

PIECEWISE LINEAR RISK FUNCTION AND PORTFOLIO OPTIMIZATION

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Abstract A new portfolio optimization model using a piecewise linear risk function is proposed. This model is similar to, but has several advantages over the classical Markowitz's quadratic risk model. First, it is much easier to generate an optimal portfolio since the problem to be solved is a linear program instead of a quadratic program. Second, integer constraints associated with real transaction can be incorporated without making the problem intractable. Third, it enables us to distinguish two distributions with the same first and second moment but with different third moment. Fourth, we can generate the capital-market line and derive CAPM type equilibrium relations. We compared the piecewise linear risk model with the quadratic risk model using historical data of Tokyo Stock Market, whose results partly support the claims stated above.

1 Introduction

Harry Markowitz, in his seminal work [7], formulated the portfolio optimization problem as a quadratic programming problem in which variance of return out of the portfolio is minimized subject to the constraint on the average return. This formulation which has long served as the basis of financial theory is known to be valid if an investor's utility function is quadratic and/or the distribution of the rate of return of stocks is multivariate normal.

It turned out, however that solving a large scale quadratic programming problem is not easy if not impossible. This computational burden was one of the reasons why this model was not used in practice to determine an optimal portfolio consisting of more than a few hundred stocks. Also this model suggests investors to purchase a large number of different stocks. This is rather inconvenient since managing a portfolio with a large number of different stocks is weary and expensive. Yet even more computational difficulty arises if we take into account the constraint on the minimal unit associated with each transaction. In fact, we have to solve an intractable integer quadratic programming problem particularly when the amount of fund is small compared to the number of stocks.

Also, the detailed study [4] of the historical data in the stock market shows that the distribution of the rate of return of stocks is neither normal nor even symmetric. In addition, recent studies on portfolio insurance show that most investor do not purchase "efficient" portfolio implied by Markowitz's model. They usually buy a portfolio apart from the efficient frontier. This means that quadratic utility model need not apply to all investors.

These observations motivated us to introduce L_1 risk function [5] which has the following

advantages over its counterpart.

First, the associated optimization problem is an easy linear program instead of a "not so easy" quadratic program. Typically, the model with several hundred stocks may be solved on a real time basis by using current state-of-the-art technique. Second, this model is expected to suggest investors to purchase significantly fewer number of different stocks than its counterpart. Also it is much easier to incorporate integer constraints associated with minimum transaction units. Third, this model can incorporate investors' subjective perception against risk and hence is more operational than the traditional model. Finally, our model is essentially equivalent to the traditional model if the rate of return of stocks are multivariate normally distributed. Thus our model can be used as a good proxy of Markowitz model.

In Section 2, we introduce compound L_1 risk model and derive some of its important properties. In Section 3, we derive the "capital-market line" and equilibrium conditions for compound L_1 risk model. It will be shown that the classical Sharpe-Lintner-Mossin type relation between individual stock and market portfolio holds for our model as well. Section 4 will be devoted to the generalization of the model in which the risk function can distinguish positive skewness and negative skewness of the underlying distribution of the rate of return. Finally, some computational results using the historical data of Tokyo Stock Market will be presented in Section 5.

2 Compound L_1 Risk Functions and Portfolio Optimization

Let there be n stocks denoted by S_j ($j = 1, \dots, n$) and let R_j be the random variable representing the rate of return of S_j . Also let x_j be the amount of money invested in S_j out of his total fund M_0 .

The average rate of return associated with this investment is given by

$$(2.1) \quad r(x_1, \dots, x_n) = E\left[\sum_{j=1}^n R_j x_j\right] = \sum_{j=1}^n E[R_j] x_j$$

where $E[\cdot]$ stands for the expected value of a random variable in the bracket. An investor wants to make $r(x_1, \dots, x_n)$ as large as possible. At the same time, he wants to minimize his "risk".

Markowitz [7] employed the variance of return

$$(2.2) \quad v(x_1, \dots, x_n) = E\left[\left\{\sum_{j=1}^n R_j x_j - E\left[\sum_{j=1}^n R_j x_j\right]\right\}^2\right]$$

as the measure of risk and formulated the portfolio optimization problem as a quadratic programming problem

$$(2.3) \quad \begin{cases} \text{minimize} & v(x_1, \dots, x_n) \\ \text{subject to} & r(x_1, \dots, x_n) \geq \rho M_0 \\ & \sum_{j=1}^n x_j = M_0 \\ & x_j \geq 0, \quad j = 1, \dots, n \end{cases}$$

where ρ is a parameter representing the minimal rate of return specified by the investor.

This model has several nice properties from the theoretical point of view, but it has not been used extensively in practice by practitioners. Among the more important reasons are (i) solving a large scale quadratic programming problem on a real time basis was difficult, at least until very recently, (ii) many investors were not convinced of the validity of the quadratic risk function [6], (iii) the solution of (2.3) may contain too many nonzero variables to be practical particularly when M_0 is relatively small.

