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STOCHASTIC BOUNDS FOR GENERALIZED SYSTEMS IN RELIABILITY THEORY

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Abstract Practically, it is not feasible to obtain the precise reliability of systems in a reasonable time, when the systems are large and complex. In this paper, we present some stochastic bounds on generalized systems of which state spaces are mathematically partially ordered sets. In the first place we introduce a notion of generalized systems and then present some stochastic bounds on the system reliability by using maximal and minimal elements of the structures of the systems. The bounds are generalization of the well-known max-min bounds on binary-state system reliability. Furthermore, we present the other stochastic bounds when systems are decomposed into several modules and satisfy a condition which is called MC (Maximal Coincidence) condition. We show that these bounds are tighter than the former. For a few simple systems, we give numerical examples and estimations of computational complexity for obtaining these stochastic bounds.

1. Introduction

In reliability theory it is important to obtain the precise reliability of systems. However, for the purpose of determining the precise system reliability, the indispensable computational complexity increases exponentially with an increase in the number of both the components of the system and the states of them. For example, consider a multi-state system composed of n components. Supposing that the system and all the components have m states, for convenience, then the computational complexity to the precise reliability becomes $O(m^n)$. For this reason, L.D.Bodin [3] and D.A.Butler [4] have been proposed some stochastic bounds on the system reliability which can be obtained in a little computational complexity. [3] and [4] established stochastic bounds on binary-state systems and on multi-state systems, respectively.

Recently many authors treat multi-state systems, e.g., R.E.Barlow and A.S.Wu [2], E.El-Neweihi et al[5], W.S.Griffith [7], D.A.Butler [4] and F.Ohi and T.Nishida [8,9], S.Shinmori et al [11]. Roughly speaking, almost all works on the multi-state systems have the restriction that all the state spaces are finite totally ordered sets. Therefore, we can not use the notion of multi-state systems defined by the above authors for treating the system composed of units whose state spaces are partially ordered sets. Practically, there exist systems of which state spaces are not totally ordered sets, for example, the state of a system may be directly indicated by the two distinct factors, as temperature and humidity.

In this paper, we discuss stochastic bounds on more generalized systems that the state spaces of both the systems and all the components are defined only as partially ordered sets. The results of associated probability measures proposed by F.Ohi, S. Shinmori and T.Nishida [10] are applied in order to derive stochastic bounds. Furthermore, we present stochastic bounds in case that all the state spaces are defined as finite partially ordered sets and systems are decomposed into several modules.

In Section 2, we present the definition of generalized systems and some notation. A theorem which plays a crucial role in the sequence is presented. In Section 3, we discuss the precise system reliability and some stochastic bounds. In Section 4, we consider stochastic bounds in case that the systems are decomposed into modules, and present a few simple examples. In Section 5, we give a comparison of the computational complexity for stochastic bounds developed in sections 3 and 4.

2. Preliminaries

J.D. Esary et al [6] have introduced the notion of "association" which is very useful in discussing stochastic bounds on system reliability;

Random variables $T_1,\dots,T_n$ are said to be "associated" if

$\mathrm{Cov}[f(T_1,\dots,T_n), g(T_1,\dots,T_n)] \geqq 0$ for any nondecreasing functions f and g.

In many physical reliability situations, the notion of association means that the functioning (failure) of a component contributes to the functioning (failure) of the remaining components and so on (see Section 2 of Chapter 2 in [1]). Furthermore, F.Ohi et al in [10] have presented a definition of associated probability measures on general partially ordered sets, and showed the necessary and sufficient condition for a probability $\mathbf{P}$ to be associated as follows:

Definition 0. Let Ω be a partially ordered set and let $\mathcal{F}$ be the σ-field generated by the class of all the increasing subsets of Ω. A probability $\mathbf{P}$ on $(\Omega,\mathcal{F})$ is called associated if and only if for any real-valued increasing measurable functions f and g such that f, g and $f\circ g$ are integrable with respect to $\mathbf{P}$,

$$(2.1)\qquad \int_\Omega f\circ g\, d\mathbf{P} \geqq \int_\Omega f\, d\mathbf{P} \int_\Omega g\, d\mathbf{P}$$

