume for $j = m-2, m-3, \cdots, 0$, it holds $y^{(j-1)} \leq y(x) < y^{(j-2)}$ for every x with $y^{(j)} \leq x < y^{(j-1)}$. Suppose $x < y^{(m-2)}$ and $y^{(m-2)} \leq y(x) < y^{(m-3)}$. By the definition of $y^{(m-2)}$, we have $y^{(m-2)} \leq y(y^{(m-2)} - 1) < y^{(m-3)}$. Assume $y^{(m-2)} \leq y(w) < y^{(m-3)}$ for $w = z+1, \cdots, y^{(m-2)} - 1$, where $x \leq z$. Let's check if $y^{(m-2)} \leq y(z) < y(m-3)$. Let

$$f_2(t) \equiv f(z,t) + U(t) - \min\{f(z,u) + U(u) : y^{(m-2)} \leq u < y^{(m-3)}\}.$$

If $t \geq y^{(m-3)}$ then $f_2(t) \geq 0$ since $z < y^{(m-2)}$. If $y^{(m-2)} \leq t < y^{(m-3)}$ then $f_2(t) \geq 0$ by the definition of $f_2(t)$. If $y^{(m-2)} > t > z$, then

$$\begin{aligned} f_2(t) &\geq f(z,t) + [f(x,y(x)) + U(y(x)) - f(x,t)] \\ &\quad - \min\{f(z,u) + U(u) : y^{(m-2)} \leq u < y^{(m-3)}\} \\ &\geq f(z,t) + [f(x,y(x)) + U(y(x)) - f(x,t)] - [f(z,y(x)) + U(y(x))] \\ &= f_{x,z}(y(x)) - f_{x,z}(t) > 0. \end{aligned}$$

Hence $y^{(m-2)} \leq y(z) < y^{(m-3)}$.

Moreover

$$\begin{aligned} 0 &\leq f(z,y(x)) + U(y(x)) - [f(z,y(z)) + U(y(z))] \\ &\leq f(z,y(x)) + [f(x,y(z)) + U(y(z)) - f(x,y(x))] - [f(z,y(z)) + U(y(z))] \\ &= f_{x,z}(y(z)) - f_{x,z}(y(x)). \end{aligned}$$

This implies $y(z) \geq y(x)$ since $f_{x,z}$ is strictly increasing. **Q.E.D.**

In our model,

$$f_{x,z}(y) = c(z-x)/2 - c(z-x)q(1,y) + (2+c)yq(x+1,z).$$

For $y > y' \geq z$,

$$\begin{aligned} & f_{x,z}(y) - f_{x,z}(y') = (2+c)(y-y')q(x+1,z) - c(z-x)q(y'+1,y) \\ &= (y-y')(z-x)\{(2+c)q(x+1,z)/(z-x) - cq(y'+1,y)/(y-y')\} \\ &> 0 \end{aligned}$$

by (2.1) and (2.4). Thus $f_{x,z}$ is strictly increasing, and Theorem 10 applies.

An optimal search plan, $s = [\ell_1, \cdots, \ell_m] \in S(c)$, gives $U(0)$, that is, $U(0) = f(0,\ell_1) + f(\ell_1,\ell_2) + \cdots + f(\ell_{m-1},\ell_m)$.

Here we generalize the notation. Let $S(x)$ be the set of all search plans after starting at $x-1$, given that all cells labeled from $-(x-1)$ to $x-1$ were examined, the object was not detected, and he is at cell $x-1$ now. Let $S(x,c)$ be the set of all optimal search plans after starting at $x-1$, under the same assumption. Define $S_m(x)$ and $S_m(x,c)$ in the same way. Note that $S = S(0), S_m = S_m(0), S(c) = S(0,c)$, and $S_m(c) = S_m(0,c)$. Thus $U(x) = f(x,\ell_1) + \cdots + f(\ell_{m-1},\ell_m)$ for some $[\ell_1,\cdots,\ell_m] \in S(x,c)$, where $x < \ell_1 < \ell_2 < \cdots < \ell_m = n$.