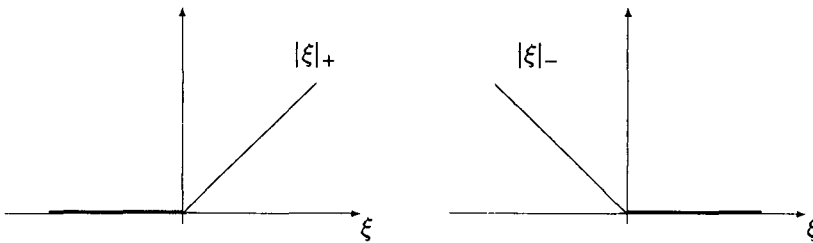


Figure 2.1

These observations led us [5] to introduce an alternative measure of risk

$$(2.4) \quad \begin{aligned} w_\alpha(x_1, \dots, x_n) = & E\left[\left| \sum_{j=1}^n R_j x_j - E\left[\sum_{j=1}^n R_j x_j \right] \right|_- \right] \\ & - \alpha E\left[\left| \sum_{j=1}^n R_j x_j - E\left[\sum_{j=1}^n R_j x_j \right] \right|_+ \right] \end{aligned}$$

where α is a positive parameter representing the degree of risk aversion of an investor and

$$(2.5) \quad |\xi|_+ = \begin{cases} \xi, & \text{if } \xi \geq 0 \\ 0, & \text{if } \xi < 0 \end{cases}$$

$$(2.6) \quad |\xi|_- = \begin{cases} 0, & \text{if } \xi \geq 0 \\ -\xi, & \text{if } \xi < 0 \end{cases}$$

Theorem 2.1 *If (R_1, \dots, R_n) are multivariate normally distributed with mean (μ_1, \dots, μ_n) and variance-covariance matrix $\Sigma \equiv (\sigma_{ij})$, then*

$$w_\alpha(x_1, \dots, x_n) = \frac{1 - \alpha}{\sqrt{2\pi}} \left\{ \sum_{i=1}^n \sum_{j=1}^n \sigma_{ij} x_i x_j \right\}^{1/2}$$

Proof. It is well known [8] that the random variable $Y = \sum_{j=1}^n R_j x_j$ is normally distributed with mean $\mu \equiv \sum_{j=1}^n \mu_j x_j$ and variance $\sigma^2 \equiv \sum_{i=1}^n \sum_{j=1}^n \sigma_{ij} x_i x_j$. Thus

$$\begin{aligned} w_\alpha(x_1, \dots, x_n) &= E[|Y - E[Y]|_-] - \alpha E[|Y - E[Y]|_+] \\ &= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^0 u \exp -\frac{u^2}{2\sigma^2} du - \alpha \frac{1}{\sqrt{2\pi}\sigma} \int_0^{\infty} u \exp -\frac{u^2}{2\sigma^2} du \\ &= \frac{\sigma}{\sqrt{2\pi}}(1 - \alpha) \quad \square \end{aligned}$$

This theorem implies that minimizing $w_\alpha(x_1, \dots, x_n)$ is equivalent to minimizing $v(x_1, \dots, x_n)$ if $\alpha < 1$ and (R_1, \dots, R_n) are multivariate normally distributed.

Let us next proceed to the representation of the function $w_\alpha(x_1, \dots, x_n)$ using historical data or projected data. Let r_{jt} be the realization of random variable R_j during period t ($t = 1, \dots, T$), which we assume to be available from historical data or from some future projection. We also assume that the expected value of the random variable can be approximated by the average derived from these data.

In particular, let

$$(2.7) \quad r_j \equiv E[R_j] = \sum_{t=1}^T r_{jt}/T$$

Then

$$(2.8) \quad r(x_1, \dots, x_n) = E[\sum_{j=1}^n R_j x_j] = \sum_{j=1}^n r_j x_j$$

$$\begin{aligned} (2.9) \quad w_\alpha(x_1, \dots, x_n) &= E[|\sum_{j=1}^n R_j x_j - E[\sum_{j=1}^n R_j x_j]|_-] - \alpha E[|\sum_{j=1}^n R_j x_j - E[\sum_{j=1}^n R_j x_j]|_+] \\ &= \sum_{t=1}^T \{|\sum_{j=1}^n (r_{jt} - r_j)x_j|_- - \alpha |\sum_{j=1}^n (r_{jt} - r_j)x_j|_+\}/T \end{aligned}$$

Let

$$(2.10) \quad \xi_t = \sum_{j=1}^n (r_{jt} - r_j)x_j, \quad t = 1, \dots, T$$

and let

$$(2.11) \quad g_\alpha(\xi) \equiv |\xi|_- - \alpha |\xi|_+$$

Then

$$(2.12) \quad w_\alpha(x_1, \dots, x_n) = \sum_{t=1}^T \{|\xi_t|_- - \alpha |\xi_t|_+\}/T = \sum_{t=1}^T g_\alpha(\xi_t)/T$$

It can be seen from Fig. 2.2 that $\alpha = -1$ corresponds to absolute value (L_1) risk. Also $\alpha = 0$ is associated with an investor who only cares about "below the average" return. Moreover, $\alpha > 0$ represents an investor whose risk associated with "below the average" return is compensated to some extent by "above the average" return. In fact, an investor whose α is greater than one may be viewed as a risk prone investor.

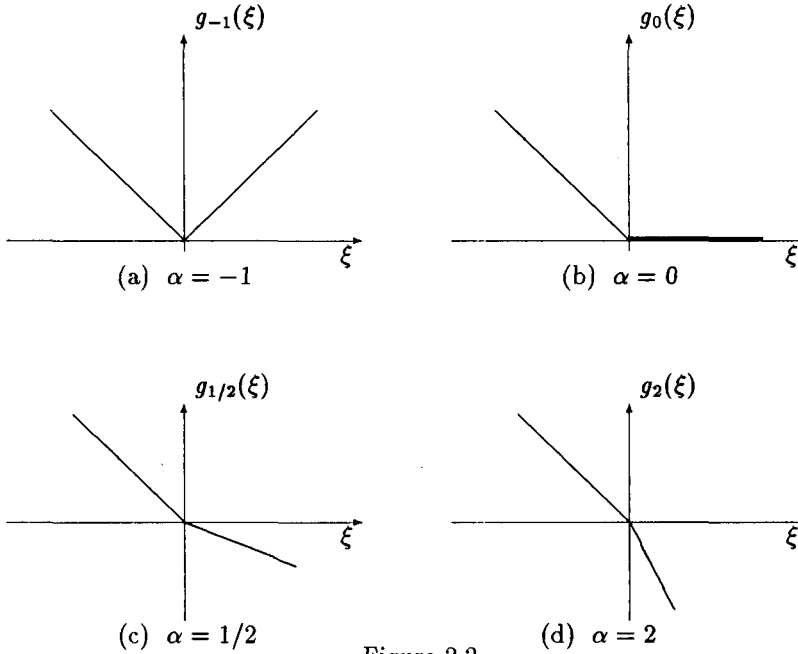


Figure 2.2

Replacing quadratic risk function of (2.2) by compound L_1 risk function (2.4), we obtain an alternative class of portfolio optimization problems $P(\alpha)$:

$$(2.13) \quad \begin{cases} \text{minimize} & \sum_{t=1}^T \{ |\sum_{j=1}^n a_{jt} x_j|_- - \alpha |\sum_{j=1}^n a_{jt} x_j|_+ \} \\ \text{subject to} & \sum_{j=1}^n r_j x_j \geq \rho M_0 \\ & \sum_{j=1}^n x_j = M_0 \\ & x_j \geq 0, \quad j = 1, \dots, n \end{cases}$$

where

$$a_{jt} = r_{jt} - r_j, \quad j = 1, \dots, n; \quad t = 1, \dots, T$$

Theorem 2.2 *The class of optimization problems $P(\alpha)$ have the same optimal solution for all $\alpha \in (0, 1)$. Also they have the same optimal solution for all α greater than one.*

Proof. Let us note the identity

$$|\xi|_+ = \frac{1}{2}\{|\xi| + \xi\}$$

$$|\xi|_- = \frac{1}{2}\{|\xi| - \xi\}$$

Then

$$\begin{aligned} w_\alpha(x_1, \dots, x_n) &= \left\{ \frac{1-\alpha}{2} \sum_{t=1}^T \left| \sum_{j=1}^n a_{jt} x_j \right| - \frac{1+\alpha}{2} \sum_{t=1}^T \sum_{j=1}^n a_{jt} x_j \right\} / T \\ &= \frac{1-\alpha}{2} \sum_{t=1}^T \left| \sum_{j=1}^n a_{jt} x_j \right| / T \end{aligned}$$

since

$$\sum_{t=1}^T \sum_{j=1}^n a_{jt} x_j = \sum_{j=1}^n \sum_{t=1}^T (r_{jt} - r_j) x_j = 0$$

Thus for fixed T , minimizing $w_\alpha(x_1, \dots, x_n)$ is equivalent to minimizing $\sum_{t=1}^T \left| \sum_{j=1}^n a_{jt} x_j \right|$ for all α less than one. Also for fixed T it is equivalent to maximizing $\sum_{t=1}^T \left| \sum_{j=1}^n a_{jt} x_j \right|$ for all α greater than one. \square

We thus need to solve the following two problems:

$$(2.14) \quad \left\{ \begin{array}{l} \text{minimize} \quad \sum_{t=1}^T \left| \sum_{j=1}^n a_{jt} x_j \right| \\ \text{subject to} \quad \sum_{j=1}^n r_j x_j \geq \rho M_0 \\ \quad \quad \quad \sum_{j=1}^n x_j = M_0 \\ \quad \quad \quad x_j \geq 0, \quad j = 1, \dots, n \end{array} \right.$$

$$(2.15) \quad \left\{ \begin{array}{l} \text{maximize} \quad \sum_{t=1}^T \left| \sum_{j=1}^n a_{jt} x_j \right| \\ \text{subject to} \quad \sum_{j=1}^n r_j x_j \geq \rho M_0 \\ \quad \quad \quad \sum_{j=1}^n x_j = M_0 \\ \quad \quad \quad x_j \geq 0, \quad j = 1, \dots, n \end{array} \right.$$

It is well known [2] that (2.14) is equivalent to the following linear program:

$$\begin{aligned}
 (2.16) \quad & \left. \begin{aligned}
 & \text{minimize} && \sum_{t=1}^T y_t \\
 & \text{subject to} && y_t + \sum_{j=1}^n a_{jt} x_j \geq 0, \quad t = 1, \dots, T \\
 & && y_t - \sum_{j=1}^n a_{jt} x_j \geq 0, \quad t = 1, \dots, T \\
 & && \sum_{j=1}^n r_j x_j \geq \rho M_0 \\
 & && \sum_{j=1}^n x_j = M_0 \\
 & && x_j \geq 0, \quad j = 1, \dots, n
 \end{aligned} \right\}
 \end{aligned}$$

Remark. The dual of the problem (2.16) is:

$$\begin{aligned}
 (2.17) \quad & \left. \begin{aligned}
 & \text{maximize} && \rho M_0 z_1 + M_0 z_2 \\
 & \text{subject to} && 2 \sum_{t=1}^T a_{jt} \xi_t + r_j z_1 + z_2 \leq 0, \quad j = 1, \dots, n \\
 & && 0 \leq \xi_t \leq 1, \quad t = 1, \dots, T \\
 & && z_1 \geq 0
 \end{aligned} \right\}
 \end{aligned}$$

Thus, we had better solve this problem instead of (2.16) if n is less than T . \square

Problem (2.15), on the other hand cannot be converted into a linear program.

Theorem 2.3 *There exists an optimal solution x_j^* ($j = 1, \dots, n$) of (2.15) for which at most two indices j satisfy $x_j^* > 0$.*

Proof. The objective function of (2.15) is convex. Thus there exists an optimal solution among extreme points of the feasible region [2], which has the stated property. \square

This theorem implies that the optimal portfolio can be obtained by enumerating all extreme points in $O(n^2)$ steps.

