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THE BALANCED LOT SIZE FOR A SINGLE-MACHINE MULTI-PRODUCT LOT SCHEDULING PROBLEM

Hirokazu Kono Zentaro Nakamura
Keio University

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Abstract The basic nature of feasible schedules and their lot sizes are investigated for a single-machine multi-product problem, under the assumptions of deterministic demand on infinite time horizon and cyclic production sequence. Feasible schedule is obtained under the balanced lot size which is in proportion to the demand rate of each product and independent of production sequence. It is determined uniquely if the production rate is higher than the sum of demand rate, while there are infinitely many feasible schedules if the production rate is just balanced to the sum of demand rate. The quantitative relation between such related factors as demand rate, production rate, machine idle time, production lot size, can be a practical guide for designing a small lot size production system, as well as determining a feasible schedule on long-term finite planning period.

1. Introduction

This paper is concerned with lot sizing and scheduling of production runs for multiple items of products to be processed on a single machine. Since the machine can produce only one product at a time, prior production to the demand and holding inventory are inevitable for all products out of that being produced. If some products are produced with too large lot sizes, it will result in shortage to the demand of other products. If production is to be interrupted by setup time, production lot sizes must be adjusted so that no shortage occurs during the setup periods. Thus, we must carefully determine production schedule (lot sizes, timing and sequence) in order to supply the demand for every product without any shortage.

The single-machine multi-product problem has been studied by many authors [1]–[5] as an economic lot scheduling problem (ELSP) to minimize the sum of changeover cost and inventory holding cost. Most of those researches start from determination of economic production lot size for each product, and then proceed to lot size ammendment to obtain a feasible schedule. A problem underlying there is that it becomes necessary to adopt try and error process or feasibility check algorithm in order to find a feasible schedule, despite some studies do not even guarantee feasibility. Another problem to be pointed out is that these researches implicitly assume enough production capacity against the total demand so that frequent changeover is feasible. However, in the sense that the net production capacity decreases as the machine idle time caused by changeover increases, the balance between the total demand and the net production capacity must be a critical factor to affect the lot sizes in feasible schedules.

This paper firstly investigates the problem to find all feasible schedules, taking into consideration such related factors as demand rate, production rate, machine idle time, production lot size, production sequence, and clarifies the necessary condition to assure existence of a feasible schedule. Then the paper examines the properties of feasible schedules and their lot sizes, which can be utilized for designing a small lot size production system as well as determining a production schedule. The whole discussion will be done on a model of a stationary case with deterministic demand on infinite time horizon and cyclic repetition of a production sequence.

2. Model Formulation

2.1 Basic assumptions and notations

(1) On infinite time horizon $[0, \infty)$, product demand rates are deterministically given by constant $D_i > 0$ for product $i, i = 1, 2, \cdots, r$.

(2) Only one product can be produced at a time. Production rate of product i is given by constant $K_i > 0, i = 1, 2, \cdots, r$. For the sake of simplification, each D_i and K_i shall be counted by its own production rate as one unit, so that

$$K_i = 1, i = 1, 2, \cdots, r, \tag{2.1}$$

and the following condition is assumed.

$$D_i < 1, i = 1, 2, \cdots, r. \tag{2.2}$$

(3) r products are produced by the lot with cyclic repetition of a production sequence $\pi = (i_1, i_2, \cdots, i_r)$.

(4) Production lot size of product i in the k-th cycle is denoted by $x_{ik} > 0, i = 1, 2, \cdots, r, k = 0, 1, 2, \cdots, \infty$, where x_{i0} represents initial inventory of product i held at time 0.

(5) There is no unexpected machine downtime. From the condition of (2.1), x_{ik} also represents its production time.

(6) Machine idle time immediately before production of product i is represented by $\tau_i \geq 0, i = 1, 2, \cdots, r$. Each τ_i is the sum of necessary setup time and intentionally placed idle time.

(7) From the reason that inventory capacity is limited, production lot sizes are bounded by a finite value M so that

$$0 < x_{ik} \leq M, i = 1, 2, \cdots, r, k = 0, 1, 2, \cdots, \infty. \tag{2.3}$$

(8) Production lot x_{ik} is supplied continuously from its production starting time to meet the demand without causing any shortage nor any avoidable (surplus) inventory. Supply starting time of lot x_{ik} is denoted by z_{ik}.

2.2 Demand and Supply balance equation

Figure 1 shows in Gantt chart form the k-th and $k+1$-st cycles of a production schedule under the sequence $\pi = (1, 2, \cdots, r)$. Vertical arrows in the figure represent supply starting time of product i in each cycle. The k-th cycle is represented by a set of r pieces of production time and machine idle time as

$$(x_{1k}, \tau_2, x_{2k}, \cdots, \tau_i, x_{ik}, \cdots, \tau_r, x_{rk}, \tau_1).$$

From assumption (8), each production lot x_{ik} must meet the demand on time interval $[z_{ik}, z_{i,k+1}]$. Therefore, the next statement is satisfied.

$$x_{ik} = (z_{i,k+1} - z_{ik}) D_i, i = 1, 2, \cdots, r, k = 1, 2, \cdots, \infty. \tag{2.4}$$

Since the left hand side means the production (supply) quantity and the right hand side refers to the demand satisfied by lot x_{ik}, we shall call (2.4) as the "demand and supply balance equation" for product i in cycle k. Representing $z_{i,k+1}-z_{ik}$ with $x_{ik}, x_{i,k+1}, \tau_i, i=1,2,\cdots,r$, the following equations can be obtained (refer to Figure 1).

$$\begin{aligned}
x_{1k} &= \{(x_{1k}+x_{2k}+\cdots+x_{rk})+(\tau_2+\cdots+\tau_r+\tau_1)\}D_1,\\
x_{2k} &= \{(x_{2k}+\cdots+x_{rk}+x_{1,k+1})+(\tau_3+\cdots+\tau_r+\tau_1+\tau_2)\}D_2,\\
&\vdots\\
x_{ik} &= \{(x_{ik}+x_{i+1,k}+\cdots+x_{rk}+x_{1,k+1}+x_{2,k+1}+\cdots+x_{i-1,k+1})\\
&\quad +(\tau_{i+1}+\cdots+\tau_r+\tau_1+\tau_2+\cdots+\tau_i)\}D_i,\\
&\vdots\\
x_{rk} &= \{(x_{rk}+x_{1,k+1}+\cdots+x_{r-1,k+1})+(\tau_1+\tau_2+\cdots+\tau_r)\}D_r,\\
&k=1,2,\cdots,\infty.
\end{aligned} \tag{2.5}$$

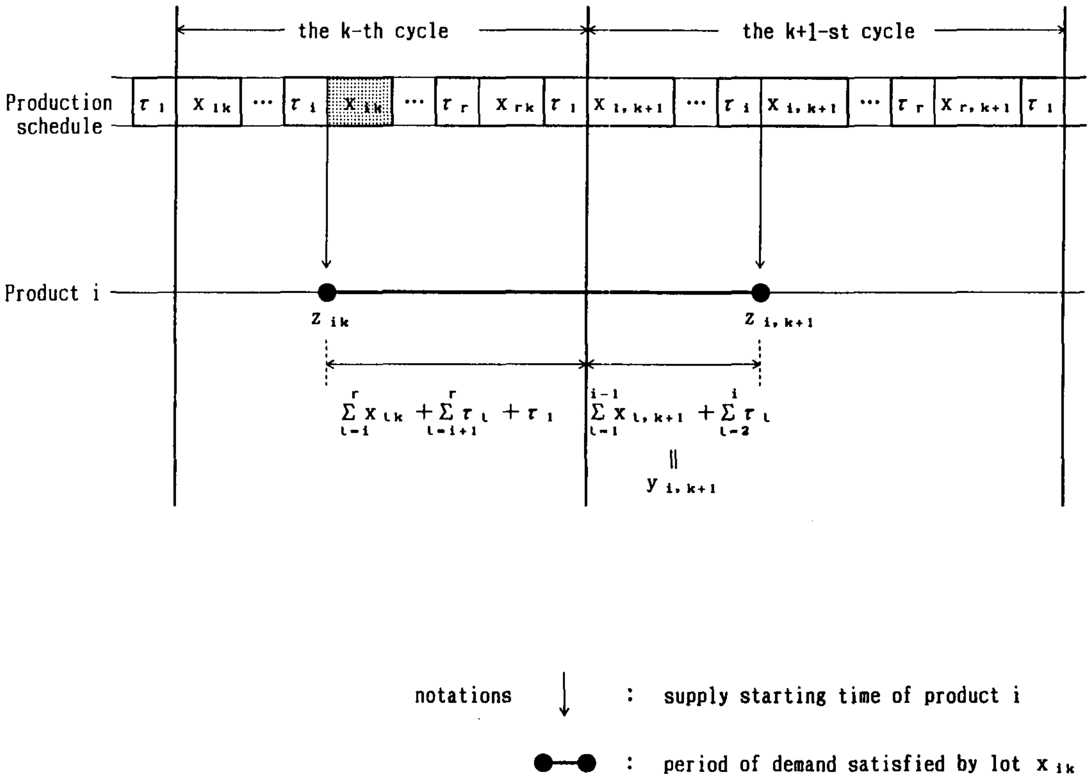

Figure 1 Production schedule and supply timing

In (2.5), shifting the terms representing the demand in the k-th cycle to the left hand side, we obtain the following equations.