Theorem 0. Let Ω be a partially ordered set. A probability $\mathbf{P}$ on $(\Omega,\mathcal{F})$is associated if and only if for every increasing subsets A and B of $\mathcal{F}$,

$$(2.2)\qquad \mathbf{P}(A\cap B) \geqq \mathbf{P}(A)\mathbf{P}(B)$$

Note that a subset A of a partially ordered set Ω is called increasing (decreasing) if and only if $x\in A$ and $x\leqq y$ $(y\leqq x)$ imply $y\in A$. The concept of increasing and decreasing subsets will play an important role in our study. We denote by $(\Pi_{i=1}^n\Omega_i, \otimes_{i=1}^n\mathcal{F}_i)$ the product measurable space of $(\Omega_i,\mathcal{F}_i)$, $i=1,\dots,n$. From Theorem 0 we obtain the next theorem which is useful for obtaining stochastic bounds on reliability of systems.

Theorem 2.1. Let Ω_i $(i=1,\dots,n)$ be a partially ordered set and let $\mathcal{F}_i$ be the σ-field generated by the class of all the increasing subsets of Ω_i. If $\mathbf{P}$ is an associated probability on $(\Pi_{i=1}^n\Omega_i, \otimes_{i=1}^n\mathcal{F}_i)$, then we have

$$(2.3)\qquad 1-\Pi_{i=1}^n\{1-\mathbf{P}_i(A_i)\} \geqq \mathbf{P}(\Pi_{i=1}^n A_i) \geqq \Pi_{i=1}^n \mathbf{P}_i(A_i)$$

for any increasing sets $A_i\in\mathcal{F}_i\,(i=1,\dots,n)$, where $\mathbf{P}_i$ is the restriction of $\mathbf{P}$ to Ω_i.

Proof: Noticing that $\Pi_{i=1}^n A_i = \cap_{j=1}^n(A_j\times\Pi_{i=1,i\neq j}^n\Omega_i)$ for any increasing sets $A_i\in\mathcal{F}_i$ $(i=1,\dots,n)$, from Thoerem 0, it follows that

(2.4) $\quad \mathbf{P}(\Pi_{i=1}^{n} A_i) = \mathbf{P}\{\cap_{j=1}^{n}(A_j \times \Pi_{i=1, i\neq j}^{n}\Omega_i)\} \geqq \Pi_{j=1}^{n}\mathbf{P}(A_j \times \Pi_{i=1,i\neq j}^{n}\Omega_i)$

$= \Pi_{j=1}^{n}\{\mathbf{P}_j(A_j)\,\Pi_{i=1,i\neq j}^{n}\mathbf{P}_i(\Omega_i)\} = \Pi_{i=1}^{n}\mathbf{P}_i(A_i).$

Therefore, we have $\mathbf{P}(\Pi_{i=1}^{n} A_i) \geqq \Pi_{i=1}^{n}\mathbf{P}_i(A_i)$ for any increasing sets $A_i \in \mathcal{F}_i (i=1,\ldots,n)$.

It follows that $(\Pi_{i=1}^{n}\Omega_i)\setminus\{\Pi_{i=1}^{n}(\Omega_i\setminus A_i)\} \supset \Pi_{i=1}^{n} A_i$ for any increasing sets $A_i \in \mathcal{F}_i$ $(i=1,\ldots,n)$, so that

(2.5) $\quad \mathbf{P}[(\Pi_{i=1}^{n}\Omega_i)\setminus\{\Pi_{i=1}^{n}(\Omega_i\setminus A_i)\}] = \mathbf{P}(\Pi_{i=1}^{n}\Omega_i) - \mathbf{P}\{\Pi_{i=1}^{n}(\Omega_i\setminus A_i)\}$

$\leqq 1 - \Pi_{i=1}^{n}\mathbf{P}_i(\Omega_i\setminus A_i) = 1 - \Pi_{i=1}^{n}\{1-\mathbf{P}_i(A_i)\},$

since $\Omega_i\setminus A_i$ is a decreasing set. Therefore, we have

$1 - \Pi_{i=1}^{n}\{1-\mathbf{P}_i(A_i)\} \geqq \mathbf{P}(\Pi_{i=1}^{n} A_i)$ for any increasing sets $A_i \in \mathcal{F}_i (i=1,\ldots,n)$.