Lemma 11. Suppose $[\ell_1,\cdots,\ell_m] \in S_m(x)$, where $x < \ell_1 < \ell_2 < \cdots < \ell_m = n$. Assume

(i) $f(x,\ell_1) + \cdots + f(\ell_{m-1},\ell_m) \leq f(x,r_1) + \cdots + f(r_{j-1},r_j)$ for all $[r_1,\cdots,r_j] \in S_{m+1}(x)$, and

(ii) $\ell_1 \geq y^{(m-2)}$.

Then $U(x) = f(x,\ell_1) + \cdots, + f(\ell_{m-1},\ell_m)$. That is, $[\ell_1,\cdots,\ell_m] \in S(x,c)$.

Proof: From assumption (i) we see $x < y^{(m-2)}$. $\ell_1 < y^{(m-3)}$ by the definition of $y^{(m-3)}$,

$\cdots, y^{(0)}$. Let $z = y^{(m-2)} - 1$. Then $y^{(m-1)} \le z < y^{(m-2)}$, and there is $[r_1, \cdots, r_m] \in S_m(z, c)$. Let $z = y^{(m-2)} - 2$. Suppose $x \le z$. Assumption (ii) implies $z < \ell_1$. Thus

$$\begin{aligned}
&f(z, z+1) + U(z+1) - [f(z, \ell_1) + f(\ell_1, \ell_2) + \cdots + f(\ell_{m-1}, \ell_m)] \\
&\ge f(z, z+1) + [f(x, \ell_1) + \cdots + f(\ell_{m-1}, \ell_m) - f(x, z+1)] \\
&\quad - [f(z, \ell_1) + f(\ell_1, \ell_2) + \cdots + f(\ell_{m-1}, \ell_m)] \\
&= f(z, z+1) + f(x, \ell_1) - f(x, z+1) - f(z, \ell_1) \\
&= f_{x,z}(\ell_1) - f_{x,z}(z+1) > 0.
\end{aligned}$$

This implies

$$f(z, z+1) + U(z+1) > \min\{f(z, w) + U(w) : y^{(m-2)} \le w < y^{(m-3)}\},$$

which implies $y^{(m-1)} \le z < y^{(m-2)}$, and there is $[r_1, \cdots, r_m] \in S_m(z, c)$.

Assume for $y = z+1, \cdots, y^{(m-2)} - 1$,

$$f(y, w) + U(w) \ge \min\{f(y, w') + U(w') : y^{(m-2)} \le w' < y^{(m-3)}\}.$$

Let $y = z$. For w with $z + 1 \le w < y^{(m-2)}$,

$$\begin{aligned}
&f(z, w) + U(w) - [f(z, \ell_1) + f(\ell_1, \ell_2) + \cdots + f(\ell_{m-1}, \ell_m)] \\
&\ge f(z, w) + [f(x, \ell_1) + \cdots + f(\ell_{m-1}, \ell_m) - f(x, w)] \\
&\quad - [f(z, \ell_1) + f(\ell_1, \ell_2) + \cdots + f(\ell_{m-1}, \ell_m)] \\
&= f(z, w) + f(x, \ell_1) - f(x, w) - f(z, \ell_1) = f_{x,z}(\ell_1) - f_{x,z}(w) > 0.
\end{aligned}$$

Hence
$$\begin{aligned}
f(z, w) + U(w) &> f(z, \ell_1) + f(\ell_1, \ell_2) + \cdots + f(\ell_{m-1}, \ell_m) \\
&\ge \min\{f(z, w') + U(w') : y^{(m-2)} \le w' < y^{(m-3)}\}.
\end{aligned}$$

This implies $y^{(m-2)} \le y(z) < y^{(m-3)}$ and $y^{(m-1)} \le z < y^{(m-2)}$. By letting $z = x$, we have the desired result. **Q.E.D.**

The converse of Lemma 11 is clearly seen by the definition of $U(\cdot)$. Thus we have

Theorem 12. $[\ell_1, \cdots, \ell_m] \in S(x, c)$ if and only if (i) and (ii) in Lemma 11 hold.

