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A POLYNOMIAL TIME INTERIOR POINT ALGORITHM FOR MINIMUM COST FLOW PROBLEMS

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Abstract This paper deals with a minimum cost flow problem. We propose a polynomial time algorithm for the problem. The algorithm is based on an interior point algorithm for a general linear programming problem. Using some features of the minimum cost flow problem, we decrease the running time. We show that the algorithm requires at most $O(|E|^{0.5}\log(|V|M))$ iterations, $O(|V|^3)$ arithmetic operations in each iteration, and $O(|V|^3|E|^{0.5}\log(|V|M))$ arithmetic operations in total. Here $|V|$, $|E|$ and M denote the number of nodes, that of arcs, and the maximum absolute value of input data, respectively.

1. Introduction

This paper deals with a minimum cost flow problem on a connected digraph $G = (V, E)$, where V is a set of nodes and E is a set of arcs. The problem is formulated as a linear programming problem in the following way.

$$
\begin{array}{llll}
(P) & \text{min} & \sum_{(i,j)\in E} c_{ij}x_{ij}, & \\
 & \text{subject to} & \sum_{(i,v)\in E} x_{iv} - \sum_{(v,j)\in E} x_{vj} = b_v & \text{for } v \in V, \\
 & & x_{ij} + w_{ij} = u_{ij} & \text{for } (i,j) \in E, \\
 & & x_{ij} \geq 0 & \text{for } (i,j) \in E, \\
 & & w_{ij} \geq 0 & \text{for } (i,j) \in E,
\end{array}
$$

where, for each $(i,j) \in E$ or $v \in V$, c_{ij} denotes the cost, u_{ij} the upper capacity, b_v the demand, x_{ij} the flow variable, and w_{ij} the slack variable. This problem is equivalent to a minimum cost circulation problem.

Throughout this paper, we assume that all the c_{ij} and b_v are integers, all the u_{ij} are positive integers, and $\sum_{v\in V} b_v = 0$. We also assume that $u_{ij} \geq 2$ for each $(i,j) \in E$ so that we can get an integer in the interval $(0, u_{ij})$. In the case of some $u_{ij} = 1$, we multiply all the $u_{ij}((i,j) \in E)$ and $b_v(v \in V)$ by 2 and substitute x for $2x$ so that the assumption above holds.

For a general linear programming problem, a polynomial time algorithm was proposed by Khachiyan [8] in 1979 for the first time. Then Karmarkar [6] proposed an $O(n^{3.5}L)$ interior point algorithm. Here n denotes the number of variables and L the input size of the problem. After the presentation of Karmarkar, many polynomial time algorithms called interior point algorithms have been developed. Vaidya [17] and Gonzaga [5] presented the least running time algorithms which require at most $O(n^{0.5}L)$ iterations and $O(n^3L)$ arithmetic operations in total. Kojima, Mizuno, and Yoshise [9] and Monteiro and Adler [12] also proposed the same running time algorithms. If we apply these algorithms to the minimum cost flow problem (P), it requires $O(|E|^{0.5}L)$ iterations and $O(|E|^3L)$ arithmetic operations in total, where

$$L = \lceil \sum_{(i,j)\in E} \log(|c_{ij}|+1) + \sum_{(i,j)\in E} \log(u_{ij}+1) + \sum_{v\in V} \log(|b_v|+1) + 2|E| \rceil .$$

In this paper, we demonstrate a lower polynomial time algorithm. We show that the algorithm requires at most $O(|E|^{0.5}\log(|V|M))$ iterations, $O(|V|^3)$ arithmetic operations in each iteration, and $O(|V|^3|E|^{0.5}\log(|V|M))$ arithmetic operations in total. Here

$$M = \max \{|c_{ij}| \ ((i,j)\in E), \ u_{ij} \ ((i,j)\in E), \ |b_v| \ (v\in V)\} .$$

Since we have $L = O(|E|\log M)$, the number of iterations $O(|E|^{0.5}\log(|V|M))$ is much less than $O(|E|^{0.5}L)$ in most cases. The running time we evaluate here is the same as that of the interior point algorithm of Mizuno and Masuzawa [10] for a transportation problem.