Q.E.D.

Now we give a definition of generalized systems which includes almost all the multi-state systems.

Definition 2.1. A generalized system is a triplet $(\Pi_{i=1}^{n}\Omega_i, S, \varphi)$ satisfying the following conditions:

(i) $\Omega_i (i=1,\ldots,n)$ and S are partially ordered sets.

(ii) φ is an increasing surjective mapping from the product ordered set $\Pi_{i=1}^{n}\Omega_i$ to S.

We use "increasing" in place of "nondecreasing" and the common symbol "$\leqq$" in order to indicate the "order" on $\Omega_i (i=1,\ldots,n)$ and S. Further we use the following notation for a system $(\Pi_{i=1}^{n}\Omega_i, S, \varphi)$ throughout this paper, where $s \in S$, and $\underline{x} \in \Pi_{i=1}^{n}\Omega_i$.

(1) $C = \{1, \ldots, n\}$, that is, C denotes the set of all the components.

(2) $\varphi^{-1}(s\leqq) = \{\underline{x}\,;\, s \leqq \varphi(\underline{x})\}$. $\quad$ (3) $\varphi^{-1}(s\not\leqq) = \{\underline{x}\,;\, s \not\leqq \varphi(\underline{x})\}$.

(4) $\varphi^{-1}(\leqq s) = \{\underline{x}\,;\, \varphi(\underline{x}) \leqq s\}$. $\quad$ (5) $\varphi^{-1}(s) = \{\underline{x}\,;\, \varphi(\underline{x}) = s\}$.

(6) $M[A]$ is the set of all the maximal elements of a set A.

(7) $N[A]$ is the set of all the minimal elements of a set A.

(8) $\mathrm{pro.}_{\Omega_j}$ is the projection map from $\Pi_{i=1}^{n}\Omega_i$ to Ω_j $(j \in C)$.

(9) $\underline{x}^A$ denotes $\mathrm{pro.}_{\Pi_{i\in A}\Omega_i}(\underline{x})$, where $A(\neq\phi)\subset C$.

3. Stochastic Bounds by Maximal and Minimal Elements.

In this section, we present stochastic bounds on the generalized system $(\Pi_{i=1}^{n}\Omega_i, S, \varphi)$ given in Definition 2.1, and apply them to some concrete systems. Applying stochastic bounds obtained in this section to an actual system, the point is whether the assumption of association holds or not. If all the components, whose state spaces are the same finite totally ordered sets, are mutually independent, the assumption of association always holds. As stated in [10], it is notable that the assumption of association may be also satisfied for the generalized system composed of independent components.

3.1 Stochastic bounds

Let $(\Pi_{i=1}^{n}\Omega_i, S, \varphi)$ be a generalized system. We define $\mathcal{W}_s$, $\mathcal{V}_s (s\in S)$ as follows:

$\mathcal{W}_s = \{w\,;\, w$ is an increasing subset of $\varphi^{-1}(s\leqq)$ such that $w = \Pi_{i=1}^{n}\mathrm{pro.}_{\Omega_i}(w)\}$,

$\mathcal{V}_s = \{v\,;\, v$ is a decreasing subset of $\varphi^{-1}(s\not\leqq)$ such that $v = \Pi_{i=1}^{n}\mathrm{pro.}_{\Omega_i}(v)\}$.

Then, for $\mathcal{W}_s$ and $\mathcal{V}_s$ $(s\in S)$, the next property holds.

Property 3.1. $\mathcal{W}_s$ and $\mathcal{V}_s$ are inductive with respect to the inclusion relation between sets as an order.

Since it is obvious from the definition of $\mathcal{W}_s$ and $\mathcal{V}_s$, the proof is omitted.