3 Capital Market Line and Equilibrium Model for L_1 Risk Function

We showed in Section 2 that compound L_1 risk minimization is equivalent to the L_1 risk minimization problem (2.14) for all α less than one. In this section, we will derive capital market line and the equilibrium relation between individual stock and market portfolio [9] for L_1 risk model.

For this purpose, let us first assume without loss of generality that $M_0 = 1$. Also let us remove the nonnegativity constraint on each variable in problem (2.14). This means that an investor is allowed to purchase each stock as much as he wants. Also, he is allowed to sell each stock short (i.e., regardless of whether he owns it or not). Under these assumptions, L_1 risk minimization problem (2.14) is substantially simplified as follows:

$$(3.1) \quad \begin{cases} \text{minimize} & \sum_{t=1}^T \left| \sum_{j=1}^n a_{jt} x_j \right| \\ \text{subject to} & \sum_{j=1}^n r_j x_j \geq \rho \\ & \sum_{j=1}^n x_j = 1 \end{cases}$$

Let $x_j(\rho)$ ($j = 1, \dots, n$) be an optimal solution of this problem. Also let

$$(3.2) \quad f(\rho) = \sum_{t=1}^T \left| \sum_{j=1}^n a_{jt} x_j(\rho) \right|$$

Theorem 3.1 $f(\rho)$ is a non-decreasing piecewise linear convex function.

Proof. $f(\rho)$ is obviously non-decreasing. That $f(\rho)$ is piecewise linear convex function follows from the standard result of linear programming [2] by noting that (3.1) is equivalent to the following linear programming problem:

$$(3.3) \quad \begin{cases} \text{minimize} & \sum_{t=1}^T y_t \\ \text{subject to} & y_t + \sum_{j=1}^n a_{jt} x_j \geq 0, \quad t = 1, \dots, T \\ & y_t - \sum_{j=1}^n a_{jt} x_j \geq 0, \quad t = 1, \dots, T \\ & \sum_{j=1}^n r_j x_j \geq \rho \\ & \sum_{j=1}^n x_j = 1 \quad \square \end{cases}$$

Fig. 3.1 shows the piecewise linear convex function $f(\rho)$, which we call the efficient frontier.

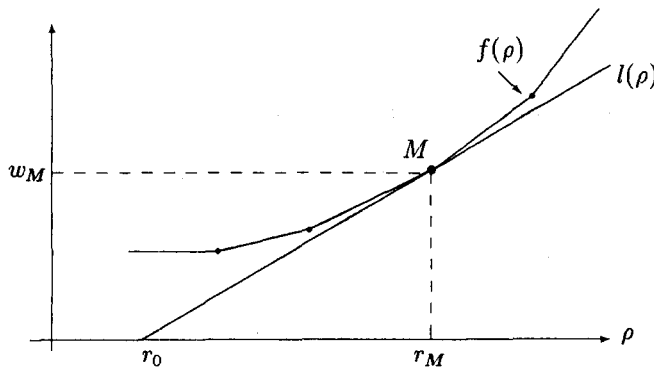


Figure 3.1

Next, we introduce a special asset S_0 whose rate of return is constant throughout the entire period, i.e.,

$$(3.4) \quad r_{0t} = r_0, \quad t = 1, \dots, T$$

S_0 is called a risk-free asset.

By virtue of Theorem 3.1, we can draw a tangent line $l(\rho)$ to the efficient frontier $f(\rho)$ passing through point $(r_0, 0)$. Let us assume in this section that $M = (r_M, w_M)$ is the unique point of tangent and let P_M be the portfolio corresponding to this point. P_M is called the market portfolio associated with L_1 risk function, which can be obtained by solving the following fractional program:

$$(3.5) \quad \left\{ \begin{array}{l} \text{minimize} \quad \frac{\sum_{t=1}^T |\sum_{j=1}^n a_{jt} x_j|}{\sum_{j=1}^n r_j x_j - r_0} \\ \text{subject to} \quad \sum_{j=1}^n x_j = 1 \end{array} \right.$$

This problem is equivalent to the linear fractional program:

$$(3.6) \quad \left\{ \begin{array}{l} \text{minimize} \quad \frac{\sum_{t=1}^T y_t}{\sum_{j=1}^n r_j x_j - r_0} \\ \text{subject to} \quad y_t + \sum_{j=1}^n a_{jt} x_j \geq 0, \quad t = 1, \dots, T \\ \quad \quad \quad y_t - \sum_{j=1}^n a_{jt} x_j \geq 0, \quad t = 1, \dots, T \\ \quad \quad \quad \sum_{j=1}^n x_j = 1 \end{array} \right.$$

so that it can be solved by a variant of simplex method [1].

Theorem 3.2 *All L_1 risk minimizing investors purchase the combination of market portfolio and risk-free asset and nothing else.*

Proof. Let λ be the amount of fund invested into risk free asset S_0 and let $(1 - \lambda)$ be the amount of fund invested into the market portfolio P_M . Simple arithmetic shows that L_1 risk of this portfolio is given by $(1 - \lambda)w_M$. This means that the L_1 risk of this portfolio lies on the line $l(\rho)$ which lies everywhere below $f(\rho)$. Thus we conclude that the combination of risk-free asset and market portfolio is better than any combination of risky assets S_1, \dots, S_n . \square

We will next proceed to the derivation of the equilibrium relations between individual stocks and the market portfolio. For this purpose, let us define a function

$$(3.7) \quad h(x_1, \dots, x_n) = \sum_{t=1}^T \left| \sum_{j=1}^n a_{jt} x_j \right|$$