$$\begin{aligned}
&x_{1k}-(x_{1k}+x_{2k}+\cdots+x_{rk})D_1-(\tau_2+\cdots+\tau_r+\tau_1)D_1\\
&\quad =0,\\
&x_{2k}-(x_{2k}+\cdots+x_{rk})D_2-(\tau_3+\cdots+\tau_r+\tau_1)D_2\\
&\quad =x_{1,k+1}D_2+\tau_2D_2,\\
&\quad \vdots\\
&x_{ik}-(x_{ik}+x_{i+1,k}+\cdots+x_{rk})D_i-(\tau_{i+1}+\cdots+\tau_r+\tau_1)D_i\\
&\quad =(x_{1,k+1}+x_{2,k+1}+\cdots+x_{i-1,k+1})D_i+(\tau_2+\cdots+\tau_i)D_i,\\
&\quad \vdots\\
&x_{rk}-x_{rk}D_r-\tau_1D_r\\
&\quad =(x_{1,k+1}+\cdots+x_{r-1,k+1})D_r+(\tau_2+\cdots+\tau_r)D_r,\\
&k=1,2,\cdots,\infty.
\end{aligned}\tag{2.6.i}$$

Since there is no demand before time 0, the second and the third terms on the left hand side are erased when $k=0$, and then x_{i0} is given by

$$\begin{aligned}
x_{10}&=0,\\
x_{20}&=x_{11}D_2+\tau_2D_2,\\
&\vdots\\
x_{i0}&=(x_{11}+x_{21}+\cdots+x_{i-1,1})D_i+(\tau_2+\cdots+\tau_i)D_i,\\
&\vdots\\
x_{r0}&=(x_{11}+x_{21}+\cdots\cdot+x_{r-1,1})D_r+(\tau_2+\cdots\cdot+\tau_r)D_r.
\end{aligned}\tag{2.6.ii}$$

In (2.6.i), the left hand side means the carried over inventory at the end of the k-th cycle, and the right hand side means the quantity produced in advance in the k-th cycle for the consumption in the $k+1$-st cycle. We shall call the amount in the right hand side as the "prior production quantity" of product i for cycle $k+1$, and denote by $y_{i,k+1}$. y_{i1} means prior production quantity of product i in advance to time 0, which is equivalent to the initial inventory x_{i0}.

We shall represent the production lot size (initial inventory when $k=0$) and the machine idle time in each cycle by vectors as

$$\begin{aligned}
\mathbf{x}_k&=[x_{1k},x_{2k},\cdots,x_{rk}]^T, k=0,1,2,\cdots,\infty, \text{ and}\\
\boldsymbol{\tau}&=[\tau_2,\tau_3,\cdots,\tau_r,\tau_1]^T.
\end{aligned}$$

And we shall denote coefficient matrix of $\mathbf{x}_k$ on the left hand side and that of $\mathbf{x}_{k+1}$ on the right hand side in (2.6.i) by $\mathbf{E}$ and $\mathbf{F}$ respectively. Then the equations in (2.6.i) can be represented by

$$\mathbf{E}\mathbf{x}_k+(\mathbf{E}-\mathbf{I})\boldsymbol{\tau}=\mathbf{F}\mathbf{x}_{k+1}+\mathbf{F}\boldsymbol{\tau}, k=1,2,\cdots,\infty, \tag{2.7}$$

where $\mathbf{I}$ is the unit matrix, and

$$\mathbf{E}=\begin{bmatrix}1-D_1 & -D_1 & -D_1 & \cdots\cdots & -D_1\\ 0 & 1-D_2 & -D_2 & \cdots\cdots & -D_2\\ 0 & 0 & 1-D_3 & -D_3\cdots & -D_3\\ \vdots & \vdots & & \ddots & \vdots\\ 0 & 0 & 0 & \cdots\cdots & 1-D_r\end{bmatrix}, \mathbf{F}=\begin{bmatrix}0 & 0 & 0 & \cdots\cdots & 0\\ D_2 & 0 & 0 & \cdots\cdots & 0\\ D_3 & D_3 & 0 & \cdots\cdots & 0\\ \vdots & \vdots & & \ddots & \\ D_r & D_r & D_r & \cdots\cdots & 0\end{bmatrix}.$$

Value 1 in diagonal elements in matrix $\mathbf{E}$ means production of a lot, and $-D_i$ in matrix $\mathbf{E}$ means consumption in the k-th cycle, while D_i in matrix $\mathbf{F}$ means consumption in the $k+1$-st cycle. Shifting $(\mathbf{E}-\mathbf{I})\boldsymbol{\tau}$ in (2.7) to the right hand side, we get

$$\mathbf{E}\mathbf{x}_k = \mathbf{F}\mathbf{x}_{k+1} + \mathbf{D}\boldsymbol{\tau}, k = 1, 2, \cdots, \infty, \tag{2.8.i}$$

and from (2.6.ii),

$$\mathbf{x}_0 = \mathbf{F}\mathbf{x}_1 + \mathbf{F}\boldsymbol{\tau}, \tag{2.8.ii}$$

$$\text{where } \mathbf{D} = \begin{bmatrix} D_1 & D_1 \cdots\cdots & D_1 \\ D_2 & D_2 \cdots\cdots & D_2 \\ \vdots & \vdots & \vdots \\ D_r & D_r \cdots\cdots & D_r \end{bmatrix}.$$

Denoting prior production quantities by

$$\mathbf{y}_k = [y_{1k}, y_{2k}, \cdots, y_{rk}]^T, k = 1, 2, \cdots, \infty,$$

the relation between $\mathbf{y}_k$ and $\mathbf{x}_k$ is represented from (2.7) by

$$\mathbf{y}_k = \mathbf{F}\mathbf{x}_k + \mathbf{F}\boldsymbol{\tau}, k = 1, 2, \cdots, \infty. \tag{2.9}$$

And from (2.8.ii) and (2.9), obviously

$$\mathbf{y}_1 = \mathbf{x}_0. \tag{2.10}$$

The problem to be investigated in this paper is to find all solutions $\{\mathbf{x}_k\}$, $k = 0, 1, 2, \cdots, \infty$, in equations (2.8.i) and (2.8.ii) under the condition of (2.3).

2.3 Fundamental properties of matrices E and F

Matrix $\mathbf{E} = [e_{ij}]$ is a triangular matrix with

$$\begin{aligned} e_{ij} &= -D_i \quad &, i < j, \\ e_{ij} &= 1 - D_i &, i = j, \\ e_{ij} &= 0 &, i > j. \end{aligned} \tag{2.11}$$

Therefore, matrix $\mathbf{E}$ satisfies the next property.

<u>Property 1.</u> Matrix $\mathbf{E}$ is full rank, and $\mathbf{E}^{-1}$ is a non-negative matrix.

Matrix $\mathbf{F} = [f_{ij}]$ is also a triangular matrix with

$$\begin{aligned} f_{ij} &= 0, \quad i \leq j, \\ f_{ij} &= D_i, i > j. \end{aligned} \tag{2.12}$$

Rank of matrix $\mathbf{F}$ is obviously $r-1$.

Furthermore, the relation between (2.7) and (2.8.i) yields the next property.

Property 2. The next statement is satisfied on the relation between $\mathbf{E}$ and $\mathbf{F}$.

$$\mathbf{E} - \mathbf{F} = \mathbf{I} - \mathbf{D}. \tag{2.13}$$

Equation (2.13) is true because each lot x_{ik} is consumed during successive r lot production periods in the k-th and $k+1$-st cycles, and there is value 1 in each diagonal element of matrix $\mathbf{E}$ which means production of a lot.