From Property 3.1 and lemma due to Zorn, we see that $\mathcal{W}_s$ and $\mathcal{V}_s$ have some maximal elements. Hence let w_γ^s $(\gamma \in \Gamma)$ and v_δ^s $(\delta \in \Delta)$ be all the maximal elements of $\mathcal{W}_s$ and $\mathcal{V}_s$, respectively. Then the following property and lemma hold.

Property 3.2. For every s of S, we have

(3.1) $\quad \varphi^{-1}(s\leq) = \cup_{\gamma \in \Gamma} w_\gamma^s, \quad$ and $\quad \varphi^{-1}(s\not\leq) = \cup_{\delta \in \Delta} v_\delta^s.$

Since it is obvious from property 3.1 and the notation of $\varphi^{-1}(s\leq)$ and $\varphi^{-1}(s\not\leq)$, the proof is omitted.

Lemma 3.1. For every s of S, we have

(3.2) $\quad \varphi^{-1}(s\leq) = \cup_{\gamma \in \Gamma} w_\gamma^s = (\Pi_{i=1}^n \Omega_i) \setminus \cup_{\delta \in \Delta} v_\delta^s,$ and

(3.3) $\quad \cap_{\gamma \in \Gamma} w_\gamma^s \subset \varphi^{-1}(s\leq) \subset (\Pi_{i=1}^n \Omega_i) \setminus \cap_{\delta \in \Delta} v_\delta^s.$

Furthermore, for every s of S, any γ of Γ and δ of Δ, we have

(3.4) $\quad w_\gamma^s \subset \varphi^{-1}(s\leq) \subset (\Pi_{i=1}^n \Omega_i) \setminus v_\delta^s.$

Proof: From (3.1), it follows that

$$\{\cup_{\gamma \in \Gamma} w_\gamma^s\} \cup \{\cup_{\delta \in \Delta} v_\delta^s\} = \varphi^{-1}(s\leq) \cup \varphi^{-1}(s\not\leq) = \Pi_{i=1}^n \Omega_i.$$

Since $\varphi^{-1}(s\leq) \cap \varphi^{-1}(s\not\leq) = \phi$ (the empty set), we have

$$\varphi^{-1}(s\leq) = \cup_{\gamma \in \Gamma} w_\gamma^s = (\Pi_{i=1}^n \Omega_i) \setminus \cup_{\delta \in \Delta} v_\delta^s.$$

(3.3) and (3.4) are obvious from (3.2). Q.E.D.

Since $(\Pi_{i=1}^n \Omega_i, S, \varphi)$ is a generalized system, we see that φ is a measurable function from $(\Pi_{i=1}^n \Omega_i, \otimes_{i=1}^n \mathcal{F}_i)$ to $(S, \mathcal{F})$, where $\mathcal{F}$ is the σ-field generated by the class of all the increasing subsets of S. Letting $\mathbf{P}$ be a probability on $(\Pi_{i=1}^n \Omega_i, \otimes_{i=1}^n \mathcal{F}_i)$, $\mathbf{P}\{\varphi^{-1}(s)\}$ means the system reliability that the performance of the system becomes state s, where s belongs to S. If we obtain the system reliability $\mathbf{P}\{\varphi^{-1}(s)\}$ for any s of S, the another probability $\mathbf{P}\{\varphi^{-1}(s\leq)\}$, of which the state is greater than or equal to s, can be directly computed as $\sum_{s\leq t} \mathbf{P}\{\varphi^{-1}(t)\}$. Hence we see that the probability $\mathbf{P}\{\varphi^{-1}(s\leq)\}$ is equivalent to the system reliability $\mathbf{P}\{\varphi^{-1}(s)\}$. Further, throughout all the results of this section, we may replace $\mathbf{P}\{\varphi^{-1}(s\leq)\}$ with $\mathbf{P}\{\varphi^{-1}(A)\}$, where A is an increasing subset of S, since $\{y \in S ; s\leq y\}$ belongs to the class of all the increasing subsets of partially ordered set S. From a physical point of view, however, we conclude that $\mathbf{P}\{\varphi^{-1}(s\leq)\}$ is noteworthy and clearly easy to treat. For the above reasons, we consider the probability $\mathbf{P}\{\varphi^{-1}(s\leq)\}$ as the system reliability in the following.