Corollary 13. For each $x : 1 \le x \le n-1, U(x) = f(x, n)$ if and only if

$$f(x, n) \le f(x, y) + f(y, n) \text{ for } y : x+1 \le y \le n. \tag{4.8}$$

Proof: $\ell_1 = n$. Thus Assumption (ii) in Lemma 11 is satisfied. (4.8) is just Assumption (i) in Lemma 11 **Q.E.D.**

If we let $x = 0$ in Corollary 13, we have the next proposition.

Corollary 14. $[n]$ is optimal if and only if

$$q(1, y) \le \frac{4y + cy}{4n + 4y + 2nc} \text{for } y = 1, \cdots, n.$$

Proof: (4.8) is rewritten. **Q.E.D.**

In particular, if $c = +\infty$, then $q(1,y) \le \frac{y}{2n}$ for $y = 1,\cdots,n$. This, (2.3), and (2.4) imply $p_i = \frac{1}{2n}$ for $i = 1,\cdots,n$ (See Corollary 9).

Example 7. (Continued). Assume $c = 0$. Then $U(x) = f(x,5)$ for $2 \le x \le 5, U(1) = f(1,4) + U(4)$ and $U(0) = f(0,3) + U(3)$. Thus $y^{(0)} = 2$.

Example 15. Suppose $p_i = b - ai$ for $i = 1,\cdots,n$. Assume $c = 0$. Let $T(y) \equiv nU(y)$ for all y. Then $T(n) = nf(n,n) = 0$ and $T(n-1) = nf(n-1,n) = n - an^2(n-1)$. (4.5) becomes $T(x) = \min_{x+1\le y\le n}\{v(x,y)\}$, where $v(x,y) \equiv y(n-y)(1-ay) + y(n-x)(1/n - ax) + T(y)$. Assume $T(y) = nf(y,n)$ for all $y : x+1 \le y \le n$. Thus $v(x,y) = (n-y)(1-nay)(n+y) + y(n-x)(1-nax)$. After simple calculation, we see $v(x,n) \le v(x,y) + v(y,n)$ for all $y : x+1 \le y \le n-1$ if and only if

$$x^2 + y^2 - (\frac{1}{na} + n)x + (n - \frac{1}{na})y \le 0 \text{ for all } y : x+1 \le y \le n-1. \tag{4.9}$$

This is equivalent to:

$$x^2 + (n-1)^2 - (\frac{1}{na} + n)x + (n - \frac{1}{na})(n-1) \le 0. \tag{4.10}$$

since the left hand side of (4.9) is incresing in y if $y \ge n - 1/(na)$. Here $a < 1/(n^2 - n)$ implies $n - 1/(na) < 1$. From (4.10) we have $x \ge [n + 1/(na) - \sqrt{\eta(n,a)}]/2$, where $\eta(n,a) \equiv -7n^2 + 12n - 4 + 6/a - 4/(na) + 1/(n^2a^2)$. Thus

$$y^{(0)} = \text{Int}[\frac{n}{2} + \frac{1}{2na} - \frac{\sqrt{\eta(n,a)}}{2}] + 1. \tag{4.11}$$

Assume

$$\lim_{n\to\infty} n^2 a = t. \tag{4.12}$$

$0 \le a < 1/(n^2 - n)$ implies $0 \le t \le 1$. From (4.11) and (4.12) we have

$$\lim_{n\to\infty} \frac{y^{(0)}}{n} = \lambda(t) \equiv \frac{4t-2}{t+1+\sqrt{1+6t-7t^2}}. \tag{4.13}$$

$t < 1/2$ implies $\lambda(t) < 0$. $\lambda(1/2) = 0$, and $\lambda(1) = 1$. $\lambda'(t) = \frac{6+10/\sqrt{1+6t-7t^2}}{(t+1+\sqrt{1+6t-7t^2})^2} > 0$. Hence $\lambda(\cdot)$ is increasing in t.

