Journal of the Operations Research Society of Japan
Vol. 33, No. 2, June 1990

A HIDE AND SEEK GAME WITH TRAVELING COST

Kensaku Kikuta
Toyama University

(Received July 10, 1989; Revised January 22, 1990)

Abstract There are n neighboring cells in a straight line. A man hides among one of all cells and stays there. The searcher examines each cell until he finds the hider. Associated with the examination by the searcher are a traveling cost dependent on the distance from the last cell examined and a fixed examination cost. The searcher wishes to minimize the expectation of cost of finding the hider. On the other hand the hider wishes to maximize it. This is formulated as a two-person zero-sum game and it is solved.

1. Introduction.

Gluss [4] analyzed a model in which there are n+1 neighboring cells in a straight line, labeled from 0 to n in that order. An object is in one of them except for Cell 0 with a priori probabilities $p_1, \ldots, p_n$. At the beginning of the search the searcher is at Cell 0 that is next to Cell 1. It is required to determine a strategy that will minimize the statistical expectation of the cost of finding the object. Associated with the examination of each cell is the examination cost. The only difference between his model and the previous one (See [1]) is that while that cost is constant in the latter, it varies through time, that is, a traveling cost is added in the former (See [4]). Gluss treated two cases : $p_1 \geq \ldots \geq p_n$ and $p_1 \leq \ldots \leq p_n$. He showed that the former case is trivial, the searcher should examine each cell in the order of 1, 2, . . . ,n, and in the latter case he found approximately optimal search strategies when p_i is proportional to i. These strategies are written by one parameter.

While Gluss treats a one decision-maker problem, in this note we assume a hider with his will instead of the object and take the game theoretical point of view. Thus, there are a hider and a searcher. While the searcher wishes to minimize the cost of finding the hider, the hider chooses a cell so as to maximize it. We have a two-person zero-sum game.

For example, this new model can be regarded as a description of a military situation. Alternatively we can imagine a one decision-maker problem in which he has no information on the probability of existence. The decision-maker may usually assume the uniform distribution. However, assume he is very pessimistic or conservative. Then he may imagine a game theoretical situation.

The unique optimal strategy for the hider obtained in this note can be compared with that in Gluss [4] given as an a priori probability. The latter is proportional and the former is hyperbolic.

Another variant of the model of Gluss is in Kikuta[5], where the searcher is at the cell that locates at the center of all cells at the beginning of the search. But, it is, still, a one decision-maker problem. [6] and [7] are surveys on the search theory.

In the next section our model is stated in detail. Most part of Section 3 is spent for solving the game. In Section 4 some remarks are given.

2. The Model and Notation.

There are n+1 neighboring cells in a straight line, labeled from 0 to n in that order. Player 1 (the hider) hides among one of all cells except for Cell 0, and stays there. Player 2 (the searcher) examines each cell until he finds Player 1.

0	1		i		n

Figure 1.

Associated with the examination of Cell i ($1 \leq i \leq n$) is the examination cost that consists of two parts: (i) a traveling cost $d|i-j|$ ($d > 0$) of examining Cell i after having examined Cell j, and (ii) a fixed examination cost $c \geq 0$. (i) means that the examination cost varies through the search and is a function of which cell was last examined. There is not a probability of overlooking Player 1 given that the right cell is searched. It is assumed that at the beginning of the search Player 2 is at Cell 0. Before searching (hiding) Player 2 (Player 1) must determine a strategy so as to make the cost of finding Player 1 as small (large) as possible.

A (*pure*) *strategy* for Player 1 is to choose an element, say i, of $N \equiv \{1,2, . . . ,n\}$, which means he determines on hiding in Cell i. This is denoted by $\underline{i}$ ($i \in N$). The set of all strategies for Player 1 is denoted by $\underline{N} \equiv \{\underline{1},\underline{2}, . . . ,\underline{n}\}$. A strategy for Player 2 is defined by a permutation on N. The set of all permutations on N is denoted by $\underline{M} \equiv \{\underline{1},\underline{2}, . . . ,\underline{m}\}$, where $m = n!$. Thus under a strategy $\underline{j}$, Player 2 examines Cells $\underline{j}(1)$, $\underline{j}(2)$, ..., $\underline{j}(n)$ in this order. In particular, $\underline{1}$ expresses the identity and $\underline{m}$ expresses the permutation such that $\underline{m}(i) = n-i+1$ for $i = 1, . . . ,n$.

For a strategy pair $(\underline{i},\underline{j})$ ($\underline{i} \in \underline{N}$, $\underline{j} \in \underline{M}$), let $k = \underline{j}^{-1}(i)$. Then the cost of finding Player 1, written as $f(\underline{i},\underline{j})$, is :

$$f(\underline{i},\underline{j}) = d\{\underline{j}(1) + |\underline{j}(2)-\underline{j}(1)| + . . . + |\underline{j}(k)-\underline{j}(k-1)|\} + kc. \tag{2.1}$$

Thus we have a two-person zero-sum game, which is denoted by $(f;\underline{N},\underline{M})$.
Since both $\underline{N}$ and $\underline{M}$ are finite sets this game is expressed by a matrix whose (i,j)-component is $f(\underline{i},\underline{j})$ ($\underline{i} \in \underline{N}$, $\underline{j} \in \underline{M}$). The numbers of rows and columns are n and n!

respectively. We see that this matrix does not always have a saddle point, by checking the cases of 2-cell and 3-cell. Indeed suppose n = 2. Let $\underline{1}$ and $\underline{2}$ be the identity permutation and the other one. Then $f(\underline{1},\underline{1}) = c+d$, $f(\underline{1},\underline{2}) = 3d+2c$, $f(\underline{2},\underline{1}) = 2d+2c$, and $f(\underline{2},\underline{2}) = 2d+c$. This 2x2 matrix has no saddle point if $c > 0$. Thus, we need to have the mixed extension of $(f;\underline{N},\underline{M})$ in order to have a solution for any game. Let (f;P,Q) be the mixed extension. The elements of P and Q are called *mixed* strategies, or simply, strategies, without confusion. For a strategy pair $(p,q) \in P\times Q$, f(p,q) is the expected cost of finding Player 1. From (2.1), without loss of generality we assume d = 1 in this note. Then alternatively we can interpret the constant c as the ratio of fixed examination cost to traveling cost.

Our problem is to solve this matrix game.

3. Optimal Strategies.

The purpose of this section is to give optimal strategies and examine their properties. Let $b = \frac{c}{2+c}$. Define 1-vector p^1, 2-vector p^2, . . . , (n-1)-vector p^{n-1} and n-vector p^n inductively as follows:

$$(3.1) \qquad p^k = \frac{1}{1+\frac{b}{k-1}}\left(\frac{b}{k-1}, p^{k-1}\right),\ k = 2, \ldots, n, \text{ and } p^1 = (1).$$

For each k, $1 \le k \le n$, p^k is a probability vector. All components of p^n are positive if $c > 0$. Let $v_n \equiv n + \frac{n+1}{2} c = \frac{n+b}{1-b}$. The following proposition gives properties of p^n, which are referred later.