We adopt the interior point algorithm of Mizuno [11] for a base of our algorithm because the algorithm requires at most $(n^{0.5}L)$ iterations for an initial point in wider area than the other $O(n^{0.5}L)$ iteration algorithms ([5], [9], [12], [17]). The number of iterations depends on an initial point and a convergence criterion. We construct an artificial minimum cost flow problem which has a trivial initial interior point, and we get a convergence criterion of an interior point such that we can compute an exact solution from the interior point in $O(|V|^3)$ arithmetic operations. Then we show that the algorithm requires at most $O(|E|^{0.5}\log(|V|M))$ iterations. In each iteration, we have to solve a linear system of equations of $O(|E|)$ variables. If the matrix is dense, it requires $O(|E|^3)$ arithmetic operations to solve the system. In the case of the minimum cost flow problem, however, the matrix is sparse. Using the sparsity, we show that it only requires $O(|V|^3)$ arithmetic operations.

For the minimum cost flow problem (P), some polynomial time algorithms, which are not interior point algorithms, have been developed. In 1972, Edmonds and Karp [1] proposed a polynomial time algorithm which requires at most $O(|E|(\log U)(|E|+|V|\log|V|))$ arithmetic operations, where U is the maximum value of arc capacity. Röck [14] also proposed polynomial time algorithms later on. In the case of $|E| = |V|^2$, the running time of the algorithms is almost the same to ours. Tardos [16] devised a strongly polynomial time algorithm for a minimum circulation problem. The running time is $O(|E|^4)$. Fujishige [2] and Goldberg and Tarjan [3, 4] proposed other strongly polynomial time algorithms later on. Our algorithm is not strongly polynomial. Most of the strongly polynomial time algorithms use an idea of scaling. It seems to be difficult to construct a strongly polynomial time algorithm based on an interior point algorithm without using the idea of scaling.

In Section 2, we show the algorithm under the conditions that we have a known initial interior point and a convergence criterion. In Section 3, we construct an artificial minimum cost flow problem by using the so-called Big M method which was also used in Katsura, Fukushima, and Ibaraki [7]. The artificial problem has a trivial initial solution. In order to see the size of the initial solution, we evaluate the size of the Big M correctly such that the artificial problem is equivalent to the original problem. In Section 4, we obtain a convergence criterion such that we can compute an exact solution in $O(|V|^3)$ arithmetic operations. In Section 5, we show that each iteration requires at most $O(|V|^3)$ arithmetic operations and the total running time is $O(|V|^3|E|^{0.5}\log(|V|M))$.

2. The algorithm

Here we show our algorithm that is based on the algorithm of Mizuno [11]. We represent the problem (P) and its dual problem (D) by the following matrix forms.

$$\begin{array}{lll} (P) & \min & c^t x, \\ & \text{subject to} & Ax = b, \\ & & x + w = u, \\ & & x \geq 0, \\ & & w \geq 0, \end{array}$$

$$(D) \quad \begin{array}{ll} \max & b^t y - u^t q, \\ \text{subject to} & A^t y - q + p = c, \\ & y_1 = 0, \\ & p \geq 0, \\ & q \geq 0, \end{array}$$

where A is the node-arc incidence matrix of the graph G, and w and p are slack variable vectors. Since one of the constraints of (P) is redundant, we impose the constraint $y_1 = 0$ on (D). Let S_P and S_D denote the primal and dual interior feasible regions, i.e.,

$$S_P = \{(x, w) : Ax = b, x + w = u, x > 0, w > 0\},$$

$$S_D = \left\{(y, p, q) : A^t y - q + p = c, y_1 = 0, p > 0, q > 0\right\}.$$

We call the pair of primal and dual feasible solutions a feasible point or simply a point. Assume that we have an initial point $(x^0, w^0, y^0, p^0, q^0) \in S_P \times S_D$ in advance. Let $\epsilon = (|V| + |E|)^{-2}$ and $e = (1, 1, \ldots, 1) \in R^{2|E|}$. Then the algorithm consists of the following steps.