For example, suppose $a = \frac{1}{n^2+n}$. Then from(4.11), $y^{(0)} = \text{Int}[n + \frac{1}{2} - \sqrt{4n-7}/4] + 1$. From (4.12), we have $t = 1$, and $\lambda(1) = 1$. $n = 10$ implies $y^{(0)} = 5$. Remember that [7, 10] is optimal (Table 2).

From Corollary 14, [n] is optimal if and only if $0 \le a \le \frac{2}{4n^2+cn^2-2n}$.

Proposition 16. Assume c is sufficiently large. Then for $y = 0,\cdots,n-1, U(y+1) - U(y) = f(y,y+1)$.

Proof: We can put $f(x,y) = cyq(x+1,n) - cxq(y+1,n)$. Then $f(x,x+1) + f(x+1,x+2) = c(x+1)q(x+1,n) - cxq(x+2,n) + c(x+2)q(x+2,n) - c(x+1)q(x+3,n) = c(x+2)q(x+1,n) -$

$cxq(x+3,n)-cq(x+1,n)-cq(x+3,n)+2cq(x+2,n) = f(x,x+2)-cp_{x+1}+cp_{x+2} \le f(x,x+2)$. That is, $f(x,x+1)+f(x+1,x+2) \le f(x,x+2)$. Applying this inequality repeatedly, we have the desired result. **Q.E.D.**

Let's assume $c = 0$. That is, only the traveling cost is taken into account. Actually we want to minimize the expected time that it takes to find the object. We use the D.P. equation (4.5) and (4.6) repeatedly. By letting $c = 0$ in Corollaries 13 and 14 we have

Corollary 17. Assume $c = 0$. $U(x) = f(x,n)$. If and only if

$$q(y+1,n)/q(x+1,n) \ge (n-y)/(n+y) \text{ for } y : x+1 \le y \le n-1. \tag{4.14}$$

In particular by letting $x = 0, s_1 \equiv [n]$ is optimal if and only if $q(1,y) \le y/(n+y)$ for $y : 1 \le y \le n$.

For example, if $p_i = b - ai$ for $i = 1,\cdots,n$. Then $[n]$ is optimal if and only if $0 \le a \le \frac{1}{2n^2-n}$.

5. Approximately Optimal Search Plan.

In general it is difficult to deduce exactly optimal search plans. In this section we rewrite (4.5) and (4.6) in order to find 3–optimal search plans. A search plan in S_3 is written as $s = [x,y,n]$ where $1 \le x \le n$ and $x+1 \le y \le n$. Since we are concerned on 3–optimal search plans, we rewrite $U(\cdot)$ as $U_0(\cdot), U_1(\cdot), U_2(\cdot)$, and $U_3(\cdot)$, after reviewing the definitions of $W(\cdot), V(\cdot)$ and $U(\cdot)$. Then we have

$$\begin{aligned} U_1(x) &= f(x,n) + U_0(n) = f(x,n) \text{ for } 0 \le x \le n, \\ U_2(x) &= \min_{x+1\le y\le n} \{f(x,y) + U_1(y)\} \text{ for } 0 \le x \le n, \text{and} \\ U_3(0) &= \min_{1\le x\le n} \{f(0,x) + U_2(x)\}. \end{aligned} \tag{5.1}$$

Of course, Theorem 10 and Lemma 11 apply to $U_i (i = 2,3)$. The following proposition is an analogue of Theorem 10.