Proposition 1. (i) $\sum_{i=1}^{n} i p_i^n = \frac{v_n}{1+c}$.

(ii) $p_1^n < p_2^n < \ldots < p_n^n$.

(iii) $(n-i)p_i^n = (n-i-1)p_{i+1}^n + bp_{i+1}^n$ for i = 1, . . . ,n-1.

Proof: From (3.1), for i = 1, . . . ,n-1,

$$p_i^n = b(b+1)(b+2) \cdots (b+n-i-1)p_n^n/(n-i)! \quad \text{and}$$

$$p_n^n = (n-1)!/[(b+1)(b+2)\cdots(b+n-1)].$$

From these, $p_i^n = \frac{b+n-i-1}{n-i} p_{i+1}^n$ for i = 1, . . . , n-2, and $p_{n-1}^n = bp_n^n$. These are just (iii).

Further we have (ii) since b < 1. Let's see (i). When n = 1, (i) is true by (3.1). Assume n ≥ 2 and (i) is true for 1, . . . ,n-1.

$$\sum_{i=1}^{n} i p_i^n = p_1^n + \sum_{i=2}^{n} i p_i^n.$$

By (3.1), we have,

$$= p_1^n + \sum_{i=2}^{n} i \frac{n-1}{n-1+b} p_{i-1}^{n-1} = \frac{b}{n-1+b} + \frac{n-1}{n-1+b} \sum_{i=1}^{n-1} (i+1)\, p_i^{n-1}$$

$$= 1 + \frac{1}{1+\frac{b}{n-1}} \sum_{i=1}^{n-1} i\, p_i^{n-1}.$$

By the induction hypothesis,

$$= 1 + \frac{1}{1+\frac{b}{n-1}} \frac{1}{1+c} (n-1 + \frac{n}{2} c) = \frac{1}{1+c} (n + \frac{n+1}{2} c). \quad \textbf{Q.E.D.}$$

For any $\underline{j} \in \underline{M}$, define $\underline{\rho j} \in \underline{M}$ by

(3.2) $\underline{\rho j}(i) = \underline{j}(n+1-i)$ for all i = 1, . . . ,n.

$\underline{\rho j}$ reverses the order of examination under $\underline{j}$. Thus, if $\underline{j}$ is expressed as an n-vector, that is, $\underline{j} = [\underline{j}(1),\underline{j}(2), \ldots ,\underline{j}(n)]$, then $\underline{\rho j} = [\underline{j}(n), \ldots ,\underline{j}(1)]$. We can assume $\underline{j}$ is as follows :

$$\underline{j}(1) < \underline{j}(2) < \ldots < \underline{j}(i_1),$$
$$\underline{j}(i_1) > \underline{j}(i_1+1) > \ldots > \underline{j}(i_2),$$
$$\underline{j}(i_2) < \underline{j}(i_2+1) < \ldots < \underline{j}(i_3),$$

•

•

$$\underline{j}(i_{2k-1}) > \underline{j}(i_{2k-1}+1) > \ldots > \underline{j}(n).$$

Thus, $\underline{j}$ has k peaks and we say $\underline{j}$ is a *k-peaked* strategy. In particular, we set $\underline{j}(n+1) = \underline{j}(0) = 0$ for convenience.

Example 1. The next figure indicates a 3-peaked strategy when n = 14. $\underline{j}$ = [1,2,5,10,6,3,8,4,9,11,14,13,12,7].

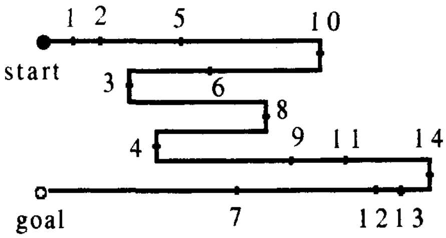


Figure 2.

If $\underline{j} \in \underline{M}$ is k-peaked then $\rho\underline{j}$ is also k-peaked. In particular 1-peaked strategies are interesting since less traveling costs are required under them. Let for any 1-peaked strategy $\underline{j} \in \underline{M}$, $q(\underline{j})$ is a mixed strategy such that the searcher chooses $\underline{j}$ and $\rho\underline{j}$ with probability 1/2 respectively.

Our main result is :

Theorem 1. The value of the game is v_n. p^n is the unique optimal strategy for Player 1. Optimal strategies for Player 2 are $\{q(\underline{j}) : \underline{j}$ is 1-peaked$\}$.

By (3.1) we see that if c = 0, then $p^n = (0,0, \ldots ,0,1)$, and $\lim_{c\to+\infty} p^n = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$. These are intuitively acceptable. Further, $\lim_{n\to+\infty} \frac{v_n}{n} = \frac{1}{1-b} = 1+\frac{c}{2}$. Theorem 1 only gives optimal strategies for player 2, and it says nothing about the set of all optimal strategies for Player 2. Suppose n = 3. Suppose $\underline{1}$ and $\underline{2}$ are strategies such that $\underline{1}(1) = 1$, $\underline{1}(2) = 2$, $\underline{1}(3) = 3$, $\underline{2}(1) = 1$, $\underline{2}(2) = 3$, and $\underline{2}(3) = 2$. All optimal strategies for Player 2 are mixed strategies such that the searcher chooses $q(\underline{1})$ and $q(\underline{2})$ with probabilities t and 1-t respectively for all t with $0 \le t \le 1$.

Example 2. A 1-peaked strategy in the 5-cell case. Let $\underline{j}$ = [2,3,5,4,1]. Then $\rho\underline{j}$ = [1,4,5,3,2].

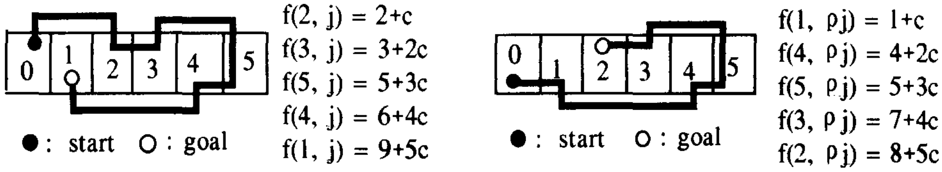
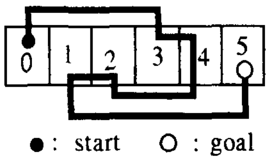


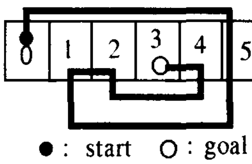
Figure 3.

We see that $f(\underline{i}, \underline{j}) + f(\underline{i}, \rho\underline{j}) = 2(5 + 3c)$ for i = 1,2,3,4,5. By (3.1), $f(p^5, \underline{j}) = 5 + 3c$. Indeed the value of the game is $v_5 = 5 + 3c$.