Step 1: Generate a sequence $\{v^k \in R^{2|E|} : k = 0, 1, \ldots, m\}$ for a positive integer m such that

$$(1) \quad \begin{cases} v^0 = \begin{bmatrix} X_0 p^0 \\ W_0 q^0 \end{bmatrix}, \\ v^m \leq \frac{1}{2}\epsilon e, \\ \|V_k^{-0.5}(v^{k+1} - v^k)\| \leq 0.3\sqrt{v^k_{min}} \quad \text{for } k = 0, 1, \ldots, m-1, \end{cases}$$

where $X_0 = \text{diag}(x^0)$, $W_0 = \text{diag}(w^0)$, $V_k = \text{diag}(v^k)$, and v^k_{min} denotes the minimum component of v^k.

Step 2: Let $k = 0$.

Step 3: If the point $(x, w, y, p, q) = (x^k, w^k, y^k, p^k, q^k)$ satisfies the following condition then stop:

$$(2) \quad \begin{bmatrix} Xp \\ Wq \end{bmatrix} < \epsilon e,$$

where $X = \text{diag}(x)$ and $W = \text{diag}(w)$.

Step 4: Compute the next point $(x^{k+1}, w^{k+1}, y^{k+1}, p^{k+1}, q^{k+1})$ by one iteration of Newton's method from the k-th point to the following system:

$$(3) \quad \begin{cases} \begin{bmatrix} Xp \\ Wq \end{bmatrix} = v^{k+1} \\ Ax = b, \\ x + w = u, \\ A^t y - q + p = c, \\ y_1 = 0. \end{cases}$$

Step 5: Increase k by 1 and go to Step 3.

In the next section, we construct an artificial minimum cost flow problem which has a trivial initial interior point and is equivalent to the original problem. In section 4, we show that an optimal solution is computed in $O(|V|^3)$ arithmetic operations from any feasible point $(x, w, y, p, q) \in S_P \times S_D$ which satisfies (2).

The algorithm above computes a sequence $\left\{(x^k, w^k, y^k, p^k, q^k)\right\}$ of interior points such that

$$(4) \qquad \left\| V_k^{-0.5} \left(\begin{bmatrix} X_k p^k \\ W_k q^k \end{bmatrix} - v^k \right) \right\| \le 0.3\sqrt{v_{min}^k}.$$

When $k = 0$, the condition above obviously holds. Mizuno [11] proved that if the k-th feasible point $(x^k, w^k, y^k, p^k, q^k)$ satisfies (4) and the sequence $\{v^k\}$ satisfies (1), the next point $(x^{k+1}, w^{k+1}, y^{k+1}, p^{k+1}, q^{k+1})$ computed at Step 4 is feasible and satisfies (4). The next point computed at Step 4 is expressed as

$$(x^{k+1}, w^{k+1}, y^{k+1}, p^{k+1}, q^{k+1}) = (x^k, w^k, y^k, p^k, q^k) - (\Delta x, \Delta w, \Delta y, \Delta p, \Delta q),$$

where $(\Delta x, \Delta w, \Delta y, \Delta p, \Delta q)$ is the solution of the linear system

$$(5) \qquad \begin{cases} P_k \Delta x + X_k \Delta p = \mu^{k+1}, \\ Q_k \Delta w + W_k \Delta q = \nu^{k+1}, \\ A\Delta x = 0, \\ \Delta x + \Delta w = 0, \\ A^t \Delta y - \Delta q + \Delta p = 0, \\ \Delta y_1 = 0. \end{cases}$$

Here $P_k = \mathrm{diag}(p^k), Q_k = \mathrm{diag}(q^k)$, and μ^{k+1} and ν^{k+1} denote the first and last n-vectors of $\left((X_k p^k)^t, (W_k q^k)^t\right)^t - v^{k+1}$, respectively.

Now we shall show that the stopping criterion (2) holds for $k \le m$ in the algorithm. If (2) does not hold until $k = m - 1$, the algorithm computes m-th point $(x^m, w^m, y^m, p^m, q^m)$. Then from (1) and (4), we have

$$\begin{aligned} \begin{bmatrix} X_m p^m \\ W_m q^m \end{bmatrix} & \le v^m + 0.3\sqrt{v_{min}^m} V_m^{0.5} e \\ & \le 1.3 v^m \\ & \le \tfrac{1.3}{2} \epsilon e. \end{aligned}$$

Hence $(x^m, w^m, y^m, p^m, q^m)$ satisfies (2).