Note that this function is differentiable at (x_1^M, \dots, x_n^M) if

$$(3.8) \quad \sum_{j=1}^n a_{jt} x_j^M \neq 0, \quad t = 1, \dots, T$$

Theorem 3.3 Let $h(x_1, \dots, x_n)$ be differentiable at (x_1^M, \dots, x_n^M) . Then there exists a constant θ_i such that

$$(3.9) \quad r_i = r_M + \theta_i(r_M - r_0), \quad i = 1, \dots, n$$

Proof. Market portfolio x_j^M ($j = 1, \dots, n$) is by definition an optimal solution of the problem (3.5). Thus there exists a constant λ such that

$$\frac{\partial}{\partial x_i} \frac{h(x_1, \dots, x_n)}{\sum_{j=1}^n r_j x_j - r_0} \Big|_{x_j=x_j^M} - \lambda = 0, \quad i = 1, \dots, n$$

from which we obtain

$$(3.10) \quad \frac{\eta_i - r_i w_M / (r_M - r_0)}{r_M - r_0} = \lambda, \quad i = 1, \dots, n$$

where

$$\eta_i = \frac{\partial}{\partial x_i} h(x_1, \dots, x_n) \Big|_{x_j=x_j^M}$$

Multiplying x_i^M on both sides of (3.10) and summing for all i , we obtain

$$(3.11) \quad \lambda = \frac{\sum_{i=1}^n \eta_i x_i^M - r_M w_M / (r_M - r_0)}{r_M - r_0}$$

Equating both sides of (3.10) and (3.11), we obtain the relation

$$(3.12) \quad r_i = r_M + \frac{1}{w_M} (\eta_i - \sum_{l=1}^n \eta_l x_l^M) (r_M - r_0)$$

Thus we obtain (3.9) by defining

$$(3.13) \quad \theta_i = [\eta_i - \sum_{l=1}^n \eta_l x_l^M] / w_M \quad \square$$

θ_i will be called "Theta" of the stock S_j .

Corollary 3.2 Let $P = (S_1 x_1, \dots, S_n x_n)$ be an arbitrary portfolio where $\sum_{i=1}^n x_i = 1$. Then the theta of the portfolio is given by $\sum_{i=1}^n \theta_i x_i$.

Proof. The average return of P is given by

$$r_p = \sum_{i=1}^n r_i x_i = r_M + \left(\sum_{i=1}^n \theta_i x_i \right) (r_M - r_0) \quad \square$$

Let us note the remarkable similarity of (3.9) with the Sharpe-Lintner-Mossin relation

$$(3.14) \quad r_i = r_0 + \beta_i (r_M - r_0)$$

for the quadratic risk formulation [3] where β_i is the so called "Beta" of S_i .

Extension of our result to the non-differentiable situation will need more complicated analysis and will be postponed to the subsequent paper.

4 Piecewise Linear Risk Function with Three Linear Pieces

In this section, we introduce a more general risk function with three linear components depicted in Fig. 4.1.

This risk function can be expressed as follows:

$$(4.1) \quad g(\xi) = p|\xi - \alpha\rho| - (\xi - \rho) + q|\xi - \beta\rho|_+ \\ = \frac{p}{2}|\xi - \alpha\rho| + \frac{q}{2}|\xi - \beta\rho| + \left(\frac{-p+q}{2} - 1\right)\xi + \left(\frac{\alpha p - \beta q}{2} + 1\right)\rho$$

where p, q, α, β are some positive constants.

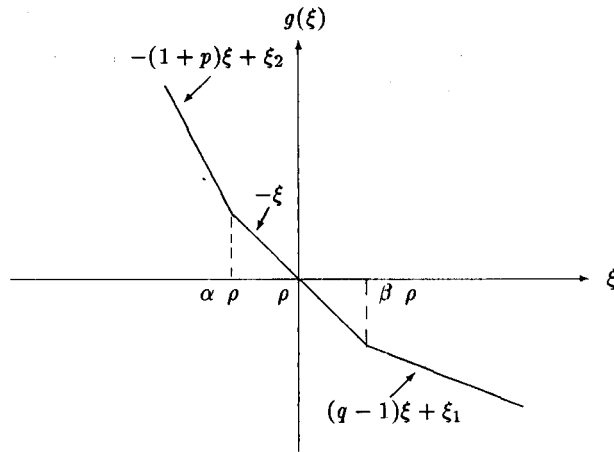


Figure 4.1

Example. Let us consider the special case in which

$$(p, q, \alpha, \beta) = (1, 1/2, 1/2, 3/2)$$

Corresponding risk function is given by

$$g(\xi) = \frac{1}{2}|\xi - \frac{1}{2}\rho| + \frac{1}{4}|\xi - \frac{3}{2}\rho| - \frac{5}{4}\xi + \frac{7}{8}\rho$$

Let us consider two discrete random variables X_1 and X_2 whose density functions are given by:

$$\text{Prob}\{X_1 = \xi\} = \begin{cases} 0.2, & \xi = 0 \\ 0.1, & \xi = 1 \\ 0.4, & \xi = 2 \\ 0.3, & \xi = 7 \\ 0, & \text{otherwise} \end{cases}$$

$$\text{Prob}\{X_2 = \xi\} = \begin{cases} 0.3, & \xi = -1 \\ 0.4, & \xi = 4 \\ 0.1, & \xi = 5 \\ 0.2, & \xi = 6 \\ 0, & \text{otherwise} \end{cases}$$

Simple arithmetic shows that $E[X_1] = E[X_2] = 3$ and $V[X_1] = V[X_2] = 7.4$. Hence two random variables X_1 and X_2 are identical from the viewpoint of Markowitz's model. It turns out, however that

$$E\{X_1 - E[X_1]\}^3 = 6.2 \qquad E\{X_2 - E[X_2]\}^3 = -6.2$$

There is a good reason to believe that a random variable with larger third moment is preferred since it has a longer tail to the right of the mean.