The system reliability $\mathbf{P}\{\varphi^{-1}(s\leq)\}$ and some stochastic bounds on the system reliability are derived from Lemma 3.1 as follows:

Theorem 3.1. Suppose that $(\Pi_{i=1}^n \Omega_i, S, \varphi)$ is a generalized system and $\mathbf{P}$ isa probability on $(\Pi_{i=1}^n \Omega_i, \otimes_{i=1}^n \mathcal{F}_i)$. Then for each s of S, we have

(3.5) $\quad \mathbf{P}\{\cap_{\delta \in \Delta} (\Pi_{i=1}^n \Omega_i \setminus v_\delta^s)\} = \mathbf{P}\{\varphi^{-1}(s\leq)\} = 1 - \mathbf{P}\{\cap_{\gamma \in \Gamma} (\Pi_{i=1}^n \Omega_i \setminus w_\gamma^s)\}.$

Proof: Noticing that $(\Pi_{i=1}^{n}\Omega_i)\setminus \cup_{\delta\in\Delta} v_\delta^S = \cap_{\delta\in\Delta}(\Pi_{i=1}^{n}\Omega_i\setminus v_\delta^S)$ and $\cup_{\gamma\in\Gamma} w_\gamma^S = (\Pi_{i=1}^{n}\Omega_i)\setminus \cap_{\gamma\in\Gamma}(\Pi_{i=1}^{n}\Omega_i\setminus w_\gamma^S)$, from (3.2), we see that

$$\varphi^{-1}(s_S) = \cap_{\delta\in\Delta}(\Pi_{i=1}^{n}\Omega_i\setminus v_\delta^S) = (\Pi_{i=1}^{n}\Omega_i)\setminus \cap_{\gamma\in\Gamma}(\Pi_{i=1}^{n}\Omega_i\setminus w_\gamma^S).$$

Since $\mathbf{P}\{(\Pi_{i=1}^{n}\Omega_i)\setminus \cap_{\gamma\in\Gamma}(\Pi_{i=1}^{n}\Omega_i\setminus w_\gamma^S)\} = 1-\mathbf{P}\{\cap_{\gamma\in\Gamma}(\Pi_{i=1}^{n}\Omega_i\setminus w_\gamma^S)\}$, the proof is complete. Q.E.D.

Theorem 3.2. Under the same assumption of Theorem 3.1, the following inequality holds:

$$\text{(3.6)}\qquad \mathbf{P}(\cap_{\gamma\in\Gamma} w_\gamma^S) \leqq \sup_{\gamma\in\Gamma}\mathbf{P}(w_\gamma^S) \leqq \mathbf{P}\{\varphi^{-1}(s_S)\}$$
$$\leqq 1-\sup_{\delta\in\Delta}\mathbf{P}(v_\delta^S) \leqq 1-\mathbf{P}(\cap_{\delta\in\Delta} v_\delta^S).$$

Further, under the additional assumption that $\mathbf{P}$ is associated, we have

$$\text{(3.7)}\qquad \Pi_{i=1}^{n}\mathbf{P}_i\{\mathrm{pro.}_{\Omega_i}(\cap_{\gamma\in\Gamma} w_\gamma^S)\} \leqq \sup_{\gamma\in\Gamma}\Pi_{i=1}^{n}\mathbf{P}_i\{\mathrm{pro.}_{\Omega_i}(w_\gamma^S)\}$$
$$\leqq \mathbf{P}\{\varphi^{-1}(s_S)\}\quad 1-\sup_{\delta\in\Delta}\Pi_{i=1}^{n}\mathbf{P}_i\{\mathrm{pro.}_{\Omega_i}(v_\delta^S)\}$$
$$\leqq 1-\Pi_{i=1}^{n}\mathbf{P}_i\{\mathrm{pro.}_{\Omega_i}(\cap_{\delta\in\Delta} v_\delta^S)\}.$$