3. Feasible solutions

In this chapter, we shall examine the problem to find all feasible solutions $\{\mathbf{x}_k\}$ in (2.8), taking into consideration the relative size of the total demand $\sum_{i=1}^{r} D_i$ against the production rate $K_i = 1$. If $\sum_{i=1}^{r} D_i > 1$, the production capacity is in shortage to meet the given total demand on infinite time horizon and obviously there exists no feasible solution. Therefore, the other two cases, $\sum_{i=1}^{r} D_i < 1$ and $\sum_{i=1}^{r} D_i = 1$, are investigated.

3.1 Feasible solution in the case $\sum_{i=1}^{r} D_i < 1$

3.1.1 Existence of a stationary solution under $\sum_{i=1}^{r} \tau_i > 0$

We assume that there exists a stationary solution $\{\mathbf{x}_0^*, \mathbf{x}^*, \cdots, \mathbf{x}^*, \cdots\}$ in (2.8). Then

$$\mathbf{E}\mathbf{x}^* = \mathbf{F}\mathbf{x}^* + \mathbf{D}\boldsymbol{\tau}. \tag{3.1.i}$$

$$\mathbf{x}_0^* = \mathbf{F}\mathbf{x}^* + \mathbf{F}\boldsymbol{\tau}. \tag{3.1.ii}$$

Applying (2.13) in Property 2 to (3.1.i),

$$(\mathbf{I} - \mathbf{D})\mathbf{x}^* = \mathbf{D}\boldsymbol{\tau}. \tag{3.2}$$

Since matrix $(\mathbf{I} - \mathbf{D})$ is full rank in the case $\sum_{i=1}^{r} D_i < 1, \mathbf{x}^*$ and $\mathbf{x}_0^*$ are determined by the following equations (refer to Appendix A).

$$\mathbf{x}^* = t^*\mathbf{d}, \tag{3.3.i}$$

$$\mathbf{x}_0^* = t^*\mathbf{F}\mathbf{d} + \mathbf{F}\boldsymbol{\tau}, \tag{3.3.ii}$$

where

$$t^* = \sum_{i=1}^{r} \tau_i / (1 - \sum_{i=1}^{r} D_i), \tag{3.4}$$

and $\mathbf{d} = [D_1, D_2, \cdots, D_r]^T$.

The lot size determined by (3.3.i) shall be called as the "balanced lot size". The prior production quantity $\mathbf{y}^*$ under the balanced lot size is given from (2.9) by

$$\mathbf{y}^* = t^*\mathbf{F}\mathbf{d} + \mathbf{F}\boldsymbol{\tau}. \tag{3.5}$$

3.1.2 Uniqueness of the solution

In this section, we shall prove that $\{\mathbf{x}_0^*, \mathbf{x}^*, \cdots, \mathbf{x}^*, \cdots\}$ is a unique solution of (2.8) in the case that $\sum_{i=1}^{r} D_i < 1$ and $\sum_{i=1}^{r} \tau_i > 0$. We assume that there exists another solution $\{\mathbf{x}_k^\circ\}, k = 0, 1, 2, \cdots, \infty$. The difference between the two solutions $\{\mathbf{x}_k^\circ\}$ and $\{\mathbf{x}_0^*, \mathbf{x}^*, \cdots, \mathbf{x}^*, \cdots\}$ shall be denoted by

$$\overline{\mathbf{x}}_0 = \mathbf{x}_0^\circ - \mathbf{x}_0^*, \tag{3.6.i}$$

$$\overline{\mathbf{x}}_k = \mathbf{x}_k^\circ - \mathbf{x}^*, k = 1, 2, \cdots, \infty. \tag{3.6.ii}$$

It follows from (2.8.i) and (2.8.ii) that

$$\overline{\mathbf{x}}_0 = \mathbf{F}\overline{\mathbf{x}}_1, \tag{3.7.i}$$
$$\mathbf{E}\overline{\mathbf{x}}_k = \mathbf{F}\overline{\mathbf{x}}_{k+1}, k = 1, 2, \cdots, \infty. \tag{3.7.ii}$$

We can easily confirm from the structure of (3.7.i) and (3.7.ii) that

$$\overline{\mathbf{x}}_0 = \mathbf{0} \rightarrow \overline{\mathbf{x}}_1 = \mathbf{0} \rightarrow \overline{\mathbf{x}}_2 = \mathbf{0} \rightarrow \cdots$$

Therefore, under the assumption of the existence of $\{\mathbf{x}_k^o\}$, the next statement must be satisfied.

$$\overline{\mathbf{x}}_0 \neq \mathbf{0}. \tag{3.8}$$

We put

$$\overline{\mathbf{y}}_k = \mathbf{F}\overline{\mathbf{x}}_k, k = 1, 2, \cdots, \infty, \tag{3.9}$$

and[1]

$$\mathbf{G} = \mathbf{F} \cdot \mathbf{E}^{-1}. \tag{3.10}$$

Multiplied by matrix $\mathbf{G}$ from the left and applying (3.9), (3.7.ii) can be represented as

$$\overline{\mathbf{y}}_k = \mathbf{G}\overline{\mathbf{y}}_{k+1}, k = 1, 2, \cdots, \infty. \tag{3.11}$$

Further

$$\overline{\mathbf{y}}_1 = \mathbf{G}^N \overline{\mathbf{y}}_{N+1}, N \geq 1. \tag{3.12}$$

From (3.7.i) and (3.9), condition (3.8) is equivalent to the next statement.

$$\overline{\mathbf{y}}_1 \neq \mathbf{0} \tag{3.13}$$

Under the condition of (2.3), $\{\overline{\mathbf{y}}_k\}$ is obviously bounded by finite value vector $\mathbf{M}$ so that

$$|\overline{\mathbf{y}}_k| \leq \mathbf{M}, k = 1, 2, \cdots, \infty. \tag{3.14}$$

Therefore

$$|\overline{\mathbf{y}}_1| = \mathbf{G}^N |\overline{\mathbf{y}}_{N+1}| \leq \mathbf{G}^N \mathbf{M}, N \geq 1. \tag{3.15}$$

Here, if it can be proven that

$$\lim_{N \rightarrow \infty} \mathbf{G}^N = \mathbf{0}, \tag{3.16}$$

then a contradictory statement to (3.13) that $|\overline{\mathbf{y}}_1| = \mathbf{0}$ can be derived. Therefore, we shall below investigate the convergency of $\lim_{N \rightarrow \infty} \mathbf{G}^N$.

<u>Lemma 1.</u> The next statement holds for the sum of column elements in $\mathbf{G} = [g_{ij}]$.

$$\sum_{i=1}^{r} D_i < 1 \rightarrow \sum_{i=1}^{r} g_{ij} < \sum_{i=1}^{r} D_i < 1, 1 \leq j \leq r.$$

Proof. Refer to Appendix B.

Lemma 2. The next statement holds about the powers of matrix $\mathbf{G}$.

$$\sum_{i=1}^{r} g_{ij} < 1, 1 \leq j \leq r \rightarrow \lim_{N\rightarrow\infty} \mathbf{G}^N = \mathbf{0}.$$

Proof. Refer to Appendix C.

From lemma 1 and lemma 2, the next statement is satisfied.

$$\sum_{i=1}^{r} D_i < 1 \rightarrow \lim_{N\rightarrow\infty} \mathbf{G}^N = \mathbf{0}.$$

It follows from (3.15) and (3.16) that

$$|\bar{\mathbf{y}}_1| \leq \lim_{N\rightarrow\infty} \mathbf{G}^N\mathbf{M} = \mathbf{0}. \tag{3.17}$$

The result in (3.17) is contradictory to the assumption in (3.13).

Summarizing the above analysis, we have the next theorem.

Theorem 1. In the case that $\sum_{i=1}^{r} D_i < 1$ and $\sum_{i=1}^{r} \tau_i > 0$, the balanced lot size determined by (3.3.i) is a unique solution in (2.8).