Let us compute the risk of X_1 and X_2 using risk function $g(\xi)$:

$$g_1 = \frac{1}{2}E|X_1 - \frac{1}{2}\rho| + \frac{1}{4}E|X_1 - \frac{3}{2}\rho| - \frac{5}{4}E[X_1] + \frac{7}{8}\rho$$

$$g_2 = \frac{1}{2}E|X_2 - \frac{1}{2}\rho| + \frac{1}{4}E|X_2 - \frac{3}{2}\rho| - \frac{5}{4}E[X_2] + \frac{7}{8}\rho$$

Table 4.1:

ρ	g_1	g_2	$g_1 - g_2$
0	-1.500	-1.025	-0.475
1	-1.013	-0.363	-0.650
2	-0.275	0.300	-0.575
3	0.613	1.063	-0.450
4	1.500	2.075	-0.575
5	2.663	3.263	-0.600
6	3.975	4.450	-0.475

Table 4.1 shows the value of g_1 and g_2 for various value of ρ . It turns out that the risk of X_1 is smaller than that of X_2 for all values of ρ . Thus X_1 is preferred to X_2 in our model. This is compatible with our statement about the third moment of these variables. \square

The optimization problem using the risk function (4.1) can be written as follows:

$$(4.2) \quad \begin{cases} \text{minimize} & \frac{p}{2} \sum_{t=1}^T \left| \sum_{j=1}^n r_{jt} x_j - \alpha \rho M_0 \right| \\ & + \frac{q}{2} \sum_{t=1}^T \left| \sum_{j=1}^n r_{jt} x_j - \beta \rho M_0 \right| + \left(\frac{\alpha p - \beta q}{2} + 1 \right) \rho M_0 \\ \text{subject to} & \sum_{j=1}^n r_j x_j \geq \rho M_0 \\ & \sum_{j=1}^n x_j = M_0 \\ & x_j \geq 0, \quad j = 1, \dots, n \end{cases}$$

This is a convex minimization problem whose optimal solution can be obtained by solving the following linear program:

$$(4.3) \quad \left\{ \begin{array}{l} \text{minimize} \quad \frac{p}{2} \sum_{t=1}^T y_t + \frac{q}{2} \sum_{t=1}^T z_t + \left(\frac{\alpha p - \beta q}{2} + 1 \right) \rho M_0 \\ \text{subject to} \quad y_t - \sum_{j=1}^n r_{jt} x_j \geq -\alpha \rho M_0, \\ \quad \quad \quad y_t + \sum_{j=1}^n r_{jt} x_j \geq \alpha \rho M_0, \\ \quad \quad \quad z_t - \sum_{j=1}^n r_{jt} x_j \geq -\beta \rho M_0, \\ \quad \quad \quad z_t + \sum_{j=1}^n r_{jt} x_j \geq \beta \rho M_0, \\ \quad \quad \quad \sum_{j=1}^n r_j x_j \geq \rho M_0 \\ \quad \quad \quad \sum_{j=1}^n x_j = M_0 \\ \quad \quad \quad x_j \geq 0, \quad j = 1, \dots, n \end{array} \right.$$

The dual of this problem is

$$(4.4) \quad \left\{ \begin{array}{l} \text{maximize} \quad -2\alpha \rho \sum_{t=1}^T \xi_t - 2\beta \rho \sum_{t=1}^T \mu_t + \rho z_1 + z_2 \\ \text{subject to} \quad 2 \sum_{t=1}^T r_{jt} \xi_t + 2 \sum_{t=1}^T r_{jt} \mu_t - r_j z_1 - z_2 \geq \frac{p+q}{2} \sum_{t=1}^T r_{jt} \quad j = 1, \dots, n \\ \quad \quad \quad 0 \leq \xi_t \leq \frac{p}{2}, \quad 0 \leq \mu_t \leq \frac{q}{2}, \quad z_1 \geq 0 \quad t = 1, \dots, T \end{array} \right.$$

Note that this problem has exactly the same number of constraints as its counterpart (2.17), so that it can be solved in much the same time as (2.17) by using upper bounding simplex method.

5 Comparison of Models Using Historical Data

We compared the models proposed in this paper with the Markowitz's quadratic risk model using historical data of Tokyo Stock Market. Monthly data of fifty stocks were collected for five years from November 1983 through October 1988. These stocks are the ones incorporated in the future index called Osaka 50.

Fig. 5.1 shows the efficient frontier of Markowitz's L_2 -risk model. Also Fig. 5.2 shows the efficient frontiers of L_1 -risk model (2.16) and piecewise linear risk model (4.2), whose parameters are chosen as follows:

$$(p, q, \alpha, \beta) = \left(1, \frac{1}{2}, \frac{1}{2}, \frac{3}{2} \right)$$

All the problems were solved on SUN IV system.