Finally we shall show one of the ways of computing the sequence $\{v^k\}$. The method is almost equivalent to Mizuno [11]. At first we compute v^0 by (1). Let v_{min}^0 and v_{max}^0 be the least and greatest components of v^0, respectively. For each $\mu > 0$, we define

$$v(\mu) = \left(\max(v_1^0, \mu), \max(v_2^0, \mu), \ldots, \max(v_{2|E|}^0, \mu)\right)^t.$$

We set $\mu^0 = v_{min}^0$ at the beginning and compute

$$\begin{aligned} \mu^{k+1} &= \max\{\mu : \|V_k^{-0.5}(v(\mu) - v(\mu^k))\| \le 0.3\sqrt{\mu^k} \text{ and } \mu \le v_{max}^0\}, \\ v^{k+1} &= v(\mu^{k+1}), \end{aligned}$$

for $k = 0, 1, \ldots$ until $\mu^{k+1} = v_{max}^0$. Suppose that $\mu^\ell = v_{max}^0$. Then it is easy to see that $v^\ell = v_{max}^0 e$ and

$$\ell \leq 1 + \frac{\log(v^0_{max}/v^0_{min})}{\log(1 + 0.3/\sqrt{2|E|})}.$$

For $k = \ell + 1, \ell + 2, \ldots$, we compute

$$\begin{aligned} \mu^{k+1} &= \left(1 - 0.3/\sqrt{2|E|}\right)\mu^k, \\ v^{k+1} &= \mu^{k+1}e \end{aligned}$$

until $\mu^{k+1} \leq \epsilon/2$. Suppose that $\mu^{\ell+\ell'} \leq \epsilon/2$. Then we have

$$\ell' \leq 1 + \frac{\log(2v^0_{max}/\epsilon)}{-\log(1 - 0.3/\sqrt{2|E|})}.$$

Hence we obtain

$$m = \ell + \ell' = O\left(|E|^{0.5}\log\left(\frac{(v^0_{max})^2}{\epsilon v^0_{min}}\right)\right).$$

3. An artificial problem having an initial interior point

In order to find an initial point, we construct an artificial minimum cost flow problem.

First of all, we appropriately assign an integer in $(0, u_{ij})$ to x^0_{ij} for each $(i,j) \in E$. Since $u_{ij} \geq 2$, such x^0 exists. Let

$$(6) \quad r_v = b_v - \sum_{(i,v)\in E} x^0_{iv} + \sum_{(v,j)\in E} x^0_{vj}$$

for each $v \in V$. If $r_v = 0$ for all $v \in V$, the point (x^0, w^0), where $w^0 = u - x^0$, is an interior feasible point of (P). Otherwise, we construct an artificial digraph $\bar{G} = (\bar{V}, \bar{E})$ by

$$\begin{aligned} \bar{V} &= V \cup \{s\}, \\ \bar{E} &= E \cup \{(s,v) : r_v > 0\} \cup \{(v,s) : r_v < 0\}. \end{aligned}$$

Let M_c be an integer and $\bar{A}$ is the node-arc incidence matrix of the graph $\bar{G}$. We consider the following minimum cost flow problem $(\bar{P})$.

$$\begin{array}{lll} (\bar{P}) & \min & \bar{c}^t\bar{x}, \\ & \text{subject to} & \bar{A}\bar{x} = \bar{b}, \\ & & \bar{x} + \bar{w} = \bar{u}, \\ & & \bar{x} \geq 0, \\ & & \bar{w} \geq 0, \end{array}$$

where

$$(7) \quad \bar{b}_j = \begin{cases} b_j & \text{for } j \in V, \\ 0 & \text{for } j = s, \end{cases}$$

$$(8) \quad \bar{u}_{ij} = \begin{cases} u_{ij} & \text{for } (i,j) \in E, \\ r_j + 1 & \text{for } i = s, \\ -r_i + 1 & \text{for } j = s, \end{cases}$$

$$(9) \quad \bar{c}_{ij} = \begin{cases} c_{ij} & \text{for } (i,j) \in E, \\ M_c & \text{for } (i,j) \in \bar{E}\backslash E. \end{cases}$$

The next lemma is used to prove the theorem 2.