Example 3. A 2-peaked strategy in the 5-cell case. Let $j = [3, 4, 2, 1, 5]$. Then $\rho j = [5, 1, 2, 4, 3]$.



$f(3, j) = 3+c$
$f(4, j) = 4+2c$
$f(2, j) = 6+3c$
$f(1, j) = 7+4c$
$f(5, j) = 11+5c$



$f(5, \rho j) = 5+c$
$f(1, \rho j) = 9+2c$
$f(2, \rho j) = 10+3c$
$f(4, \rho j) = 12+4c$
$f(3, \rho j) = 13+5c$

Figure 4.

$f(i,j) + f(i,\rho j) = 16 + 6c$ for $i = 1, \ldots, 5$. $f(p^5,j) > 5 + 3c$.

Before proving the theorem we need some lemmas.

Lemma 1. For any 1-peaked strategy $j \in \underline{M}$,
$f(p,q(j)) = v_n$, for any strategy $p \in P$.

Proof: By the definition of ρj, $j^{-1}(i) + \rho j^{-1}(i) = n+1$ for all $i = 1, \ldots, n$. From this and (2.1), noting $d = 1$ by normalizatin,

$$f(i, j) = \sum_{r=1}^{j^{-1}(i)} |j(r)-j(r-1)| + j^{-1}(i)c,$$

and

$$f(i, \rho j) = \sum_{r=1}^{n+1-j^{-1}(i)} |j(n+1-r)-j(n+2-r)|+[n+1-j^{-1}(i)]c = \sum_{r=j^{-1}(i)+1}^{n+1} |j(r)-j(r-1)|+[n+1-j^{-1}(i)]c.$$

Hence

$$f(i, j) + f(i, \rho j) = \sum_{r=1}^{n+1} |j(r)-j(r-1)| + (n+1)c = 2n + (n+1)c = 2v_n,$$

since j is 1-peaked. Hence

$$f(p,q(j)) = \frac{1}{2} \sum_{i=1}^{n} p_i[f(i,j) + f(i,\rho j)] = \frac{1}{2} \sum_{i=1}^{n} p_i 2v_n = v_n. \quad \textbf{Q.E.D.}$$

Lemma 2. For any 1-peaked strategy $j \in \underline{M}$, it holds $f(p^n,j) = v_n$.

Proof: Let $j = [i_1, i_2, \ldots, i_k, i_{k+1}, \ldots]$. Here $i_{k+1} = n$ and $i_1 < i_2 < \ldots < i_{k+1}$. Note that j is characterized by $i_1, \ldots, i_{k+1}$ since it is 1-peaked. $f(i_h,j) = hc + i_h$ for $h = 1, \ldots, k+1$. For i such that $i_h < i < i_{h+1}$ and $h = 0, \ldots, k$, $f(i,j) = (n-i+h+1)c + 2n-i$, where $i_0 = 0$. Then by Proposition 1(i),

$$f(p^n,j) - v_n = \sum_{i=1}^{n} p_i^n f(i,j) - \sum_{i=1}^{n} i(1+c)p_i^n$$

$$= \sum_{h=1}^{k+1} p_{i_h}^n f(i_h, j) + \sum_{h=0}^{k} \sum_{t=i_h+1}^{i_{h+1}-1} p_t^n f(t,j) - \sum_{h=1}^{k+1} p_{i_h}^n i_h(1+c) - \sum_{h=0}^{k} \sum_{t=i_h+1}^{i_{h+1}-1} p_t^n t(1+c)$$

$$= \sum_{h=1}^{k+1} p_{i_h}^n [hc+i_h-(i_h+ci_h)] + \sum_{h=0}^{k} \sum_{t=i_h+1}^{i_{h+1}-1} p_t^n [2n-t+(n-t+1+h)c-(t+tc)]$$

$$= c \sum_{h=1}^{k+1} (h-i_h)p_{i_h}^n + \sum_{h=0}^{k} \sum_{t=i_h+1}^{i_{h+1}-1} [(n-2t+h+1)c + 2(n-t)]p_t^n.$$

The second term becomes :

$$\sum_{h=0}^{k} \sum_{t=i_h+1}^{i_{h+1}-1} [(n+1)c + 2n + ch - 2t(1+c)]p_t^n.$$

$$= 2v_n\{1 - \sum_{h=1}^{k+1} p_{i_h}^n\} + c \sum_{h=0}^{k} h \sum_{t=i_h+1}^{i_{h+1}-1} p_t^n - 2(1+c) \sum_{h=0}^{k} \sum_{t=i_h+1}^{i_{h+1}-1} t p_t^n$$

$$= 2v_n\{1 - \sum_{h=1}^{k+1} p_{i_h}^n\} + c \sum_{h=0}^{k} h \sum_{t=i_h+1}^{i_{h+1}-1} p_t^n - 2(1+c)\{\sum_{i=1}^{n} i p_i^n - \sum_{h=1}^{k+1} i_h p_{i_h}^n\}$$

$$= - 2v_n \sum_{h=1}^{k+1} p_{i_h}^n + c \sum_{h=0}^{k} h \sum_{t=i_h+1}^{i_{h+1}-1} p_t^n + 2(1+c) \sum_{h=1}^{k+1} i_h p_{i_h}^n,$$

by Proposition 1(i). Hence $f(p^n,j) - v_n$ becomes :

$$\sum_{h=1}^{k+1} [2(i_h-n) + c(h+i_h-n-1)]p_{i_h}^n + c \sum_{h=0}^{k} h \sum_{t=i_h+1}^{i_{h+1}-1} p_t^n.$$

By applying Proposition 1(iii) repeatedly,

$$\sum_{t=i_h+1}^{i_{h+1}-1} p_t^n = \frac{1}{b}[(n-i_h)p_{i_h}^n - (n-i_{h+1}+1)p^n_{i_{h+1}-1}]$$

$$= \frac{1}{b}[(n-i_h)p_{i_h}^n - (n-i_{h+1})p_{i_{h+1}}^n - bp_{i_{h+1}}^n].$$

Hence

$$\sum_{h=0}^{k} h \sum_{t=i_h+1}^{i_{h+1}-1} p_t^n = \sum_{h=1}^{k} \frac{h}{b}(n-i_h)p_{i_h}^n - \sum_{h=1}^{k+1} \frac{h-1}{b}(n-i_h)p_{i_h}^n - \sum_{h=1}^{k+1}(h-1)p_{i_h}^n.$$

Thus, noting $c = \frac{2b}{1-b}$,

$$f(p^n,j) - v_n = \sum_{h=1}^{k+1}[2(i_h-n) + c(h+i_h-n-1) - c\frac{h-1}{b}(n-i_h) - c(h-1)]p_i^n$$

$$+ c\sum_{h=1}^{k} \frac{h}{b}(n-i_h)p_{i_h}^n$$

$$= \sum_{h=1}^{k+1}(i_h-n)\frac{2h}{1-b}p_{i_h}^n + \frac{2}{1-b}\sum_{h=1}^{k} h(n-i_h)p_{i_h}^n = 0. \quad \textbf{Q.E.D.}$$