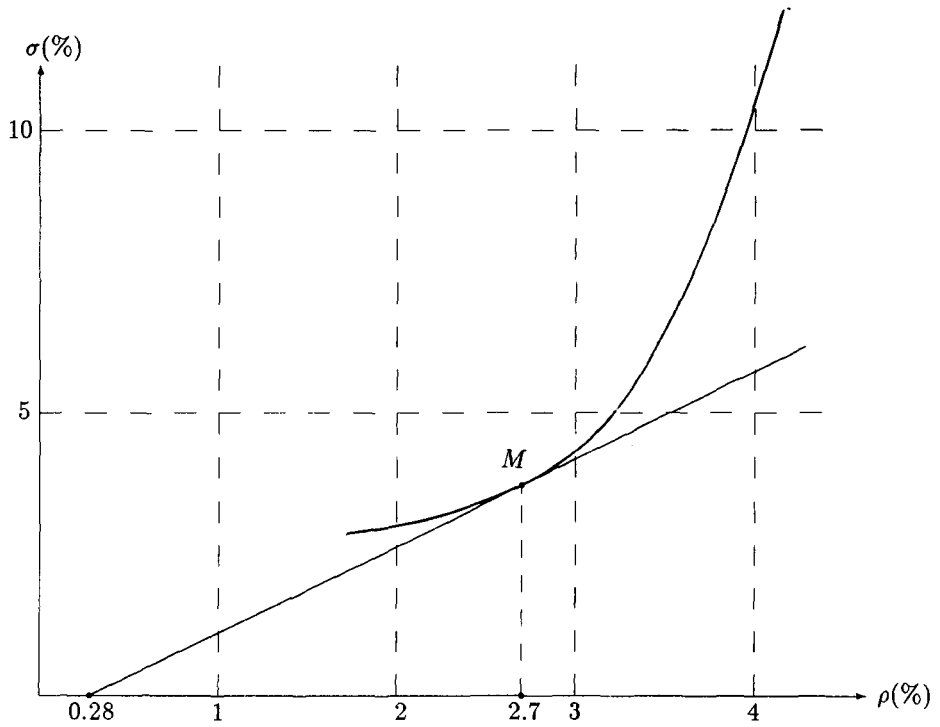


Figure 5.1. Efficient Frontier of L_2 Risk Model.

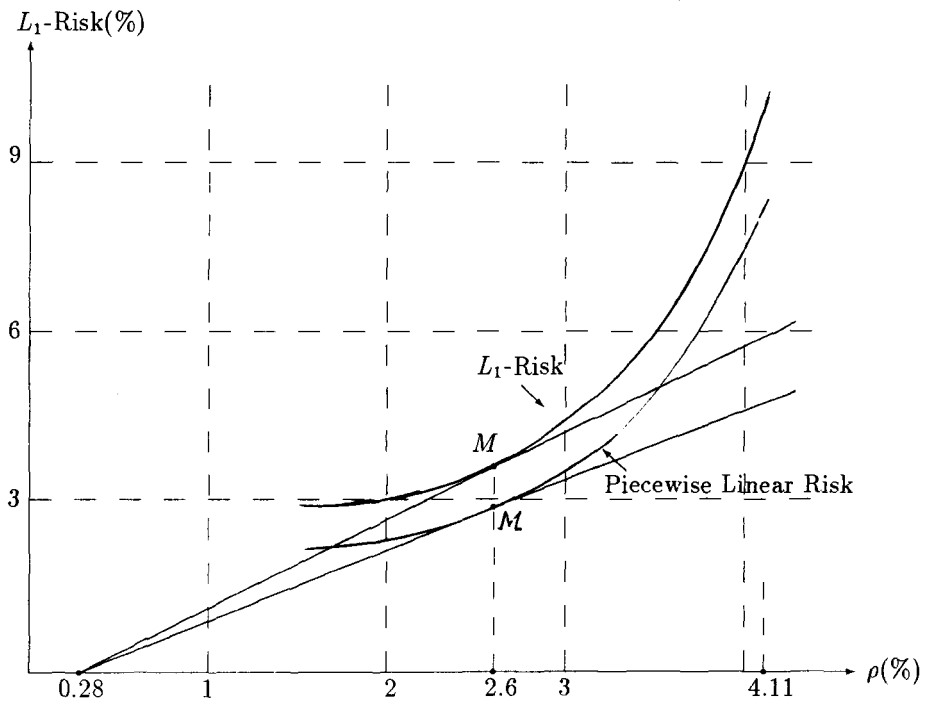


Figure 5.2. Efficient Frontier of L_1 Risk Model.