3.1.3 The condition to assure existence of a feasible solution

In the case $\sum_{i=1}^{r} D_i < 1$ in which the production rate is greater than the total demand, it is necessary to decrease the net production capacity by introducing idle time. Under the condition $\sum_{i=1}^{r} \tau_i > 0$, the total demand during cycle time t^* in (3.4) is given by

$$\sum_{i=1}^{r} D_i \times t^* = \frac{\sum_{i=1}^{r} D_i \times \sum_{i=1}^{r} \tau_i}{1 - \sum_{i=1}^{r} D_i} \text{ (units/cycle time).} \tag{3.18}$$

While the net production capacity in each cycle under the production rate $K_i = 1$ (unit/unit time) is given by

$$1 \times (t^* - \sum_{i=1}^{r} \tau_i) = \frac{\sum_{i=1}^{r} \tau_i \times \sum_{i=1}^{r} D_i}{1 - \sum_{i=1}^{r} D_i} \text{ (units/cycle time).} \tag{3.19}$$

These two statements yield that the total demand and the net production capacity are balanced under the cycle time t^*.

The right hand side of (3.19) tells that the net production capacity decreases and reaches to 0 as $\sum_{i=1}^{r} \tau_i$ decreases, and then it is predicted that there is no feasible solution under $\sum_{i=1}^{r} \tau_i = 0$. In fact, applying $\boldsymbol{\tau} = \mathbf{0}$ to (2.8),

$$\mathbf{x}_0 = \mathbf{F}\mathbf{x}_1, \tag{3.20.i}$$
$$\mathbf{E}\mathbf{x}_k = \mathbf{F}\mathbf{x}_{k+1}, k = 1, 2, \cdots, \infty. \tag{3.20.ii}$$

Since the structure of (3.20) is the same as (3.7), $\mathbf{x}_k = \mathbf{0}$ for all k can be proven in the same manner as section 3.1.2. It follows that there is no feasible solution $\{\mathbf{x}_k\}$ in (3.20).

From the above analysis, we can summarize that the condition to assure existence of a feasible solution is $\sum_{i=1}^{r} \tau_i > 0$ in the case $\sum_{i=1}^{r} D_i < 1$. Even when setup time is 0 for each product, a feasible solution can be obtained by introducing intentionally placed idle time so that $\sum_{i=1}^{r} \tau_i > 0$. The cycle time t^* and the balanced lot size for each product are determined depending upon the value of $\sum_{i=1}^{r} \tau_i$.

3.2 Feasible solution in the case $\sum_{i=1}^{r} D_i = 1$
3.2.1 Existence of a stationary solution under $\sum_{i=1}^{r} \tau_i = 0$

We assume that there exists a stationary solution $\{\mathbf{x}_0^*, \mathbf{x}^*, \cdots, \mathbf{x}^*, \cdots\}$ in (2.8). Then applying $\boldsymbol{\tau} = \mathbf{0}$,

$$\mathbf{E}\mathbf{x}^* = \mathbf{F}\mathbf{x}^*, \tag{3.21.i}$$

$$\mathbf{x}_0^* = \mathbf{F}\mathbf{x}^*, \tag{3.21.ii}$$

where $\mathbf{x}^* = [x_1^*, x_2^*, \cdots, x_{r-1}^*, x_r^*]^T$.
Applying (2.13) in Property 2 to (3.21.i),

$$(\mathbf{I} - \mathbf{D})\mathbf{x}^* = \mathbf{0}. \tag{3.22}$$

Since the rank of matrix $(\mathbf{I} - \mathbf{D})$ is $r - 1$ under the condition $\sum_{i=1}^{r} D_i = 1$, there exist infinitely many non-trivial solutions which can be represented by (refer to Appendix A),

$$x_i^* = tD_i, i = 1, 2, \cdots, r. \tag{3.23}$$

Applying the conditions $\sum_{i=1}^{r} D_i = 1$ and $x_i^* > 0$ for all i,

$$\sum_{i=1}^{r} x_i^* = t \sum_{i=1}^{r} D_i = t > 0. \tag{3.24}$$

(3.24) tells that t is a positive constant and means the length of the cycle time.

Therefore, in the case that $\sum_{i=1}^{r} D_i = 1$ and $\sum_{i=1}^{r} \tau_i = 0$, there exist infinitely many solutions which can be represented by

$$\mathbf{x}^* = t\mathbf{d}, \tag{3.25.i}$$

$$\mathbf{x}_0^* = t\mathbf{F}\mathbf{d}, \tag{3.25.ii}$$

where $t > 0$ and $\mathbf{d} = [D_1, D_2, \cdots, D_r]^T$.
The lot size determined by (3.25.i) shall be also called as the "balanced lot size". The balanced lot size in this case is determined depending upon the length of the cycle time t. The prior production quantity $\mathbf{y}^*$ under the balanced lot size is given from (2.9) by

$$\mathbf{y}^* = t\mathbf{F}\mathbf{d}, t > 0. \tag{3.26}$$

3.2.2 Inexistence of non-stationary solution

We shall below prove that there does not exist any non-stationary solution in the case that $\sum_{i=1}^{r} D_i = 1$ and $\sum_{i=1}^{r} \tau_i = 0$. We assume that there exists a solution $\{\mathbf{x}_k^\circ\}, k = 0, 1, 2, \cdots, \infty$, which is different from $\{t\mathbf{F}\mathbf{d}, t\mathbf{d}, \cdots, t\mathbf{d}, \cdots\}$. From (2.8), $\{\mathbf{x}_k^\circ\}$ satisfies

$$\mathbf{x}_0^\circ = \mathbf{F}\mathbf{x}_1^\circ, \tag{3.27.i}$$

$$\mathbf{E}\mathbf{x}_k^\circ = \mathbf{F}\mathbf{x}_{k+1}^\circ, k = 1, 2, \cdots, \infty. \tag{3.27.ii}$$

We can easily confirm from the structure of (3.27.i) and (3.27.ii) that

$$\mathbf{x}_0^\circ = t\mathbf{F}\mathbf{d} \rightarrow \mathbf{x}_1^\circ = t\mathbf{d} \rightarrow \mathbf{x}_2^\circ = t\mathbf{d} \rightarrow \cdots$$

Therefore, under the assumption of the existence of $\{\mathbf{x}_k^\circ\}$, the next statement must be satisfied.

$$\mathbf{x}_0^\circ \neq t\mathbf{Fd}, \text{ for any } t > 0. \tag{3.28}$$

Putting

$$\mathbf{y}_k^\circ = \mathbf{F}\mathbf{x}_k^\circ, k = 1, 2, \cdots, \infty, \tag{3.29}$$

and multiplying matrix $\mathbf{G}$ shown in (3.10) from the left to (3.27.ii), we obtain

$$\mathbf{y}_k^\circ = \mathbf{G}\mathbf{y}_{k+1}^\circ, k = 1, 2, \cdots, \infty. \tag{3.30}$$

Further

$$\mathbf{y}_1^\circ = \mathbf{G}^N \mathbf{y}_{N+1}^\circ, N \geq 1. \tag{3.31}$$

From (3.27.i) and (3.29), condition (3.28) is equivalent to the next statement.

$$\mathbf{y}_1^\circ \neq t\mathbf{Fd}, \text{ for any } t > 0. \tag{3.32}$$

In order to obtain a contradictory statement to (3.32), we shall below investigate the convergency of $\lim_{N\to\infty} \mathbf{G}^N$.

Lemma 3. The next statement holds for the sum of column elements in $\mathbf{G} = [g_{ij}]$.

$$\sum_{i=1}^{r} D_i = 1 \to \sum_{i=1}^{r} g_{ij} = 1, 1 \leq j \leq r.$$

Proof. The same manner as Appendix B by changing inequality sign to equality sign.

Lemma 4. The next statement holds about the powers of matrix $\mathbf{G}$.

$$\sum_{i=1}^{r} g_{ij} = 1, 1 \leq j \leq r \to \lim_{N\to\infty} \mathbf{G}^N = \overline{\mathbf{G}} \equiv \begin{bmatrix} u_1 & u_1 & \cdots & u_1 \\ u_2 & u_2 & \cdots & u_2 \\ \vdots & \vdots & & \vdots \\ u_r & u_r & \cdots & u_r \end{bmatrix},$$

where $u_1 = 0, u_i > 0, 2 \leq i \leq r$, and $\sum_{i=1}^{r} u_i = 1$.

Proof. Refer to Appendix D.

Lemma 5. $\overline{\mathbf{G}}$ is given by the next statement[2].

$$\overline{\mathbf{G}} = \frac{1}{\alpha}\mathbf{F} \cdot \mathbf{D}, \tag{3.33}$$

where α is the sum of column elements in matrix $\mathbf{F} \cdot \mathbf{D}$ and $\alpha > 0$.