Proof: Since $\cap_{\gamma\in\Gamma} w_\gamma^S \subset w_\gamma^S$ holds for any $\gamma\in\Gamma$, and from (3.4),

$$\mathbf{P}(\cap_{\gamma\in\Gamma} w_\gamma^S) \leqq \mathbf{P}(w_\gamma^S) \leqq \sup_{\gamma\in\Gamma}\mathbf{P}(w_\gamma^S) \leqq \mathbf{P}\{\varphi^{-1}(s_S)\}.$$

Similarly, for $v_\delta^S(\delta\in\Delta)$, we have

$$\mathbf{P}\{\varphi^{-1}(s_S)\} \leqq \inf_{\delta\in\Delta}\mathbf{P}\{(\Pi_{i=1}^{n}\Omega_i)\setminus v_\delta^S\} \leqq 1-\sup_{\delta\in\Delta}\mathbf{P}(v_\delta^S)$$
$$\leqq \mathbf{P}\{(\Pi_{i=1}^{n}\Omega_i)\setminus \cap_{\delta\in\Delta} v_\delta^S\} \leqq 1-\mathbf{P}(\cap_{\delta\in\Delta} v_\delta^S).$$

Let $\mathbf{P}$ be associated. Then we see that $\Pi_{i=1}^{n}\mathbf{P}_i\{\mathrm{pro.}_{\Omega_i}(w_\gamma^S)\} \leqq \mathbf{P}(w_\gamma^S)$, from Theorem 2.1, so that

$$\sup_{\gamma\in\Gamma}\Pi_{i=1}^{n}\mathbf{P}_i\{\mathrm{pro.}_{\Omega_i}(w_\gamma^S)\} \leqq \sup_{\gamma\in\Gamma}\mathbf{P}(w_\gamma^S).$$

Hence the second inequality in (3.7) follows from (3.6).

Since $\mathrm{pro.}_{\Omega_i}(\cap_{\gamma\in\Gamma} w_\gamma^S) \subset \mathrm{pro.}_{\Omega_i}(w_\gamma^S)$ holds for each i of C and γ of Γ, the first inequality follows. The third and fourth inequalities follow similarly.

Q.E.D.

Theorem 3.3. Under the same hypotheses of Theorem 3.1 and the additional assumption that $\mathbf{P}$ is associated and Γ and Δ are at most countable, the following inequality holds:

$$\text{(3.8)}\qquad \Pi_{\delta\in\Delta}\{1-\mathbf{P}(v_\delta^S)\} \leqq \mathbf{P}\{\varphi^{-1}(s_S)\} \leqq 1-\Pi_{\gamma\in\Gamma}\{1-\mathbf{P}(w_\gamma^S)\}.$$

Proof: From the hypotheses and (3.5), we have

$$\mathbf{P}\{\varphi^{-1}(s_S)\} = \mathbf{P}[\cap_{\delta\in\Delta}\{(\Pi_{i=1}^{n}\Omega_i)\setminus v_\delta^S\}]$$
$$\geqq \Pi_{\delta\in\Delta}\mathbf{P}\{(\Pi_{i=1}^{n}\Omega_i)\setminus v_\delta^S\} = \Pi_{\delta\in\Delta}\{1-\mathbf{P}(v_\delta^S)\}.$$

$$P\{\varphi^{-1}(s\leqq)\} = 1 - P\{\cap_{\gamma\in\Gamma}(\Pi_{i=1}^{n}\Omega_i)\setminus w_\gamma^s)$$
$$\leqq 1 - \Pi_{\gamma\in\Gamma} P\{(\Pi_{i=1}^{n}\Omega_i)\setminus w_\gamma^s\} = 1 - \Pi_{\gamma\in\Gamma}\{1 - P(w_\gamma^s)\}$$ Q. E. D.