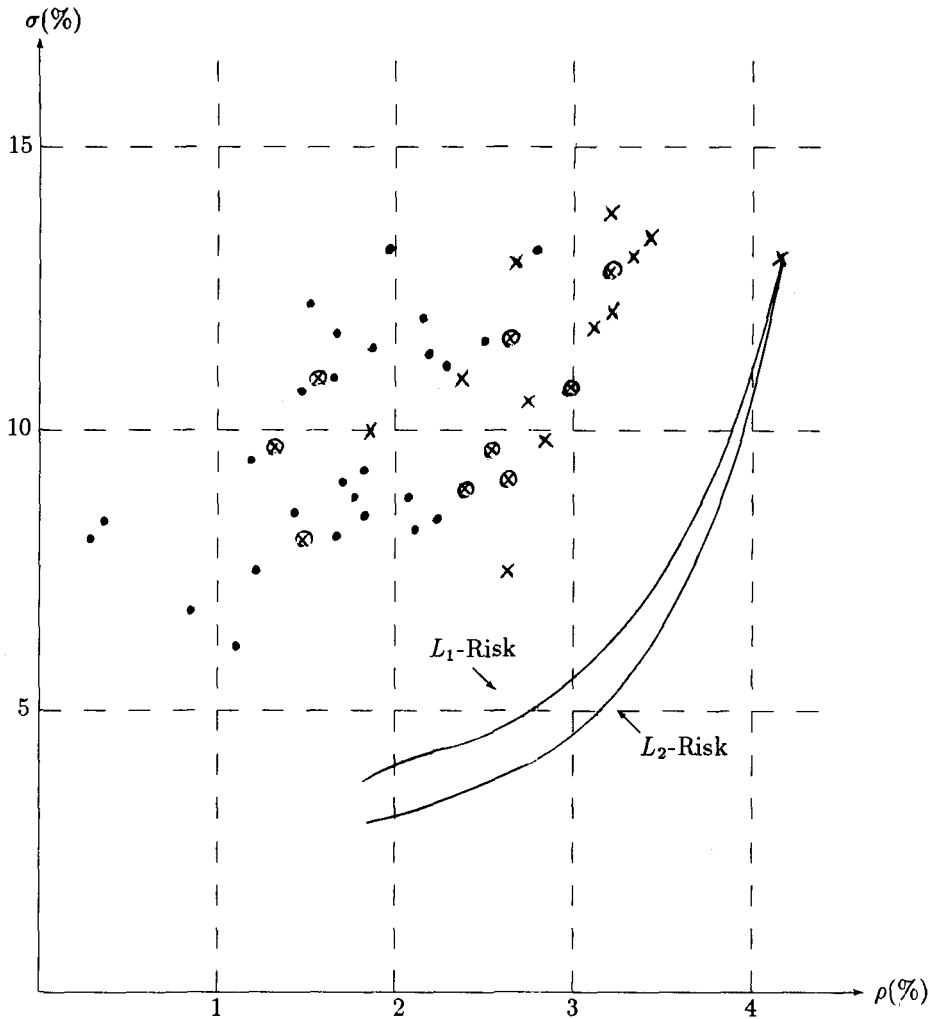


Figure 5.3. Efficient Frontier of L_2 Risk and L_1 Risk Model.

It should be noted that L_1 -risk efficient frontier and piecewise linear risk efficient frontier were generated within 20 seconds by applying parametric linear programming algorithm to the dual of the problems (2.16) and (4.2). L_2 -risk frontier, on the other hand was generated by solving 25 quadratic programs associated with 25 different ρ values, namely $\rho = 0.017, 0.018, \dots, 0.041$. Each case took a little less than one minute computing time. It could be solved in a couple of minutes if an efficient parametric programming code is available.

Market portfolios for both of these models are indicated by point M on each graph. Expected rate of return of L_2 , L_1 and piecewise linear risk models are 2.69%, 2.60%, 2.60% per month, respectively.

Fig. 5.3 shows the efficient frontiers of L_2 and L_1 models in terms of average return-standard deviation space. L_1 frontier naturally lies above L_2 frontier, and its difference

decreases as the required rate of return ρ increases. Average return and standard deviation of 50 stocks are plotted on the same graph. Stocks with circles are the ones contained in the L_1 -risk market portfolio. These stocks together with the stocks with cross marks are the ones contained in L_2 -risk market portfolio.

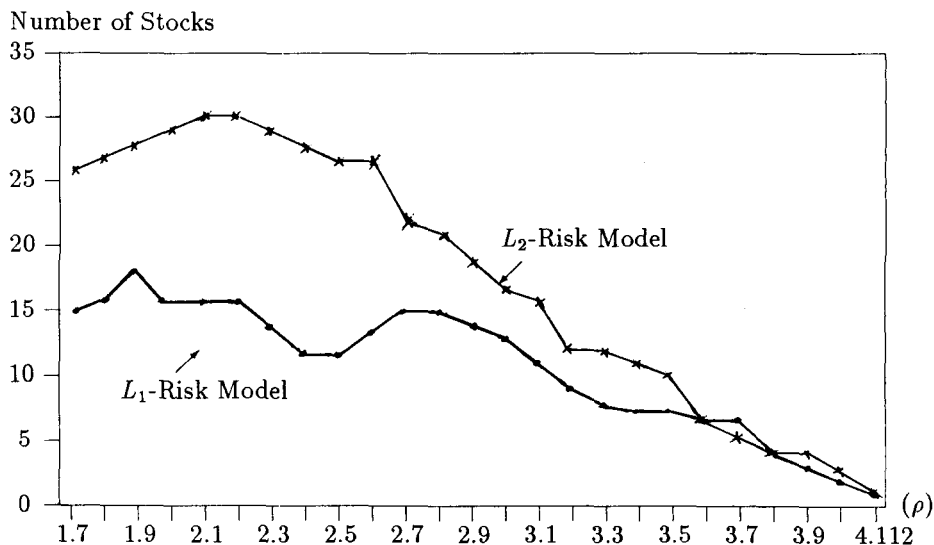


Figure 5.4. Number of Stocks with Positive Weight.

Fig. 5.4 shows the numbers of stocks with positive weight in the portfolio. Among the remarkable observations are

(i) Stocks with positive weight in L_1 -risk market portfolio are contained in the set of stocks with positive weight in L_2 -risk market portfolio for all values of ρ .

(ii) Number of stocks included in L_1 market portfolio is about one half of the number of stocks included in L_2 -risk portfolio for all values of ρ .

Table 5.1: Characteristics of the Market Portfolio

	Markowitz	L_1 Risk
# of stocks in the portfolio	22	13
average return	2.69%/mo.	2.60%/mo.
standard deviation	3.9%	4.5%
stock with maximum share	14.6%	28%
stock with minimum positive share	0.4%	2.1%
sum of the share of top 5 stocks	47.0%	66.0%
sum of the share of top 10 stocks	66.0%	92.2%

Table 5.1 shows some of the important features of L_2 -risk and L_1 -risk market portfolio. It can be seen from this that L_1 -risk model suggests one to invest substantial proportion of the

available fund into smaller number of stocks. Moreover, this property holds for all values of ρ . Massive investment into smaller number of stocks is very convenient when we consider the integral constraint imposed on real transaction. In fact, each stock has to be purchased at an integer multiple of certain minimal unit, usually 1,000 stocks. Thus to obtain an integral solution, we must either round the variables to the nearest integer multiple of minimal unit or solve a smaller integer programming problem obtained by eliminating zero variable in the optimal solution. In either case, L_1 -risk model is superior to L_2 -risk model since the problem to be solved contains fewer variables.

We observe from this preliminary experiment that L_1 -risk model and piecewise linear risk model can alleviate some of the difficulties of L_2 -risk model referred to in the Introduction. Numerical experiment using 1,100 stocks of Tokyo Stock Market is now under way, whose result will be reported subsequently.

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