Proof. Refer to Appendix E.

Lemma 6. The next statement is satisfied about the sum of elements in $\mathbf{y}_N^\circ$.

$$\lim_{N\to\infty} \sum_{i=1}^{r} y_{iN} = \beta, \ \beta \text{ is a positive constant,} \tag{3.34}$$

where $\mathbf{y}_N^\circ = [y_{1N}, y_{2N}, \cdots, y_{rN}]^T$.

Proof. Refer to Appendix F.

From lemma 3, lemma 4 and lemma 5, the next statement is satisfied.

$$\sum_{i=1}^{r} D_i = 1 \rightarrow \lim_{N\rightarrow\infty} \mathbf{G}^N = \frac{1}{\alpha}\mathbf{F}\cdot\mathbf{D}.$$

It follows from (3.31) and lemma 6 that

$$\begin{aligned}\mathbf{y}_1^\circ &= \lim_{N\rightarrow\infty} \mathbf{G}^N \mathbf{y}_{N+1}^\circ \\ &= \frac{1}{\alpha}\mathbf{F}\cdot\mathbf{D}\cdot \lim_{N\rightarrow\infty} \mathbf{y}_{N+1}^\circ \\ &= \frac{1}{\alpha}\mathbf{F}\cdot\beta\mathbf{d} \\ &= \frac{\beta}{\alpha}\mathbf{Fd}, \frac{\beta}{\alpha} > 0.\end{aligned} \tag{3.35}$$

This result in (3.35) is contradictory to the assumption in (3.32).

Summarizing the above analysis, we have the next theorem.

Theorem 2. In the case that $\sum_{i=1}^r D_i = 1$ and $\sum_{i=1}^r \tau_i = 0$, the balanced lot size given by (3.25.i) is a solution in (2.8) and there is no other non-stationary solution. There exist infinitely many solutions according to the length of the cycle time.

3.2.3 The condition to assure existence of a feasible solution

In the case that $\sum_{i=1}^r D_i = 1$ and $\sum_{i=1}^r \tau_i = 0$, the total demand $\sum_{i=1}^r D_i \times t$ (units/cycle time) and the net production capacity t (units/cycle time) are balanced regardless of the length of the cycle time t. Therefore, there exist infinitely many solutions. In this context, a feasible solution cannot be obtained if $\sum_{i=1}^r \tau_i > 0$, since the net production capacity is in shortage against the total demand. In fact, multiplying (2.8.i) by matrix $\mathbf{G}$ from the left and applying (2.9),

$$\mathbf{y}_k = \mathbf{G}\mathbf{y}_{k+1} + \mathbf{G}\boldsymbol{\tau}, k = 1, 2, \cdots, \infty. \tag{3.36}$$

Then

$$\mathbf{y}_1 = \mathbf{G}^N\mathbf{y}_{N+1} + (\mathbf{G} + \mathbf{G}^2 + \cdots + \mathbf{G}^N)\boldsymbol{\tau}, N \geq 1. \tag{3.37}$$

Since $g_{ij} > 0, 2 \leq i \leq r, 1 \leq j \leq r$, and $\boldsymbol{\tau}$ is non-negative, all elements $y_{i1}, 2 \leq i \leq r$, in $\mathbf{y}_1$ diverge for sufficiently large N. It follows from (2.9) that at least one element of $\mathbf{x}_1$ diverges and there exists no feasible solution in (2.8).

From the above analysis, we can summarize that in the case $\sum_{i=1}^r D_i = 1$, a feasible solution can be obtained under the condition of continuous operation ($\sum_{i=1}^r \tau_i = 0$). The net production capacity and the total demand are just balanced under the condition of $\sum_{i=1}^r \tau_i = 0$ in this case, and there is no room for introducing idle time τ_i between any production lot.

4. Further discussion on the balanced lot size

4.1 Properties of the balanced lot size

The analysis in chapter 3 clarified that a feasible solution in (2.8) can be obtained as the balanced lot size only when the total demand and the net production capacity are balanced. In the case that $\sum_{i=1}^r D_i < 1$ and $\sum_{i=1}^r \tau_i > 0$, the balanced lot size which is determined by (3.3.i) satisfies the next statement.

$$\frac{x_1^*}{D_1} = \frac{x_2^*}{D_2} = \cdots = \frac{x_r^*}{D_r} = \frac{\sum_{i=1}^r \tau_i}{(1 - \sum_{i=1}^r D_i)} = t^*. \tag{4.1}$$

While in the case that $\sum_{i=1}^{r} D_i = 1$ and $\sum_{i=1}^{r} \tau_i = 0$, the balanced lot size which is determined by (3.25.i) satisfies the next statement.

$$\frac{x_1^*}{D_1} = \frac{x_2^*}{D_2} = \cdots = \frac{x_{r-1}^*}{D_{r-1}} = \frac{x_r^*}{D_r} = t, \text{ for arbitrary } t > 0. \tag{4.2}$$

These two statements tell us the following properties of the balanced lot size.

1) The balanced lot size is determined uniquely for the given value of $\sum_{i=1}^{r} \tau_i$ in the case $\sum_{i=1}^{r} D_i < 1$, while there are numberless alternatives in the case $\sum_{i=1}^{r} D_i = 1$.
2) The balanced lot size in both cases is determined in proportion to the demand rate D_i of each product.
3) The balanced lot size does not depend upon production sequence.
4) The balanced lot size in the case $\sum_{i=1}^{r} D_i < 1$ is determined in proportion to the sum of machine idle time $\sum_{i=1}^{r} \tau_i$ and in inverse proportion to the production capacity surplus $1 - \sum_{i=1}^{r} D_i$.

The relation between the cases $\sum_{i=1}^{r} D_i < 1$ and $\sum_{i=1}^{r} D_i = 1$ can be interpreted as follows. In (4.2), D_r can be replaced to $(1 - \sum_{i=1}^{r-1} D_i)$ from the condition $\sum_{i=1}^{r} D_i = 1$. Then regarding product r as a dummy product where x_r^* is a machine idle time, the case $\sum_{i=1}^{r} D_i = 1$ with r products turns to that of $\sum_{i=1}^{r-1} D_i < 1$ with $r - 1$ products. Since the production lot size (i.e. production time) of the dummy product is restricted to be $\sum_{i=1}^{r} \tau_i$, the balanced lot size in the replaced $r - 1$ product problem of $\sum_{i=1}^{r-1} D_i < 1$ is uniquely determined.

The uniqueness and infiniteness of the balanced lot size affect the way to change the value of the lot size. In the case $\sum_{i=1}^{r} D_i = 1$, we can quite easily achieve small lot size production system or find a lot size to satisfy integer constraint or minimum lot size constraint, through changing the value of the cycle time t.

On the other hand, in the case $\sum_{i=1}^{r} D_i < 1$, there are two alternative ways to achieve a small lot size production system; to reduce the sum of machine idle time, and to increase $1 - \sum_{i=1}^{r} D_i$ by speeding up the production rate. The smallest production lot size can be obtained when the schedule does not include any intentional idle time. On a contrary, we can enlarge the balanced lot size by increasing intentional idle time, so that the schedule satisfies the minimum lot size constraint.

4.2 Implications to determine a production schedule

The third property of the balanced lot size tells us that the production sequence can be arbitrarily determined under the balanced lot size. We can choose a desirable sequence considering those factors as changeover efficiency or the amount of initial inventory, both of which are affected by the production sequence.

Statement (4.1) indicates that the value of $\sum_{i=1}^{r} \tau_i$ and t^* are proportional each other in the case $\sum_{i=1}^{r} D_i < 1$. It follows that in such a situation where cycle time is previously given as t° by an operation plan, the value of $\sum_{i=1}^{r} \tau_i$ is decided. And if $\sum_{i=1}^{r} \tau_i$ is less than the sum of necessary setup time for each product, a feasible schedule cannot be obtained unless setup time is decreased. If t° is large enough, $\sum_{i=1}^{r} \tau_i$ is to include intentional idle time in addition to necessary setup time.

In the situation that $\sum_{i=1}^{r} \tau_i$ includes intentional idle time, we must notice that there are numberless alternative ways to divide the sum of intentional idle time among production lots. The practical way is to equalize intentional idle time between each production lot, in order to maximize the buffer against unexpected delay in production schedule. Another significant way is to collect intentional idle time at the end of each cycle so that the initial inventory can be decreased. These alternatives enable us to choose desirable schedule due to the situation.