Lemma 3. Let $j \in \underline{M}$ be a 2-peaked strategy such that $j = [j(1), \ldots, j(i_1), j(i_1+1), \ldots, j(i_2), j(i_2+1), \ldots, j(i_2+s), \ldots, j(i_3), j(i_3+1), \ldots, j(n)]$. Let $j' = [j(1), \ldots, j(i_1), j(i_1+1), \ldots, j(i_1+r), j(i_2+s), j(i_1+r+1), \ldots, j(i_2), j(i_2+1), \ldots, j(i_3), j(i_3+1), \ldots, j(n)]$ when $j(i_1) < j(i_3)$, and $j' = [j(1), \ldots, j(i_1), j(i_1+1), \ldots, j(i_1+r), j(i_3-1), j(i_1+r+1), \ldots, j(i_2), j(i_2+1), \ldots, j(i_3), j(i_3+1), \ldots, j(n)]$ when $j(i_1) > j(i_3)$. Then $f(p^n,j) > f(p^n,j')$.

Proof: Assume $j(i_1) < j(i_3)$. Suppose $j(i_2+s) < j(i_1) < j(i_2+s+1)$ and $s \geq 1$.

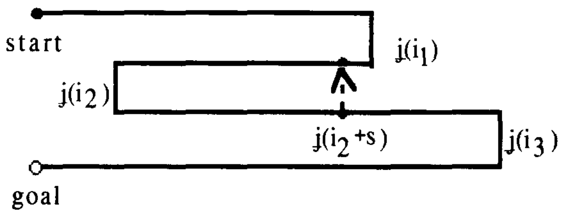


Figure 5.

$$f(p^n,j) = \ldots + \sum_{t=i_1+1}^{i_2} p^n_{j(t)} \{j(i_1)+|j(t)-j(i_1)| + tc\}$$

$$+ \sum_{t=i_2+1}^{i_2+s} p^n_{j(t)}\{j(i_1)+|j(i_2)-j(i_1)|+|j(t)-j(i_2)| + tc\} + \ldots$$

$$f(p^n,j') = \ldots + \sum_{t=i_1+1}^{i_1+r} p^n_{j(t)} \{j(i_1)+|j(t)-j(i_1)| + tc\}$$

$$+ p^n_{j(i_2+s)} \{j(i_1)+|j(i_2+s)-j(i_1)|+(i_1+r+1)c\} + \sum_{t=i_1+r+1}^{i_2} p^n_{j(t)} \{j(i_1)+|j(t)-j(i_1)| + (t+1)c\}$$

$$+ \sum_{t=i_2+1}^{i_2+s-1} p^n_{j(t)}\{j(i_1)+|j(i_2)-j(i_1)|+|j(t)-j(i_2)|+(t+1)c\} + \ldots$$

Thus,

$$f(p^n,j) - f(p^n,j') = -c \sum_{t=i_1+r+1}^{i_2+s-1} p^n_{j(t)} + \{2j(i_2+s)-2j(i_2)+(i_2+s-i_1-r-1)c\}p^n_{j(i_2+s)}$$

$$= 2[j(i_2+s)-j(i_2)]p^n_{j(i_2+s)} + \sum_{t=i_1+r+1}^{i_2+s-1} c[p^n_{j(i_2+s)} - p^n_{j(t)}] \geq 0,$$

since $j(i_2+s) \geq j(i_2)$, $j(i_2+s) \geq j(t)$ for all $t : i_1+r+1 \leq t \leq i_2+s-1$, and Proposition 1(ii).

Assume $j(i_3) < j(i_1)$.

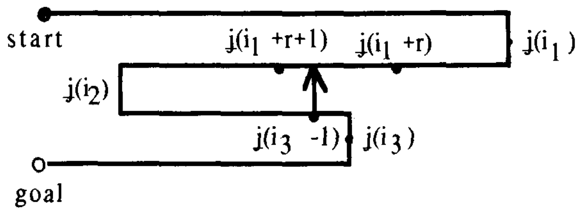


Figure 6.

$$f(p^n,j) = \ldots + \sum_{t=i_1+1}^{i_2} p^n_{j(t)} \{j(i_1)+|j(t)-j(i_1)| + tc\}$$

$$+ \sum_{t=i_2+1}^{i_3-1} p^n_{j(t)}\{j(i_1)+|j(i_2)-j(i_1)|+|j(t)-j(i_2)| + tc\} + \ldots$$

$$f(p^n,j') = \ldots + \sum_{t=i_1+1}^{i_1+r} p^n_{j(t)} \{j(i_1)+|j(t)-j(i_1)| + tc\}$$

$$+ p^n_{j(i_3-1)} \{j(i_1)+|j(i_3-1)-j(i_1)|+(i_1+r+1)c\}$$

$$+ \sum_{t=i_1+r+1}^{i_2} p^n_{j(t)} \{j(i_1)+|j(t)-j(i_1)| + (t+1)c\}$$

$$+ \sum_{t=i_2+1}^{i_3-2} p^n_{j(t)}\{j(i_1)+|j(i_2)-j(i_1)|+|j(t)-j(i_2)|+(t+1)c\} + \ldots$$

Thus,

$$f(p^n,j) - f(p^n,j') = -c \sum_{t=i_1+r+1}^{i_3-2} p^n_{j(t)} + \{2j(i_3-1)-2j(i_2)+(i_3-i_1-r-2)c\}p^n_{j(i_3-1)}$$

$$= 2[j(i_3-1)-j(i_2)]p^n_{j(i_3-1)} + \sum_{t=i_1+r+1}^{i_3-2} c[p^n_{j(i_3-1)} - p^n_{j(t)}] \geq 0,$$

since $j(i_3-1) \geq j(i_2)$, $j(i_3-1) \geq j(t)$ for all $t : i_1+r+1 \leq t \leq i_3-2$, and Proposition 1(ii). **Q.E.D.**

Corollary 1. Let $j \in \underline{M}$ be a 2-peaked strategy such that $j = [j(1), \ldots, j(i_1), j(i_1+1), \ldots, j(i_2), j(i_2+1), \ldots, j(i_2+s), \ldots, j(i_3), j(i_3+1), \ldots, j(n)]$. Let $j' = [j(1), \ldots, j(i_1), j'(i_1+1), \ldots, j'(i_2-i_1+s-1), j(i_2), j(i_2+s+1), \ldots, j(i_3), j(i_3+1), \ldots, j(n)]$, where

$\{j'(i_1+1), \ldots, j'(i_2-i_1+s-1)\} = \{j(i_1+1), \ldots, j(i_2-1), j(i_2+1), \ldots, j(i_2+s)\}$ and $j'(i_1+1) > \ldots > j'(i_2-i_1+s-1)$ when $j(i_1) < j(i_3)$, and let $j' = [j(1), \ldots, j(i_1), \ldots, j(i_1+r), j'(i_1+r+1), \ldots, j'(i_3-2), j(i_2), j(i_3), j(i_3+1), \ldots, j(n)]$, where $\{j'(i_1+r+1), \ldots, j'(i_3-2)\} = \{j(i_1+r+1), \ldots, j(i_2-1), j(i_2+1), \ldots, j(i_3-1)\}$ and $j'(i_1+r+1) > \ldots > j'(i_3-2)$ when $j(i_1) < j(i_3)$. Then $f(p^n,j) > f(p^n,j')$.