Proposition 18. Assume $f_{x,z}(y) \equiv f(x,y) - f(z,y)$ is strictly increasing in y, where $x < z \le y \le n$. Then there are $z^{(0)}$ and $z^{(1)}$ such that

(i) $0 < z^{(1)} < z^{(0)} < n$, and

(ii) $U_2(x) = f(x,n)$ if and only if $x \ge z^{(0)}$, and

(iii) $U_3(x) = f(x,n)$ if and only if $x \ge z^{(0)}$, and $U_3(x) = f(x,y(x)) + f(y(x),n)$ for some $y(x) \ge z^{(0)}$ if and only if $z^{(1)} \le x < z^{(0)}$, and $U_3(x) = f(x,y(x)) + U_2(y(x))$ for $z^{(1)} \le y(x) < z^{(0)}$ if and only if $x < z^{(1)}$.

Proof: In the same way as in Theorem 10, (i) and (iii) are shown. Also $U_2(x) = f(x,n)$ if and only if $x \ge w^{(0)}$. We must show $z^{(0)} = w^{(0)}$. If $x \ge z^{(0)}$, then $U_3(x) = f(x,n)$. This implies $U_2(x) = f(x,n)$, that is, $x \ge w^{(0)}$. Hence $w^{(0)} \le z^{(0)}$. Assume $w^{(0)} < z^{(0)}$. Suppose $w^{(0)} = z^{(0)} - 1$. Then $U_3(w^{(0)}) = f(w^{(0)}, y(w^{(0)})) + f(y(w^{(0)}),n) < f(w^{(0)},n)$. Hence $U_2(w^{(0)}) = f(w^{(0)}, y(w^{(0)})) + f(y(w^{(0)}),n)$. This contradicts the definition of $w^{(0)}$. Suppose $w^{(0)} < z^{(0)} - 1$. Let $x \equiv z^{(0)} - 1$. $U_3(x) = f(x,y(x)) + f(y(x),n) < f(x,n)$ and $U_2(x) = f(x,n)$. A contradiction. Consequently $z^{(0)} = w^{(0)}$. **Q.E.D.**

The argument in the proof of Proposition 18 applies to the case of m-optimality for $m = 2, 3, \cdots$. Thus we find $y^{(i)} = z^{(i)}$ for $i = 0, 1, \cdots$, where $y^{(i)}$ appeared in Theorem 10. In order to calculate $y^{(0)}$, Corollary 13 is useful. We must find the minimum x such that (4.8) holds. In our model (4.8) becomes

$$(2+c)(n-y)q(x+1,n) \le [2(n+y)+c(n-x)]q(y+1,n) \text{for } y \text{ with } x+1 \le y \le n. \quad (5.2)$$

Example 7 (Continued). $f(x,y) = (2y-cx)(y-6)(y-5)/60 + (2+c)y(x-5)(x-6)/60$. Let $T_i(y) \equiv 60U_i(y)$. Then $T_1(y) = 5(2+c)(y-5)(y-6), T_2(x) = \min_{x+1\le y\le 5}\{(2y-cx+10+5c)(y-5)(y-6)+(2+c)y(x-5)(x-6)\}, T_3(0) = \min_{1\le x\le 5}\{2x(x-6)(x-5)+30x(2-c)+T_2(x)\} = \min\{276+120c, 296+118c, 300+114c\}$, where the first, the second, and the third in the last blacket correspond to [3, 5], [1, 4, 5], and [2, 4, 5] respectively. Finally we have [3, 5] is 3–optimal if $0 \le c \le 4$ and [2, 4, 5] if $4 \le c$. Note that $c_1 = 4, 60e(s^{(0)}, c) = 276 + 120c$, and $60e(s^{(1)}, c) = 300 + 114c$.