4.3 Application to the problem on finite planning period

The balanced lot size, which assures a feasible schedule on infinite time horizon, can be applied to determine a schedule on long-term finite planning period. We shall consider the following numerical example.

Three items of products must be produced to meet each demand. The following values are assumed:

Production rate: 1 unit/hour for each product.
Demand rate: $D_1 = 0.1, D_2 = 0.3, D_3 = 0.4$ unit/hour.
Idle time (Setup time): $\tau_1 = 0.5, \tau_2 = 0.5, \tau_3 = 1.0$ hour.

Each day consists of 24 hour operation. The planning period is one week. A feasible production schedule under the cyclic production sequence $\pi = (1, 2, 3)$ can be obtained as follows.

Cycle time is determined as $t^* = 10$ hours from (3.4). The balanced lot size is given from (3.3.i) as

$$x_1^* = 1.0, x_2^* = 3.0, x_3^* = 4.0 \text{ units.}$$

Initial inventory is calculated from (3.3.ii) as

$$x_{10}^* = 0, x_{20}^* = 0.45, x_{30}^* = 2.2 \text{ units.}$$

One week (7 days=168 hours) consists of 16 cycles accompanied by a short cycle of 8 hours at the end. Then the lot size in the last cycle must be modified so that the inventory at the end of the week becomes 0 for each product. Under the schedule with the balanced lot size, the supply timing $z_{i,17}, 1 \le i \le 3$, in the last cycle can be easily calculated. Then the modified lot size can be calculated by

$$x_{i,17} = (168 - z_{i,17}) \times D_i, 1 \le i \le 3, \tag{4.3}$$

and machine idle time τ_i' in the last cycle can be obtained by

$$\begin{aligned} \tau_i' &= z_{i,17} - z_{i-1,17} - x_{i-1,17}, 2 \le i \le 3, \\ \tau_1' &= 168 - z_{3,17} - x_{3,17}. \end{aligned} \tag{4.4}$$

Concretely, these values are calculated as shown below.

$$\begin{aligned} x_{1,17} &= (168 - 160) \times D_1 = 0.8 \text{ unit.} \\ \tau_2' &= 161.5 - 160 - x_{1,17} = 0.7 \text{ hour.} \\ x_{2,17} &= (168 - 161.5) \times D_2 = 1.95 \text{ units.} \\ \tau_3' &= 165.5 - 161.5 - x_{2,17} = 2.05 \text{ hours.} \\ x_{3,17} &= (168 - 165.5) \times D_3 = 1.0 \text{ unit.} \\ \tau_1' &= 168 - 165.5 - x_{3,17} = 1.5 \text{ hours.} \end{aligned}$$

The obtained feasible schedule is shown in Figure 2.

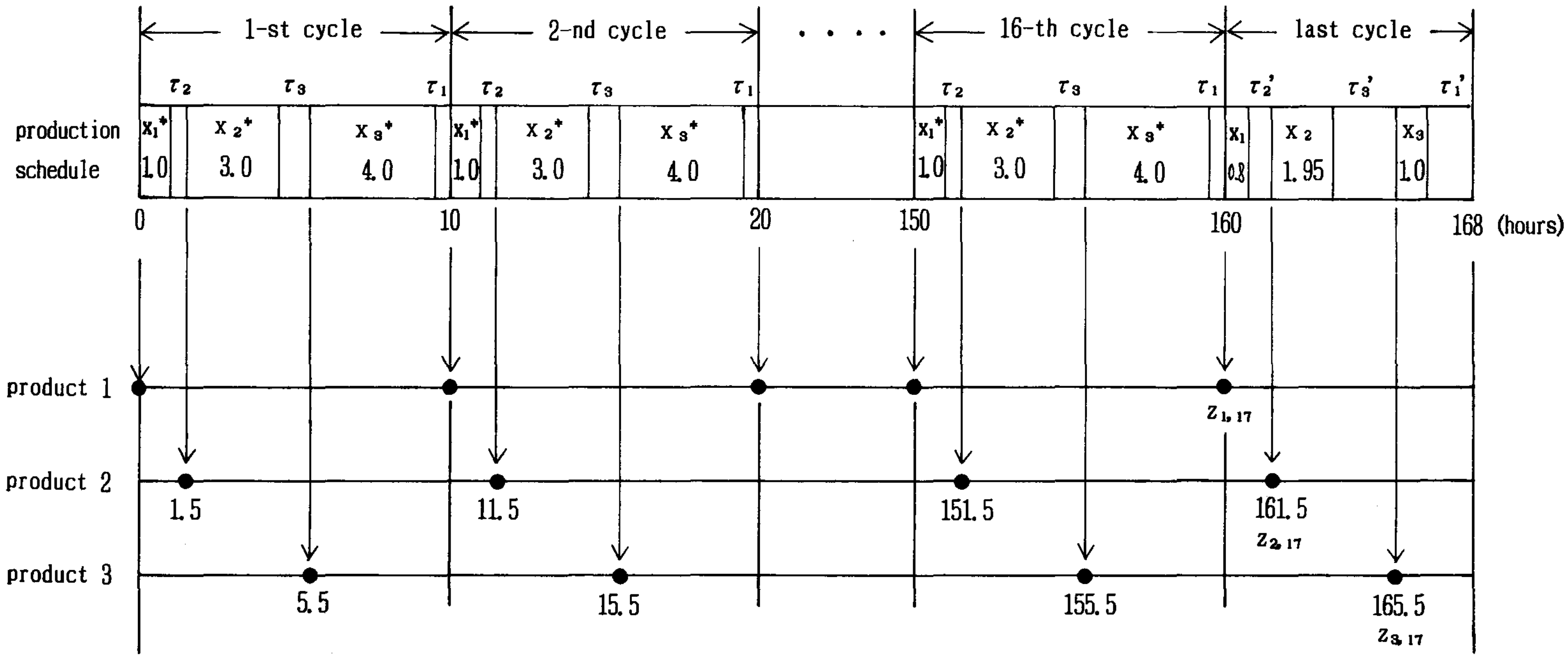

Figure 2 Production schedule for numerical example

Footnotes:
(1) Matrices $\mathbf{E}^{-1}$ and $\mathbf{G}$ for the case of $r = 3$ are shown below.

$$\mathbf{E}^{-1} = \begin{bmatrix} \frac{1}{1-D_1} & \frac{D_1}{(1-D_1)(1-D_2)} & \frac{D_1}{(1-D_1)(1-D_2)(1-D_3)} \\ 0 & \frac{1}{1-D_2} & \frac{D_2}{(1-D_2)(1-D_3)} \\ 0 & 0 & \frac{1}{1-D_3} \end{bmatrix}.$$

$$\mathbf{G} = \mathbf{F} \cdot \mathbf{E}^{-1} = \begin{bmatrix} 0 & 0 & 0 \\ \frac{D_2}{1-D_1} & \frac{D_1 D_2}{(1-D_1)(1-D_2)} & \frac{D_1 D_2}{(1-D_1)(1-D_2)(1-D_3)} \\ \frac{D_3}{1-D_1} & \frac{D_1 D_3}{(1-D_1)(1-D_2)} + \frac{D_3}{1-D_2} & \frac{D_1 D_3}{(1-D_1)(1-D_2)(1-D_3)} + \frac{D_2 D_3}{(1-D_2)(1-D_3)} \end{bmatrix}.$$

(2) Matrix $\overline{\mathbf{G}}$ for the case of $r = 3$ is shown below.

$$\overline{\mathbf{G}} = \frac{1}{\alpha}\mathbf{F} \cdot \mathbf{D} = \frac{1}{\alpha} \begin{bmatrix} 0 & 0 & 0 \\ D_2 D_1 & D_2 D_1 & D_2 D_1 \\ D_3(D_1 + D_2) & D_3(D_1 + D_2) & D_3(D_1 + D_2) \end{bmatrix},$$

where $\alpha = \sum_{1 \leq i < j \leq 3} D_i D_j$.