Proof: Assume $j(i_1) < j(i_3)$. Suppose $j(i_2+s) < j(i_1) < j(i_2+s+1)$ and $s \geq 1$.

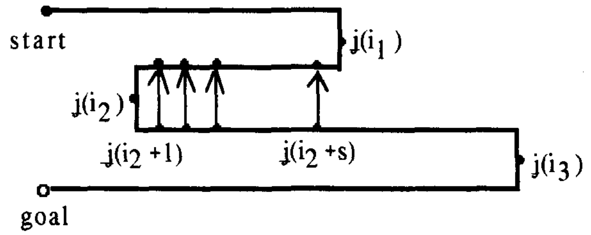


Figure 7.

Apply the first half of Lemma 3 s times, starting with $j(i_2+s)$, then $j(i_2+s-1)$, ...

Next assume $j(i_1) > j(i_3)$. Apply the second half of Lemma 3 (i_3-i_2-1) times, starting with $j(i_3-1)$, then $j(i_3-2)$, ... **Q.E.D.**

Perhaps Corollary 1 and the following lemma can be merged and shortened. But the proof will be complicate in notation if we merge. Thus we do not. Further Corollary 1 in itself says a property of a strategy for Player 2.

Lemma 4. Let $j \in \underline{M}$ be a 2-peaked strategy such that

$j = [j(1), \ldots, j(i_1), \ldots, j(i_2), j(i_2+1), \ldots, j(i_3), \ldots, j(i_3+s), \ldots, j(n)]$,

where $j(i_1) < j(i_2+1)$ and $j(i_3+s) > j(i_1) > j(i_3+s+1)$. Let

$j' = [j(1), \ldots, j(i_1), j(i_2+1), \ldots, j(i_3), \ldots, j(i_3+s), j'(i_3+s-i_2+i_1+1), \ldots, j'(n)]$, where $\{j'(i_3+s-i_2+i_1+1), \ldots, j'(n)\} = \{j(i_1+1), \ldots, j(i_2), j(i_3+s+1), \ldots, j(n)\}$ and $j'(i_3+s-i_2+i_1+1) > \ldots > j'(n)$. Then $f(p^n,j) > f(p^n,j')$.

Proof:

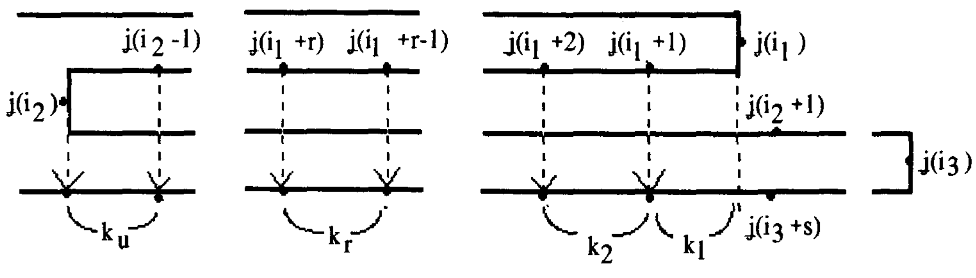


Figure 8.

Let $u = i_2 - i_1$. Observing that for t with $t \geq i_3 + 1$,

$$j(i_1) + |j(i_2)-j(i_1)| + |j(i_3)-j(i_2)| + |j(t)-j(i_3)| = 2j(i_1) - 2j(i_2) + 2n - j(t),$$

$$f(p^n,j) = \ldots + \sum_{t=i_1+1}^{i_2} p^n_{j(t)}\{j(i_1) + |j(t)-j(i_1)| + tc\}$$

$$+ \sum_{t=i_2+1}^{i_3} p^n_{j(t)}\{j(i_1) + |j(i_2)-j(i_1)| + |j(t)-j(i_2)| + tc\}$$

$$+ \sum_{t=i_3+1}^{i_3+s} p^n_{j(t)}\{2j(i_1) - 2j(i_2) + 2n - j(t) + tc\}$$

$$+ \sum_{r=1}^{u} \sum_{t=i_3+s+k_1+\ldots+k_{r-1}+1}^{i_3+s+k_1+\ldots+k_r} p^n_{j(t)}\{2j(i_1) - 2j(i_2) + 2n - j(t) + tc\} + \ldots .$$

On the other hand, seeing that j' is 1-peaked,

$$f(p^n,j') = \ldots + \sum_{t=i_2+1}^{i_3} p^n_{j(t)}\{j(t) + (t-u)c\} + \sum_{t=i_3+1}^{i_3+s} p^n_{j(t)}\{2n - j(t) + (t-u)c\}$$

$$+ \sum_{r=1}^{u} \sum_{t=i_3+s+k_1+\ldots+k_{r-1}+1}^{i_3+s+k_1+\ldots+k_r} p^n_{j(t)}\{2n - j(t) + (t-u+r-1)c\}$$

$$+ \sum_{r=1}^{u} p^n_{j(i_1+r)}\{2n - j(i_1+r) + (i_3+s-u+k_1+ \ldots + k_r+r)c\}$$

•

•

Hence,

$$f(p^n,j) - f(p^n,j')$$

$$= \sum_{t=i_2+1}^{i_3} p^n_{j(t)}\{2j(i_1) - 2j(i_2) + uc\} + \sum_{t=i_3+1}^{i_3+s} p^n_{j(t)}\{2j(i_1) - 2j(i_2) + uc\}$$

$$+\sum_{r=1}^{u}\sum_{t=i_3+s+k_1+\ldots+k_{r-1}+1}^{i_3+s+k_1+\ldots+k_r} p_{j(t)}^{n}\{2j(i_1)-2j(i_2)+(u-r+1)c\}$$

$$+\sum_{r=1}^{u} p^{n}{}_{j(i_1+r)}\{2j(i_1)-2n-(i_3+s+k_1+\ldots+k_r-i_2)c\}$$

$$=\sum_{t=i_2+1}^{i_3+s} p_{j(t)}^{n}\{2j(i_1)-2j(i_2)+uc\}+\sum_{t=i_3+s+1}^{i_3+s+k_1+\ldots+k_u} p_{j(t)}^{n}\{2j(i_1)-2j(i_2)\}$$

$$+\sum_{t=i_1+1}^{i_2} p_{j(t)}^{n}\{2j(i_1)-2n\}+\sum_{r=1}^{u}\sum_{t=i_3+s+k_1+\ldots+k_{r-1}+1}^{i_3+s+k_1+\ldots+k_r} p_{j(t)}^{n}(u-r+1)c$$