Example 19. Next we attempt to treat (5.1) in the case of $p_i = b - ai$ where $b = \frac{1}{2n} + \frac{n+1}{2}a$, and $0 \le a < \frac{1}{n^2-n}$. $f(x,y) = (y - \frac{cx}{2})(ay - \frac{1}{n})(y-n) + (1+\frac{c}{2})y(ax - \frac{1}{n})(x-n)$. $U_2(x) = \min_{x+1\le y\le n}\{g(x,y)\}, g(x,y) = (2y-cx+n+nc)(ay-1/n)(y-n)/2 + (1+c/2)y(ax-1/n)(x-n)\}$. Assume $g(x,\cdot)$ attains its minimum at $y^* = y^*(x) = \mathrm{Int}[y\#]$ where $y\#$ satisfies $\partial g/\partial y = 0$ and $\partial^2 g/\partial y^2 > 0$. Then $U_3(0) = \min_{1\le x\le n}\{h(x)\}, h(x) = x(ax-1/n)(x-n) + x(2+c)/2 + g(x,y^*)$. Assume $h(\cdot)$ attains its minimum at $x^* = \mathrm{Int}[x\#]$ where $x\#$ satisfies $dh/dx = 0$ and $d^2h/dx^2 > 0$. Then we have a system of equations of degree 2 with respect to x and y :

$$\begin{aligned} &(ay-1/n)(y-n) + (2y-cx+n+nc)(2ay-an-1/n)/2 \\ &\quad + (2+c)(ax-1/n)(x-n)/2 = 0, \text{and} \\ &3ax^2 - 2anx - 2x/n + 2 + c/2 - c(ay-1/n)(y-n)/2 \\ &\quad + y(2+c)(2ax-an-1/n)/2 = 0. \end{aligned} \quad (5.3)$$

By solving this we have a 3–optimal search plan.

Suppose $n \to \infty$. Assume $x/n \to u$ and $y/n \to v$ and $n^2a \to \alpha$. Then (5.3) becomes

$$(v-1)(\alpha v-1) + (2v-cu+1+c)(2\alpha v-\alpha-1)/2 + (1+c/2)(u-1)(\alpha u-1) = 0, \text{and}$$
$$(2\alpha u-1)(u-1) + u(\alpha u-1) + 1 + c/2 - c(v-1)(\alpha v-1)/2 + v(1+c/2)(2\alpha u-1-\alpha) = 0.$$

6. Remarks.

(i) From Table 2, we see, for example, ℓ_{m-1} becomes $7, 6, 9, 8, 9, 9, 8, \cdots$ as c becomes large. In this sequence we can find no regularity.

(ii) Limiting Case of $n \to +\infty$. If $\lim_{n\to\infty} np_i$ exists, then denote by $h(\cdot)$ the probability distribution function that is obtained after this limiting operation. If we hold $\ell_k/n = y_k (1 \le k \le m-1)$, then (3.6) reduces to an equation:

$$\begin{aligned} &2(y_k + y_{k+1}) + c(y_{k+1} - y_{k-1}) \\ &= [2(1-H(y_k)) + 2(1-H(y_{k-1})) + c(H(y_{k+1}) - H(y_{k-1}))]/h(y_k), \end{aligned} \quad (6.1)$$

where $H(\cdot)$ is the cumulative distribution function of h. Suppose $p_i = b - ai (1 \le i \le n)$ and $a = o(n^{-2})$. Then $h(t) = 1 - |t| (0 < |t| \le 1)$, and $= 0 (|t| > 1)$. Here we let $h(0) = 1$ for convenience of calculation. From (6.1) we also induce the same equations as in Example 19.

(iii) If $p_1 \le p_2 \le \cdots \le p_n$ then of all search plans in S, s_1 minimizes the expected cost, as is shown from Theorem 8. In this case, however, other search strategies should be considered. We illustrate this by a numerical example. Let $n = 2, p_1 = 1/6$, and $p_2 = 1/3$. $s_1 = [2]$. Denote by s the strategy indicating that he examines cells $-2, -1, 2$ and 1 in this order. Clearly s is not in S. $E(s_1) = (11 + 8c)/3$ and $E(s) = (13 + 7c)/3$. Hence he should use a strategy other than s_1 if $c > 2$.

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