Lemma 1 *Suppose that (P) and (D) are feasible. For any optimal solution (y', p', q') of (D), we have*

$$|y'_i - y'_j| \leq (|V|-1)C_{max} \quad \text{for} \quad \text{each } i \in V \text{ and } j \in V,$$

where $C_{max} = \max\{|c_{ij}| \ ((i,j) \in E)\}$.

Proof: Since (P) and (D) are feasible, (P) has an optimal solution. For a basic optimal solution (x', w') of (P), let B be the set of basic variables and

$$T = \left\{(i,j) : x'_{ij} \in B \text{ and } w'_{ij} \in B\right\}.$$

We first show that T forms a spanning tree of G. Since $|B| = |V| + |E| - 1$ and the dimensions of x and w are $|E|$, we see

$$|T| \geq |B| - |E| = |V| - 1.$$

Since (x', w') is a basic solution, T doesn't include a loop. Hence T is a spanning tree. For each $g \in V$ and $h \in V$, there is a path $\boldsymbol{P} = (i_1, i_2, \ldots, i_k)$ such that $i_1 = g, i_k = h$, and $(i_\ell, i_{\ell+1}) \in T$ or $(i_{\ell+1}, i_\ell) \in T$ for $\ell = 1, \ldots, k-1$. Since $p'_{ij} = 0$ and $q'_{ij} = 0$ for each $(i,j) \in T$, we have

$$\begin{aligned} |y'_{i_\ell} - y'_{i_{\ell+1}}| &= \begin{cases} |c'_{i_\ell i_{\ell+1}}| & \text{if } (i_\ell, i_{\ell+1}) \in T \\ |c'_{i_{\ell+1} i_\ell}| & \text{if } (i_{\ell+1}, i_\ell) \in T \end{cases} \\ &\leq C_{max}. \end{aligned}$$

Therefore we have

$$\begin{aligned} |y'_g - y'_h| &= |\textstyle\sum_{\ell=1}^{k-1}(y'_{i_\ell} - y'_{i_{\ell+1}})| \\ &\leq \textstyle\sum_{\ell=1}^{k-1} |y'_{i_\ell} - y'_{i_{\ell+1}}| \\ &\leq (|V|-1)C_{max}. \end{aligned}$$

□

The following theorem assures that an optimal solution of (P) is obtained from that of $(\bar{P})$.

Theorem 2 *Suppose that (P) and (D) are feasible. Let $M_c = (|V|-1)C_{max} + 1$. For any optimal solution $(\bar{x}^*, \bar{w}^*)$ of $(\bar{P})$, the $2|E|$-dimensional vector $(\bar{x}^*(E), \bar{w}^*(E))$ of $\bar{x}^*_{ij}$ and $\bar{w}^*_{ij}$ for $(i,j) \in E$ is an optimal solution of (P).*

Proof: The dual problem of $(\bar{P})$ is formulated as

$$\begin{array}{lll} (\bar{D}) & \max & \bar{b}^t\bar{y} - \bar{u}^t\bar{q}, \\ & \text{subject to} & \bar{A}^t\bar{y} - \bar{q} + \bar{p} = \bar{c}, \\ & & \bar{y}_1 = 0, \\ & & \bar{p} \geq 0, \\ & & \bar{q} \geq 0. \end{array}$$

Let (x', w') and (y', p', q') be optimal solutions of (P) and (D), respectively. Then the following complementarity condition holds:

$$x'_{ij}p'_{ij} = 0 \text{ and } w'_{ij}q'_{ij} = 0 \text{ for } \text{each } (i,j) \in E.$$