$$-\sum_{r=1}^{u} p^{n}{}_{j(i_1+r)}(i_3+s-i_2+k_1+\ \ldots\ +k_r)c$$

$$=2\sum_{t=i_2+1}^{i_3+s} p_{j(t)}^{n}[j(i_1)-j(i_2)]-2[n-j(i_1)]\sum_{t=i_1+1}^{i_2} p_{j(t)}^{n}$$

$$+2[j(i_1)-j(i_2)]\sum_{t=i_3+s+1}^{i_3+s+k_1+\ldots+k_u} p_{j(t)}^{n}+\sum_{t=i_2+1}^{i_3+s} p_{j(t)}^{n}uc$$

$$+\sum_{r=1}^{u}\sum_{t=i_3+s+k_1+\ldots+k_{r-1}+1}^{i_3+s+k_1+\ldots+k_r} p_{j(t)}^{n}(u-r+1)c-\sum_{r=1}^{u} p^{n}{}_{j(i_1+r)}(i_3+s-i_2+k_1+\ \ldots\ +k_r)c$$

$\equiv A + Bc$, where

$$A\equiv 2\sum_{t=i_2+1}^{i_3+s} p_{j(t)}^{n}[j(i_1)-j(i_2)]-2[n-j(i_1)]\sum_{t=i_1+1}^{i_2} p_{j(t)}^{n}+2[j(i_1)-j(i_2)]\sum_{t=i_3+s+1}^{i_3+s+k_1+\ldots+k_u} p_{j(t)}^{n}$$

$$\geq 2[j(i_1)-j(i_2)](i_3+s-i_2)p_{j(i_1)}^{n}-2[n-j(i_1)]up_{j(i_1)}^{n}$$

$$+2[j(i_1)-j(i_2)]\sum_{t=i_3+s+1}^{i_3+s+k_1+\ldots+k_u} p_{j(t)}^{n}\geq 0$$

since $i_3+s-i_2 = n - j(i_1)$ and $j(i_1)-j(i_2) \geq u > 0$. Further,

$$B \equiv \sum_{t=i_2+1}^{i_3+s} p^n_{j(t)} u + \sum_{r=1}^{u} \sum_{t=i_3+s+k_1+\ldots+k_{r-1}+1}^{i_3+s+k_1+\ldots+k_r} p^n_{j(t)}(u-r+1)$$

$$- \sum_{r=1}^{u} p^n_{j(i_1+r)}(i_3+s-i_2+k_1+\ .\ .\ .\ +k_r)$$

$$= u \sum_{t=i_2+1}^{i_3+s} p^n_{j(t)} - (i_3+s-i_2) \sum_{r=1}^{u} p^n_{j(i_1+r)}$$

$$+ \sum_{r=1}^{u} (u-r+1) \sum_{t=1}^{k_r} p^n_{j(i_3+s+k_1+\ldots+k_{r-1}+t)} - \sum_{r=1}^{u} p^n_{j(i_1+r)} \sum_{t=1}^{r} k_t$$

$$= u \sum_{t=i_2+1}^{i_3+s} p^n_{j(t)} - [n-j(i_1)] \sum_{t=i_1+1}^{i_2} p^n_{j(t)}$$

$$+ \sum_{r=1}^{u} \{(u-r+1) \sum_{t=1}^{k_r} p^n_{j(i_3+s+k_1+\ldots+k_{r-1}+t)} - k_r \sum_{t=r}^{u} p^n_{j(i_1+t)}\}$$

$$= \sum_{r=1}^{u} \sum_{t=i_2+1}^{i_3+s} p^n_{j(t)} - \sum_{r=1}^{i_3+s-i_2} \sum_{t=i_1+1}^{i_2} p^n_{j(t)}$$

$$+ \sum_{r=1}^{u} \{ \sum_{h=r}^{u} \sum_{t=1}^{k_r} p^n_{j(i_3+s+k_1+\ldots+k_{r-1}+t)} - \sum_{t=1}^{k_r} \sum_{h=r}^{u} p^n_{j(i_1+h)} \}$$

$$= \sum_{t=i_2+1}^{i_3+s} \sum_{r=1}^{u} [p^n_{j(t)} - p^n_{j(i_1+r)}]$$

$$+ \sum_{r=1}^{u} \sum_{h=r}^{u} \sum_{t=1}^{k_r} [p^n_{j(i_3+s+k_1+\ldots+k_{r-1}+t)} - p^n_{j(i_1+h)}] \geq 0$$

by Proposition 1(ii). **Q.E.D.**

Lemma 4'. Let $j \in \underline{M}$ be a 2-peaked strategy such that

$j = [j(1), . . . ,j(i_1), . . . ,j(i_2),j(i_3), . . . ,j(n)]$, where $j(i_1) > j(i_3)$ and $j(i_1+s) > j(i_3) > j(i_1+s+1)$. Let

$j' = [j(1), \ldots, j(i_1), \ldots, j(i_1+s), j(i_3), j'(i_1+s+2), \ldots, j'(n)]$, where $\{j'(i_1+s+2), \ldots, j'(n)\} = \{j(i_1+s+1), \ldots, j(i_2), j(i_3+1), \ldots, j(n)\}$ and $j(i_3) > j'(i_3+s+2) > \ldots > j'(n)$. Then $f(p^n,j) > f(p^n,j')$.

Proof: Let $u = i_2 - i_1$. Noting that $i_3 = i_2 + 1$,

$$f(p^n,j) = \ldots + \sum_{t=i_1+s+1}^{i_2} p^n_{j(t)}\{j(i_1) + |j(t)-j(i_1)| + tc\}$$

$$+ p^n_{j(i_2+1)}\{j(i_1) + |j(i_2)-j(i_1)| + |j(i_2+1)-j(i_2)| + (i_2+1)c\}$$

$$+ \sum_{r=0}^{u-s-1} \sum_{t=i_2+1+k_0+\ldots+k_{r-1}+1}^{i_2+1+k_0+\ldots+k_r} p^n_{j(t)}\{j(i_1) + |j(i_2)- j(i_1)|$$

$$+ |j(i_2+1)- j(i_2)| + |j(t)- j(i_2+1)| + tc\} + \ldots .$$

On the other hand, seeing that j' is 1-peaked,

$$f(p^n,j') = \ldots + p^n_{j(i_2+1)}\{j(i_1) + |j(i_2+1)- j(i_1)| + (i_1+1+s)c\}$$

$$+ \sum_{r=0}^{u-s-1} \sum_{t=i_2+1+k_0+\ldots+k_{r-1}+1}^{i_2+1+k_0+\ldots+k_r} p^n_{j(t)}\{j(i_1) + |j(t)- j(i_1)| + (t-u+s+r)c\}$$