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Appendix A. Rank and inverse of matrix $(\mathbf{I} - \mathbf{D})$

(1) Rank of matrix $(\mathbf{I} - \mathbf{D})$

We can convert matrix $(\mathbf{I} - \mathbf{D})$ into a diagonal matrix using the following fundamental operation matrices on rows and columns. For the sake of simplification, we take the case of $r = 4$ as an example.

$$\mathbf{L} \equiv \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \mathbf{R}_1 \equiv \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & -1 & -1 & 1 \end{bmatrix}, \mathbf{R}_2 \equiv \begin{bmatrix} 1 & 0 & 0 & D_1 \\ 0 & 1 & 0 & D_2 \\ 0 & 0 & 1 & D_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then

$$(\mathbf{I}-\mathbf{D})\cdot\mathbf{R}_1\cdot\mathbf{R}_2=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\-1&-1&-1&1-\sum_{i=1}^4 D_i\end{bmatrix}\equiv\mathbf{P}. \tag{A.1}$$

Since both $\mathbf{R}_1$ and $\mathbf{R}_2$ are regular, rank $(\mathbf{I}-\mathbf{D})$ = rank $\mathbf{P}$. Therefore,

$$\text{rank}(\mathbf{I}-\mathbf{D})=r,\qquad \text{if } \sum_{i=1}^{r} D_i\neq 1.$$

$$\text{rank}(\mathbf{I}-\mathbf{D})=r-1,\ \text{if } \sum_{i=1}^{r} D_i=1.$$

(2) The stationary solution $\mathbf{x}^*$ in the case that $\sum_{i=1}^r D_i<1$ and $\sum_{i=1}^r \tau_i>0$
Matrix $(\mathbf{I}-\mathbf{D})$ can be converted into a diagonal matrix by

$$\mathbf{L}\cdot\mathbf{P}=\mathbf{L}\cdot(\mathbf{I}-\mathbf{D})\cdot\mathbf{R}_1\cdot\mathbf{R}_2=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1-\sum_{i=1}^4 D_i\end{bmatrix}\equiv\mathbf{Q}. \tag{A.2}$$

It follows

$$(\mathbf{I}-\mathbf{D})=\mathbf{L}^{-1}\cdot\mathbf{Q}\cdot\mathbf{R}_2^{-1}\cdot\mathbf{R}_1^{-1}. \tag{A.3}$$

Then

$$|\mathbf{I}-\mathbf{D}|=|\mathbf{L}^{-1}||\mathbf{Q}||\mathbf{R}_2^{-1}||\mathbf{R}_1^{-1}|=1-\sum_{i=1}^{4} D_i. \tag{A.4}$$

From (A.3),

$$(\mathbf{I}-\mathbf{D})^{-1}=\mathbf{R}_1\cdot\mathbf{R}_2\cdot\mathbf{Q}^{-1}\cdot\mathbf{L}$$

$$=\frac{1}{1-\sum_{i=1}^4 D_i}\begin{bmatrix}1-\sum_{i=1}^4 D_i+D_1 & D_1 & D_1 & D_1\\ D_2 & 1-\sum_{i=1}^4 D_i+D_2 & D_2 & D_2\\ D_3 & D_3 & 1-\sum_{i=1}^4 D_i+D_3 & D_3\\ D_4 & D_4 & D_4 & 1-\sum_{i=1}^4 D_i+D_4\end{bmatrix}. \tag{A.5}$$

Therefore, for the general case of r products,

$$\begin{aligned}\mathbf{x}^*&=(\mathbf{I}-\mathbf{D})^{-1}\cdot\mathbf{D}\boldsymbol{\tau}\\&=\frac{1}{1-\sum_{i=1}^r D_i}\mathbf{D}\boldsymbol{\tau}=\frac{\sum_{i=1}^r \tau_i}{1-\sum_{i=1}^r D_i}\mathbf{d}=t^*\mathbf{d}.\end{aligned} \tag{A.6}$$

(3) The stationary solution $\mathbf{x}^*$ in the case that $\sum_{i=1}^r D_i=1$ and $\sum_{i=1}^r \tau_i=0$
We shall represent matrix $(\mathbf{I}-\mathbf{D})$ by a set of column vectors such that $(\mathbf{I}-\mathbf{D})=[\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3,\mathbf{a}_4]$. Then $(\mathbf{I}-\mathbf{D})\mathbf{x}^*=\mathbf{0}$ can be represented by

$$x_1^*\mathbf{a}_1+x_2^*\mathbf{a}_2+x_3^*\mathbf{a}_3+x_4^*\mathbf{a}_4=\mathbf{0}. \tag{A.7}$$

From (A.1),

$$\begin{aligned}&(\mathbf{I}-\mathbf{D})\cdot\mathbf{R}_1\cdot\mathbf{R}_2\\&=[\mathbf{a}_1-\mathbf{a}_4,\mathbf{a}_2-\mathbf{a}_4,\mathbf{a}_3-\mathbf{a}_4,\mathbf{a}_4]\cdot\mathbf{R}_2\\&=[\mathbf{a}_1-\mathbf{a}_4,\mathbf{a}_2-\mathbf{a}_4,\mathbf{a}_3-\mathbf{a}_4,D_1\mathbf{a}_1+\cdots+D_4\mathbf{a}_4]\end{aligned} \tag{A.8}$$

Since the last column vector in (A.8) turns out to be $\mathbf{0}$ from (A.1),

$$D_1\mathbf{a}_1 + D_2\mathbf{a}_2 + D_3\mathbf{a}_3 + D_4\mathbf{a}_4 = \mathbf{0}. \tag{A.9}$$

From (A.7) and (A.9),

$$(x_1^* - \frac{D_1}{D_4}x_4^*)\mathbf{a}_1 + (x_2^* - \frac{D_2}{D_4}x_4^*)\mathbf{a}_2 + (x_3^* - \frac{D_3}{D_4}x_4^*)\mathbf{a}_3 = \mathbf{0}. \tag{A.10}$$

Since rank $(\mathbf{I} - \mathbf{D}) = r - 1 = 3, \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ are linearly independent. Then it follows from (A.10) for the general case of r products,

$$x_i^* = \frac{D_i}{D_r}x_r^*, i = 1, 2, \cdots, r-1. \tag{A.11}$$

(A.11) yields that the solution for $(\mathbf{I} - \mathbf{D})\mathbf{x}^* = \mathbf{0}$ can be represented by

$$x_i^* = tD_i, i = 1, 2, \cdots, r, \tag{A.12}$$

where t is an arbitrary real value.

Appendix B. Proof of lemma 1

From (2.13) in Property 2,

$$\mathbf{F} = \mathbf{D} - \mathbf{I} + \mathbf{E}. \tag{B.1}$$

Multiplying this result by matrix $\mathbf{E}^{-1}$ from the right,

$$\mathbf{G} = \mathbf{D}\cdot\mathbf{E}^{-1} - \mathbf{E}^{-1} + \mathbf{I}. \tag{B.2}$$

We can describe (B.2) by row vectors as

$$\mathbf{g}_i = D_i\sum_{l=1}^{r}\tilde{\mathbf{e}}_l - \tilde{\mathbf{e}}_i + \boldsymbol{\alpha}_i, i = 1, 2, \cdots, r, \tag{B.3}$$

where $\mathbf{g}_i, \tilde{\mathbf{e}}_i, \boldsymbol{\alpha}_i$ are the i-th row vectors of matrices $\mathbf{G}, \mathbf{E}^{-1}, \mathbf{I}$ respectively.
It is obvious from (2.11) that $\mathbf{E}^{-1} = [\tilde{e}_{ij}]$ satisfies the following statements (refer to $\mathbf{E}^{-1}$ shown in Footnote (1)).

$$\tilde{e}_{ij} > 0, \qquad i < j. \tag{B.4}$$

$$\tilde{e}_{ij} = 1/(1 - D_i) > 1, i = j. \tag{B.5}$$

$$\tilde{e}_{ij} = 0, \qquad i > j. \tag{B.6}$$

It follows from (B.4) through (B.6) that

$$\tilde{\mathbf{e}}_i - \boldsymbol{\alpha}_i > \mathbf{0}, i = 1, 2, \cdots, r. \tag{B.7}$$

Applying (B.7) and the relation $\sum_{i=1}^{r} D_i < 1$ to (B.3), we can derive that

$$\begin{aligned}\sum_{i=1}^{r}\mathbf{g}_i &= (\sum_{i=1}^{r} D_i)\sum_{l=1}^{r}\tilde{\mathbf{e}}_l - \sum_{i=1}^{r}(\tilde{\mathbf{e}}_i - \boldsymbol{\alpha}_i)\\ &< (\sum_{i=1}^{r} D_i)\{\sum_{l=1}^{r}\tilde{\mathbf{e}}_l - \sum_{i=1}^{r}(\tilde{\mathbf{e}}_i - \boldsymbol{\alpha}_i)\}\\ &= (\sum_{i=1}^{r} D_i)\sum_{i=1}^{r}\boldsymbol{\alpha}_i\\ &= [\sum_{i=1}^{r} D_i, \sum_{i=1}^{r} D_i, \cdots, \sum_{i=1}^{r} D_i].\end{aligned}$$