Set

$$(10)\quad \bar{x}'_{ij} = \begin{cases} x'_{ij} & \text{for } (i,j) \in E, \\ 0 & \text{for } (i,j) \in \bar{E}\backslash E, \end{cases}$$

$$(11)\quad \bar{w}'_{ij} = \begin{cases} w'_{ij} & \text{for } (i,j) \in E, \\ \bar{u}_{ij} & \text{for } (i,j) \in \bar{E}\backslash E, \end{cases}$$

$$(12)\quad \bar{y}'_i = \begin{cases} y'_i & \text{for } i \in V, \\ 0 & \text{for } i = s, \end{cases}$$

$$(13)\quad \bar{p}'_{ij} = \begin{cases} p'_{ij} & \text{for } (i,j) \in E, \\ M_c - \bar{y}'_j & \text{for } i = s, \\ M_c + \bar{y}'_i & \text{for } j = s, \end{cases}$$

$$(14)\quad \bar{q}'_{ij} = \begin{cases} q'_{ij} & \text{for } (i,j) \in E, \\ 0 & \text{for } (i,j) \in \bar{E}\backslash E. \end{cases}$$

Since we have

$$(15)\quad \bar{p}'_{ij} > 0 \text{ for all } (i,j) \in \bar{E}\backslash E$$

from Lemma 1 and $\bar{y}_1 = 0$, $(\bar{x}', \bar{w}')$ and $(\bar{y}', \bar{p}', \bar{q}')$ are feasible solutions of $(\bar{P})$ and $(\bar{D})$, respectively. We also see that

$$\bar{x}'_{ij}\bar{p}'_{ij} = 0 \text{ and } \bar{w}'_{ij}\bar{q}'_{ij} = 0 \text{ for } (i,j) \in \bar{E}.$$

Hence $(\bar{x}', \bar{w}')$ and $(\bar{y}', \bar{p}', \bar{q}')$ are optimal solutions of $(\bar{P})$ and $(\bar{D})$, respectively. Since $(\bar{x}^*, \bar{w}^*)$ is also an optimal solution of $(\bar{P})$, we have

$$(16)\quad \bar{x}^*_{ij}\bar{p}'_{ij} = 0 \text{ and } \bar{w}^*_{ij}\bar{q}'_{ij} = 0 \text{ for } (i,j) \in \bar{E}.$$

From (15), we have

$$\bar{x}^*_{ij} = 0 \text{ for } (i,j) \in \bar{E}\backslash E.$$

Therefore $(\bar{x}^*(E), \bar{w}^*(E))$ is a feasible solution of (P). Since (y', p', q') is an optimal solution of (D) and (16) holds, $(\bar{x}^*(E), \bar{w}^*(E))$ is an optimal solution of (P). □

From the theorem above, we may solve the artificial problem $(\bar{P})$ instead of (P). We take the following point $(\bar{x}^0, \bar{w}^0, \bar{y}^0, \bar{p}^0, \bar{q}^0)$ as an initial point of $(\bar{P})$ and $(\bar{D})$:

$$(17)\quad \begin{aligned} \bar{x}^0_{ij} &= \begin{cases} x^0_{ij} & \text{for } (i,j) \in E, \\ r_j & \text{for } i = s, \\ -r_i & \text{for } j = s, \end{cases} \\ \bar{w}^0_{ij} &= \bar{u}_{ij} - \bar{x}^0_{ij} \text{ for } (i,j) \in \bar{E}, \\ \bar{y}^0 &= 0, \\ \bar{p}^0_{ij} &= \begin{cases} 1 & \text{if } \bar{c}_{ij} \le 0, \\ \bar{c}_{ij} + 1 & \text{if } \bar{c}_{ij} > 0, \end{cases} \\ \bar{q}^0_{ij} &= \begin{cases} -\bar{c}_{ij} + 1 & \text{if } \bar{c}_{ij} \le 0, \\ 1 & \text{if } \bar{c}_{ij} > 0, \end{cases} \end{aligned}$$

where $e = (1, 1, \ldots, 1)^t \in R^{2|E|}$. It is easy to see that the point $(\bar{x}^0, \bar{w}^0, \bar{y}^0, \bar{p}^0, \bar{q}^0)$ is an interior feasible point.

4. A stopping criterion

In this section, we show that if (2) holds then we can compute an optimal solution of (P) in $O(|V|^3)$ arithmetic operations.