$$+ \sum_{r=1}^{u-s} p^n_{j(i_1+s+r)}\{j(i_1) + |j(i_1+s+r)- j(i_1)| + (i_1+s+r+k_0+ \ldots + k_{r-1}+1)c\}$$

$$\vdots$$

Hence,

$$f(p^n,j) - f(p^n,j')$$

$$= p^n_{j(i_2+1)}\{2j(i_2+1) - 2j(i_2) + (u-s)c\} - \sum_{r=1}^{u-s} p^n_{j(i_1+s+r)}(k_0+\ldots+k_{r-1}+1)c$$

$$+ \sum_{r=0}^{u-s-1} \sum_{t=i_2+1+k_0+\ldots+k_{r-1}+1}^{i_2+1+k_0+\ldots+k_r} p^n_{j(t)}\{2j(i_2+1) - 2j(i_2) + (u-r-s)c\}$$

$$\equiv A + Bc.$$

$$A = 2[j(i_2+1)-j(i_2)]\ \{p^n_{j(i_2+1)} + \sum_{r=0}^{u-s-1} \sum_{t=i_2+1+k_0+\dots+k_{r-1}+1}^{i_2+1+k_0+\dots+k_r} p^n_{j(t)}\}.$$

Further,

$$B = (u-s)p^n_{j(i_2+1)} + \sum_{r=0}^{u-s-1} \sum_{t=i_2+1+k_0+\dots+k_{r-1}+1}^{i_2+1+k_0+\dots+k_r} p^n_{j(t)}(u-r-s)$$

$$- \sum_{r=1}^{u-s} p^n{}_{j(i_1+s+r)}(k_0+ \ . \ . \ . \ +k_{r-1}+1)$$

$$= (u-s)p^n_{j(i_2+1)} - \sum_{r=1}^{u-s} p^n{}_{j(i_1+s+r)} + \sum_{r=0}^{u-s-1} (u-r-s) \sum_{t=i_2+1+k_0+\dots+k_{r-1}+1}^{i_2+1+k_0+\dots+k_r} p^n_{j(t)}$$

$$- \sum_{r=1}^{u-s} p^n{}_{j(i_1+s+r)} \sum_{t=1}^{r} k_{t-1}$$

$$= \sum_{r=1}^{u-s} [p^n_{j(i_2+1)} - p^n_{j(i_1+s+r)}] + \sum_{r=1}^{u-s} (u-s-r+1) \sum_{t=1}^{k_{r-1}} p^n{}_{j(i_2+1+k_0+\dots+k_{r-2}+t)}$$

$$- \sum_{r=1}^{u-s} k_{r-1} \sum_{t=r}^{u-s} p^n{}_{j(i_1+s+t)}$$

$$= \sum_{r=1}^{u-s} [p^n_{j(i_2+1)} - p^n_{j(i_1+s+r)}]$$

$$+ \sum_{r=1}^{u-s} \sum_{x=r}^{u-s} \sum_{t=1}^{k_{r-1}} [p^n{}_{j(i_2+1+k_0+\dots+k_{r-2}+t)} - p^n{}_{j(i_1+s+x)}] \geq 0$$

by Proposition 1(ii). **Q.E.D.**

Corollary 2. For any $j \in \underline{M}$, $f(p^n,j) \geq v_n$.

Proof: Suppose j ∈ M is k-peaked. Let j' ∈ M be a 1-peaked strategy which is transferred from j by repeated operations indicated in Corollary 1, Lemma 4 and Lemma 4'. Then $f(p^n,j) \geq f(p^n,j') = v_n$. **Q.E.D.**

Lemma 5. Let j[r] = [1, . . . ,r-1,n,n-1, . . . ,r] $(2 \leq r \leq n)$ be 1-peaked strategies for Player 2. Let A_n be an n-by-n matrix whose (s,t)-component is f(t, j[n-s+1]) for s = 1, . . . ,n-1 and t = 1, . . . ,n. Further (n,t)-component is 1+c for t = 1, . . . ,n. Then the rank of A_n is equal to n.

Proof : A_n =

j[n]	1+c	2+2c	3+3c	• • •	r+rc	• • •	(n-1)x (1+c)	n+nc
j[n-1]	1+c	2+2c			r+rc		n+1+nc	n+ (n-1)c
				⋮		⋮		
j[r]	1+c	2+2c			2n-r +nc		n+1+ (r+1)c	n+rc
				⋮		⋮		
j[2]	1+c	2n-2 +nc	2n-3 +(n-1)c				n+1+3c	n+2c
	1+c	1+c			1+c		1+c	1+c

First, we sweep out the 1st row and then the 1st column by the pivot element f(1,j[n]). Then divide the n-th row by -1-c. Sweep out the n-th row and the 2nd column by the (n,2)-component. Then divide the (n-1)-th row by 2(1-n)-nc. Sweep out the (n-1)-th row and the 3rd column. Then divide the (n-2)-th row by 2(2-n)-(n-1)c. Sweep out the (n-2)-row . . . . Divide the (n+1-r)-th row by 2(r-1-n)-(n+2-r)c $(2 \leq r \leq n-1)$ . . . .Finally A_n is transformed into : A'_n =

1	0	0	• • •	0	• • •		0	0	0
0	0	0		0			0	0	1
0	0	0		0		0	0	1	0
				⋮					
0	0	0					0	0	0
0	0	1		0			0	0	0
0	1	0		0			0	0	0
0	0	0					0	0	0

It is easy to see that the rank of this matrix is equal to n. **Q.E.D.**

Proof of the theorem: By Lemma 1, Lemma 2, and Corollary 2, p^n and $q(\underline{j})$ are optimal strategies and v_n is the value. Suppose p' is an optimal strategy for Player 2. By Lemma 1, for any 1-peaked strategy $\underline{j}$, $f(p',\underline{j}) + f(p',\underline{\rho j}) = 2v_n$. $f(p',\underline{j}) \geq v_n$ and $f(p',\underline{\rho j}) \geq v_n$ since p' is optimal and v_n is the value. Hence $f(p',\underline{j}) = v_n$. That is, $f(p',\underline{j}) = v_n$ for all 1-peaked strategy $\underline{j}$. This, combined with Lemma 2 and Lemma 5, implies $p' = p^n$. **Q.E.D.**

4. Remarks.

(i) It is interesting to compare the optimal strategy p^n for Player 1 with the probability distribution given as a prior distribution in Gluss [4]. Write it as p#. That is, $p\#_i = 2i/[n(n+1)]$ for all i. Both satisfy (a) $p_i > 0$ for all i (for p^n, only if $c > 0$), and (b) $p_1 \leq p_2 \leq \ldots \leq p_n$. $f(p^n,q) \geq f(p^n,q(\underline{j})) = v_n = f(p\#,q(\underline{j}))$ for any $q \in Q$ and any 1-peaked strategy $\underline{j}$, and $f(p\#,q(\underline{j})) \geq f(p\#,\underline{j}')$ for some $\underline{j}' \in \underline{M}$.