Therefore,

$$\sum_{i=1}^{r} g_{ij} < \sum_{i=1}^{r} D_i < 1, 1 \leq j \leq r.$$

Appendix C. Proof of lemma 2

Maximum of each row is denoted by $M_i, 1 \leq i \leq r$. Applying lemma 1, we can derive that

$$\mathbf{G}^2 = [\sum_{l=1}^{r} g_{il}g_{lj}] \leq [M_i \sum_{l=1}^{r} g_{lj}] < [M_i \sum_{i=1}^{r} D_i].$$

$$\mathbf{G}^3 = \mathbf{G}^2 \cdot \mathbf{G} < [M_i(\sum_{i=1}^{r} D_i)^2].$$

Similarly

$$\mathbf{G}^N < [M_i(\sum_{i=1}^{r} D_i)^{N-1}] \equiv \mathbf{H}_N, N \geq 2.$$

Since $\lim_{N\to\infty} \mathbf{H}_N = \mathbf{0}$ under the condition $\sum_{i=1}^{r} D_i < 1$, it is clear that $\lim_{N\to\infty} \mathbf{G}^N = \mathbf{0}$.

Appendix D. Proof of lemma 4

Notations; $\mathbf{G}^N = [g_{ij}^{(N)}]$ (N is omitted when $N = 1$),

$$M_i^{(N)} = \max_{1\leq j\leq r} g_{ij}^{(N)}, m_i^{(N)} = \min_{1\leq j\leq r} g_{ij}^{(N)}, 1 \leq i \leq r.$$

The relation between $M_i^{(N-1)}$ and $M_i^{(N)}$ satisfies

$$g_{ij}^{(N)} = \sum_{l=1}^{r} g_{il}^{(N-1)} g_{lj} \leq M_i^{(N-1)} \sum_{l=1}^{r} g_{lj} = M_i^{(N-1)}, 1 \leq j \leq r. \tag{D.1}$$

Then

$$M_i^{(N-1)} \geq M_i^{(N)}, 1 \leq i \leq r. \tag{D.2}$$

Similarly

$$m_i^{(N-1)} \leq m_i^{(N)}, 1 \leq i \leq r. \tag{D.3}$$

We shall denote $\max_{1\leq j\leq r} g_{ij}^{(N)}$ and $\min_{1\leq j\leq r} g_{ij}^{(N)}$ by $g_{ia}^{(N)}$ and $g_{ib}^{(N)}$ respectively. Then we can derive that

$$\begin{aligned} M_i^{(N)} - m_i^{(N)} &= \sum_{l=1}^{r} g_{il}^{(N-1)}(g_{la} - g_{lb}) \\ &= \sum_{g_{la}>g_{lb}} g_{il}^{(N-1)}(g_{la} - g_{lb}) + \sum_{g_{la}<g_{lb}} g_{il}^{(N-1)}(g_{la} - g_{lb}) \\ &= \sum_{g_{la}>g_{lb}} g_{il}^{(N-1)}(g_{la} - g_{lb}) - \sum_{g_{la}>g_{lb}} g_{il}^{(N-1)}(g_{la} - g_{lb}) \\ &\leq (M_i^{(N-1)} - m_i^{(N-1)}) \sum_{g_{la}>g_{lb}} (g_{la} - g_{lb}) \\ &< (M_i^{(N-1)} - m_i^{(N-1)})\, \delta, 0 < \delta < 1, 1 \leq i \leq r. \end{aligned} \tag{D.4}$$

Therefore

$$\lim_{N\to\infty}(M_i^{(N)} - m_i^{(N)}) = 0, 1 \le i \le r. \tag{D.5}$$

(D.5) shows that $g_{ij}^{(N)}$ in each row converges to a constant u_i for sufficiently large N. Since $g_{1j} = 0$ and $g_{ij} > 0$ for all i and j (refer to matrix $\mathbf{G}$ shown in Footnote (1)), obviously

$$u_1 = 0 \text{ and } u_i > 0, 2 \le i \le r. \tag{D.6}$$

It is also clear from lemma 3 that

$$\boldsymbol{\gamma}\mathbf{G}^N = \boldsymbol{\gamma}\mathbf{G}\cdot\mathbf{G}^{N-1} = \boldsymbol{\gamma}\mathbf{G}^{N-1} = \cdots = \boldsymbol{\gamma}\mathbf{G} = \boldsymbol{\gamma}. \tag{D.7}$$

where $\boldsymbol{\gamma} = [1, 1, \cdots, 1]$. Then

$$\sum_{i=1}^{r} u_i = 1. \tag{D.8}$$

Appendix E. Proof of lemma 5

Multiplying (2.13) in Property 2 by matrix $\mathbf{D}$ from the right and applying $\mathbf{D}\cdot\mathbf{D} = \mathbf{D}$ in the case that $\sum_{i=1}^{r} D_i = 1$, we get

$$\mathbf{F}\cdot\mathbf{D} = \mathbf{E}\cdot\mathbf{D}. \tag{E.1}$$

Multiplied by matrix $\mathbf{G}$ from the left,

$$\mathbf{G}\cdot\mathbf{F}\cdot\mathbf{D} = \mathbf{F}\cdot\mathbf{D}. \tag{E.2}$$

Repeatedly multiplied by matrix $\mathbf{G}$ from the left and applying the relation (E.2), we get

$$\overline{\mathbf{G}}\cdot\mathbf{F}\cdot\mathbf{D} = \mathbf{F}\cdot\mathbf{D}. \tag{E.3}$$

Since elements in each row of $\mathbf{F}\cdot\mathbf{D}$ are constant, it follows

$$\overline{\mathbf{G}}\cdot\mathbf{F}\cdot\mathbf{D} = \alpha\overline{\mathbf{G}}, \tag{E.4}$$

where α is the column sum of matrix $\mathbf{F}\cdot\mathbf{D}$ and $\alpha > 0$.
From (E.3) and (E.4), we obtain

$$\overline{\mathbf{G}} = \frac{1}{\alpha}\mathbf{F}\cdot\mathbf{D}, \alpha > 0. \tag{E.5}$$

Appendix F. Proof of lemma 6

From (3.31),

$$\mathbf{y}_1^\circ = \mathbf{G}^N\mathbf{y}_{N+1}^\circ = \mathbf{G}^{N+1}\mathbf{y}_{N+2}^\circ. \tag{F.1}$$

Then

$$\mathbf{G}^{N+1}\mathbf{y}_{N+2}^\circ - \mathbf{G}^N\mathbf{y}_{N+1}^\circ = \mathbf{0}. \tag{F.2}$$

Since $\lim_{N\to\infty}\mathbf{G}^N$ converges to matrix $\overline{\mathbf{G}}$ and $\{\mathbf{y}_k^\circ\}$ is bounded from (2.9), we get

$$\begin{aligned}
&\lim_{N\to\infty}\{\mathbf{G}^{N+1}\mathbf{y}_{N+2}^\circ - \mathbf{G}^N\mathbf{y}_{N+1}^\circ\} \\
&= \lim_{N\to\infty}\{\overline{\mathbf{G}}\mathbf{y}_{N+2}^\circ - \overline{\mathbf{G}}\mathbf{y}_{N+1}^\circ\} \\
&= [u_1, u_2, \cdots, u_r]^T \lim_{N\to\infty}\{\sum_{i=1}^{r} y_{i,N+2} - \sum_{i=1}^{r} y_{i,N+1}\} = \mathbf{0}.
\end{aligned} \tag{F.3}$$

From lemma 4, $[u_1, u_2, \cdots, u_r]^T \neq \mathbf{0}$. Then (F.3) implies

$$\lim_{N\to\infty}\{\sum_{i=1}^{r} y_{iN}\} = \beta = \text{constant}, \tag{F.4}$$

where β is obviously positive.

Hirokazu Kono:
Graduate School of Business Administration,
Keio University, 2-1-1, Hiyoshi-honcho,
Kohoku-ku, Yokohama, 223, Japan.