Theorem 3 *If the condition (2) holds for feasible solutions $(x, w) \in S_P$ and $(y, p, q) \in S_D$, then there is a primal feasible solution (x^*, w^*) which satisfies*

$$(18)\quad \begin{array}{ll} x^*_{ij} = 0 & \text{for } (i,j) \in \left\{(i,j) \in E : x_{ij} < \frac{1}{|E|+|V|}\right\}, \\ w^*_{ij} = 0 & \text{for } (i,j) \in \left\{(i,j) \in E : w_{ij} < \frac{1}{|E|+|V|}\right\}, \end{array}$$

and such (x^*, w^*) is optimal.

Proof: Let m_P and m_D be the dimensions of the convex sets S_P and S_D, respectively. Since the rank of A is $|V| - 1$, we have

$$(19)\quad m_P \leq 2|E| - (|V| + |E| - 1) \leq |E| + |V| - 1,$$

$$(20)\quad m_D \leq |E| + |V| - 1.$$

From Caratheodory's theorem ([15]), we have

$$\begin{aligned} \begin{bmatrix} x \\ w \end{bmatrix} &= \sum_{h=1}^{m_P+1} \lambda_h \begin{bmatrix} f^h \\ g^h \end{bmatrix} + \xi \begin{bmatrix} f' \\ g' \end{bmatrix}, \\ \sum_{h=1}^{m_P+1} \lambda_h &= 1, \ \lambda_h \geq 0 \text{ for } h = 1, 2, \ldots, m_P + 1, \\ \xi &\geq 0, \end{aligned}$$

where $\begin{bmatrix} f^h \\ g^h \end{bmatrix}$ $(h = 1, 2, \ldots, m_P + 1)$ are vertices of S_P and $\begin{bmatrix} f' \\ g' \end{bmatrix}$ is an unbounded direction of S_P. Then there is an index ℓ such that $\lambda_\ell \geq 1/(m_P + 1)$. Since $f^h \geq 0, f' \geq 0, g^h \geq 0$ and $g' \geq 0$, we have

$$(21)\quad \begin{bmatrix} f^\ell \\ g^\ell \end{bmatrix} \leq \frac{1}{\lambda_\ell} \begin{bmatrix} x \\ w \end{bmatrix} \leq (m_P + 1) \begin{bmatrix} x \\ w \end{bmatrix}.$$

From (19) and (21), we obtain

$$(22)\quad \begin{array}{ll} f^\ell_{ij} < 1 & \text{for } (i,j) \in \left\{(i,j) \in E : x_{ij} < \frac{1}{|E|+|V|}\right\}, \\ g^\ell_{ij} < 1 & \text{for } (i,j) \in \left\{(i,j) \in E : w_{ij} < \frac{1}{|E|+|V|}\right\}. \end{array}$$

Since all the components of each vertex of S_P are integral, we have (18) for $(x^*, w^*) = (f^\ell, g^\ell)$.

In the same way, we can show that there is a dual feasible solution (y^*, p^*, q^*) which satisfies

$$(23)\quad \begin{array}{ll} p^*_{ij} = 0 & \text{for } (i,j) \in \left\{(i,j) \in E : p_{ij} < \frac{1}{|E|+|V|}\right\}, \\ q^*_{ij} = 0 & \text{for } (i,j) \in \left\{(i,j) \in E : q_{ij} < \frac{1}{|E|+|V|}\right\}. \end{array}$$

From (2),(18) and (23), the complementarity condition holds. Therefore (x^*, w^*) is optimal. □

From the theorem above, we can compute the optimal solution of (P) by solving the following feasibility problem

$$(24) \quad \begin{cases} Ax = b, \\ 0 \le x \le u, \\ x_{ij} = 0 \text{ for } (i,j) \in \left\{(i,j) \in E : x_{ij}^m < \frac{1}{|E|+|V|}\right\}, \\ x_{ij} = u_{ij} \text{ for } (i,j) \in \left\{(i,j) \in E : w_{ij}^m < \frac{1}{|E|+|V|}\right\}, \end{cases}$$

where (x^m, w^m) is the last primal feasible solution obtained in the algorithm presented in Section 2. From Theorem 3, the solution of the problem (24) exists and it is the optimal solution of (P). We can easily convert (24) into a max flow problem which can be solved in $O(|V|^3)$ arithmetic operations (see, [13]).