Gluss considered a class of 1-peaked strategies. Define r* ($1 \leq r \leq n$) and u# ($0 \leq u \leq n-1$) by (See p. 279 of Gluss [4]) :

$$r^*(i) = r+i-1 \quad \text{for } 1 \leq i \leq n-r+1 \text{ and}$$
$$= n+1-i \quad \text{for } n-r+2 \leq i \leq n.$$

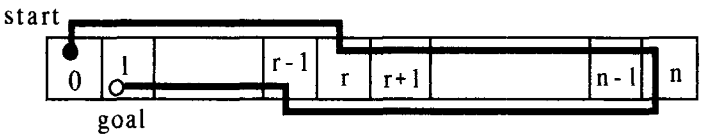


Figure 9.

$$u\#(i) = i \quad \text{for } 1 \leq i \leq u \text{ and}$$
$$= n-i+u+1 \quad \text{for } u+1 \leq i \leq n.$$

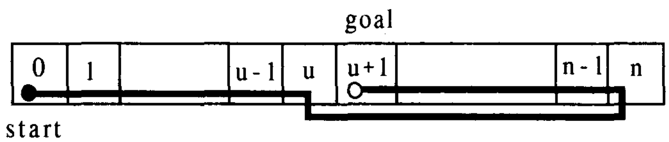


Figure 10.

Let $S^* \equiv \{r^* : 1 \leq r \leq n\}$ and $S\# \equiv \{u\# : 0 \leq u \leq n-1\}$. We see that r* reverses the order of (r-1)#. S* and S# are just sets of 1-peaked strategies. Gluss dealt with the minimization problem : Minimize $f(p\#,\underline{j}')$ subject to $\underline{j}' \in S^* \cup S\#$. If a one decision-maker problem is uncertain (that is, not risky) the decision-maker may assume the uniform distribution. Thus, let p^u be the uniform distribution on N. Then $v_n \geq$

$\mathrm{Min}\{f(p^u,\underline{j}') : \underline{j}' \in \underline{M}\} = f(p^u,\underline{1}) = (n+1)(1+c)/2$, where $\underline{1}$ corresponds to the identity permutation. In the following figure $f(p^u,\underline{1})$, v_n, and the minimum of $f(p\#,\underline{j}')$ are compared.

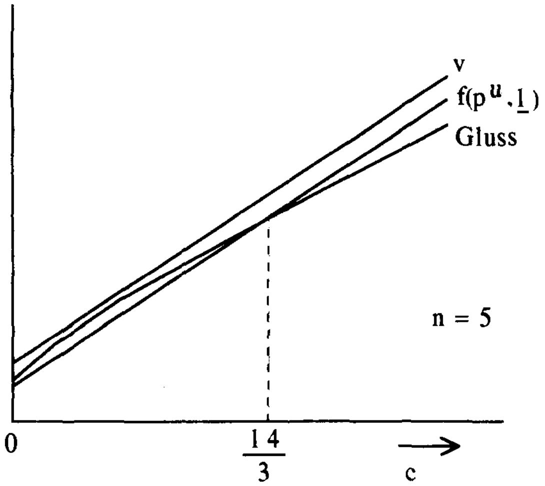


Figure 11.

(ii) It is interesting to consider a continuous version of the model dealt with in this note, in which the interval of [0,1] is given instead of the n-cells. Player 1 chooses one point in it and hides an object there. We must define strategies for Player 2 suitably before beginning the analysis (see [2] or [3]). Instead of pusueing that model, we give here a remark on the behavior of p^n when n the number of cells becomes large. The resulting probability distribution may make clear characteristics of p^n, where

$$np_i^n = \frac{1}{1+\frac{b}{n-1}} \frac{1}{1+\frac{b}{n-2}} \cdots \frac{1}{1+\frac{b}{n-i}} \frac{nb}{n-i}.$$

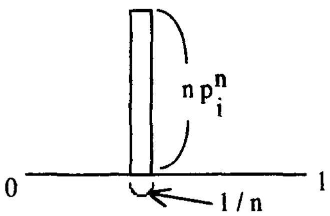


Figure 12.

Suppose for $t : 0 \le t \le 1, \frac{i-1}{n} \le t \le \frac{i}{n}$, that is, $nt \le i \le nt+1$, or $t \le \frac{i}{n} \le t + \frac{1}{n}$. Let $r = i - nt$. Then

$$(1+\frac{b}{n-1})^i < (1+\frac{b}{n-1})(1+\frac{b}{n-2}) \cdots (1+\frac{b}{n-nt-r}) < (1+\frac{b}{n-nt-r})^i .$$

Here we have $(1+\frac{b}{n-1})^i \to e^{bt}$, $(1+\frac{b}{n-nt-r})^i \to e^{bt/(1-t)}$ as $n \to \infty$.
Then let

$$(1+\frac{b}{n-1})(1+\frac{b}{n-2}) \cdots (1+\frac{b}{n-nt-r}) \to e^{bt\beta(b,t)},$$

where $1 \le \beta(b,t) \le \frac{1}{1-t}$. From this, $\beta(b,0) = 1$, and $\beta(1,t) = \frac{1}{t}\log\frac{1}{1-t}$ since $p^n_i = 1/n$ for all i when $b = 1$. If it happens that $\beta(b,t) = \beta(1,t)$ for all b, then np^n_i converges to $b(1-t)^{b-1}$, which is hyperbolic.

Both Fristedt [2] and Gal [3] treated linear search games, in which the cost is the time the searcher requires to discover the hider divided by the distance of the hider from the origin of the real line. Thus it is an interesting problem to solve the case where the cost, $f(i,j)$, is replaced by $f(i,j)/i$, and to compare with results by them.
(iii)The second variant is the one in which Player 2 is at the cell that locates at the center of all cells at the beginning of the search. The analysis of this model has not been done yet.

Acknowledgement. The author wish to thank the referees for their helpful comments and suggestions.

References.

[1] Bellman, R.:*Dynamic Programming* . Princeton University Press, Princeton, New Jersey, 1957.
[2] Fristedt, B.: Hide and Seek in a Subset of the Real Line. *Internatinal J. Game Theory* **6** (1977), 135-165.
[3] Gal, S.: *Search Games* . Math. in Sci. and Eng., **149**, Academic Press. (1980).
[4] Gluss, B.: Approximately Optimal One-Dimensional Search Policies in Which Search Costs Vary through Time. *Nav. Res. Log. Quart.* , **8** (1961), 277-283.
[5] Kikuta, K.: A One-Dimensional Search with Traveling Cost. W.P. No.92, Faculty of Economics, Toyama University, May 1988.
[6] Nakai, T.: *Tansaku Riron Tenbo* (In Japanese) mimeo. (1986).
[7] Sakaguchi, M.: Tansaku Riron.(In Japanese) *BASIC Sugaku.* **14** (1981), 61-67.

Kensaku KIKUTA : Faculty of Economics,
Toyama University,Gofuku 3190,
Toyama,930,Japan