5. Computational complexity

In this section, we evaluate the computational complexity of our algorithm.

Theorem 4 *If we solve the artificial problem $(\bar{P})$ and $(\bar{D})$ by the algorithm given in Section 2, it requires at most $O(|V|^3|E|^{0.5}\log(|V|M))$ arithmetic operations.*

This theorem follows from the two lemmas below.

Lemma 5 *The number of iterations is bounded by $m = O(|E|^{0.5}\log(|V|M))$.*

Proof: By (6), we have $|r_v| \le |V|M$ for each $v \in V$. From the definition (17) of the initial point, we see that, for each $(i,j) \in \bar{E}$,

$$\begin{aligned} &1 \le \bar{x}_{ij}^0 \le |V|M, \\ &1 \le \bar{w}_{ij}^0 \le M, \\ &1 \le \bar{p}_{ij}^0 \le M_c + 1, \\ &1 \le \bar{q}_{ij}^0 \le M_c + 1. \end{aligned}$$

Since $M_c = (|V| - 1)C_{max} + 1 \le |V|M$, we have

$$\begin{aligned} v_{max}^0 &= \max \left\{\bar{x}_{ij}^0\bar{p}_{ij}^0, \bar{w}_{ij}^0\bar{q}_{ij}^0 : (i,j) \in E\right\} \le |V|^2M^2 + |V|M, \\ v_{min}^0 &\ge 1. \end{aligned}$$

We also have $O(\frac{1}{\epsilon}) = O(|E|^2) \le O(|V|^4)$. Hence we obtain

$$\begin{aligned} m &= O\left(|E|^{0.5}\log\left(\frac{(v_{max}^0)^2}{\epsilon v_{min}^0}\right)\right) \\ &= O(|E|^{0.5}\log(|V|M)). \end{aligned}$$

□

Lemma 6 *For each $k = 0, 1, \ldots, m-1$, we can compute the point $(x^{k+1}, w^{k+1}, y^{k+1}, p^{k+1}, q^{k+1})$ from $(x^k, w^k, y^k, p^k, q^k)$ in at most $O(|V|^3)$ arithmetic operations.*

Proof: Since $O(|\bar{V}|) = O(|V|)$ and $O(|\bar{E}|) = O(|E|)$, it is enough to show the lemma for the original problem. Eliminating the variables $\Delta x, \Delta w, \Delta p$, and Δq from the system (5), we have

$$(25) \quad ADA^t\Delta y = r \text{ and } \Delta y_1 = 0,$$

where

$$D = (X_k^{-1}P_k + W_k^{-1}Q_k)^{-1} \text{ and } r = AD(W_k^{-1}\nu^{k+1} - X_k^{-1}\mu^{k+1}).$$

Since A is the incidence matrix and D is the diagonal matrix, we can compute the matrix ADA^t in $O(|E|)$ arithmetic operations. Since we solve the linear system (25) in $O(|V|^3)$ arithmetic operations and get $\Delta x, \Delta w, \Delta p$, and Δq in $O(|V||E|)$ arithmetic operations, we have the result. □

6. Conclusions

In this paper, we propose an interior point algorithm for a minimum cost flow problem. We show that the algorithm requires at most $O(|E|^{0.5}\log(|V|M))$ iterations, $O(|V|^3)$ arithmetic operations in each iteration, and $O(|V|^3|E|^{0.5}\log(|V|M))$ arithmetic operations in total. Here $|V|$ denotes the number of nodes, $|E|$ that of arcs, and M the maximum absolute value of the input data.

In the case of $|E| = O(|V|^2)$, the total running time is almost the same as that of Edmonds and Karp [1]. In the case of $|E| < O(|V|^2)$, the matrix ADA^t appeared in the linear system (25) becomes sparse. If we can adopt the sparsity effectively, we may be able to decrease the running time of each iteration.

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