Corollary 3.1. Under the same hypotheses of Theorem 3.3 and the additional assumption that $P = \otimes_{i=1}^{n} P_i$, the following inequality holds:

(3.9) $$\Pi_{\delta\in\Delta}\{1 - \Pi_{i=1}^{n} P_i(\text{pro.}_{\Omega_i}(v_\delta^s))\} \leqq P\{\varphi^{-1}(s\leqq)\}$$
$$\leqq 1 - \Pi_{\gamma\in\Gamma}[1 - \Pi_{i=1}^{n} P_i\{\text{pro.}_{\Omega_i}(w_\gamma^s)\}].$$

Proof: *From the assumption on* P, *we see that*

(3.10) $$P(v_\delta^s) = \Pi_{i=1}^{n} P_i\{\text{pro.}_{\Omega_i}(v_\delta^s)\} \text{ and } P(w_\gamma^s) = \Pi_{i=1}^{n} P_i\{\text{pro.}_{\Omega_i}(w_\gamma^s)\}.$$

Then the result follows obviously from (3.8) and (3.10). Q. E. D.

3.2 Application to some concrete systems.

We apply the results of the previous section to some concrete systems, i.e., the binary-state system, the Barlow-Wu system [2] and the multi-state system. Furthermore, we demonstrate numerical stochastic bounds for a simple generalized system composed of two components.

(1) Binary-state system

A system $(\Pi_{i=1}^{n}\Omega_i, S, \varphi)$ is called a binary-state system if Ω_i and S are totally ordered sets containing two elements, that is, $\Omega_i = S = \{0, 1\}$ $(i\in C)$. For any binary-state system, it follows that $\varphi^{-1}(1\leqq) = \varphi^{-1}(1)$ and $\varphi^{-1}(1\not\leqq) = \varphi^{-1}(0)$. Let $\underline{x}_1, \ldots, \underline{x}_t$ be all the minimal elements of $\varphi^{-1}(1)$ and let $\underline{y}_1, \ldots, \underline{y}_u$ be all the maximal elements of $\varphi^{-1}(0)$. For each $\underline{x}_k$ $(1\leqq k\leqq t)$ and each $\underline{y}_k$ $(1\leqq k\leqq u)$, A_k and B_k are defined as

$$A_k = \{i\ ;\ (\underline{x}_k)_i = 1\} \text{ and } B_k = \{i\ ;\ (\underline{y}_k)_i = 0\},$$

where $(\underline{x}_k)_i$ and $(\underline{y}_k)_i$ are the i-th coordinate of $\underline{x}_k$ and $\underline{y}_k$ respectively.

Consequently, w_γ^1 $(\gamma\in\Gamma)$ and v_δ^1 $(\delta\in\Delta)$ are given as follows:

$$\Gamma = \{1, \ldots, t\},\ w_k^1 = \{\underline{x}\ ;\ \underline{x}_k \leqq \underline{x}\}\ (k\in\Gamma),\ \Delta = \{1, \ldots, u\},\ v_k^1 = \{\underline{y}\ ;\ \underline{y} \leqq \underline{y}_k\}\ (k\in\Delta).$$

From Theorems 3.2 and 3.3, if P, a probability on $(\Pi_{i=1}^{n}\Omega_i, \otimes_{i=1}^{n}\mathcal{F}_i)$, is associated, then we obtain stochastic bounds for the binary-state system.

(3.11) $$\Pi_{i\in\cup_{j=1}^{t}A_j} P_i(1) \leqq \max_{j\in\Gamma}\Pi_{i\in A_j} P_i(1) \leqq P\{\varphi^{-1}(1)\}$$
$$\leqq 1 - \max_{j\in\Delta}\Pi_{i\in B_j} P_i(0) \leqq 1 - \Pi_{i\in\cup_{j=1}^{u}B_j} P_i(0),$$

(3.12) $$\Pi_{j\in\Delta}\{1 - P_{B_j}(\underline{0}^{B_j})\} \leqq P\{\varphi^{-1}(1)\} \leqq 1 - \Pi_{j\in\Gamma}\{1 - P_{A_j}(\underline{1}^{A_j})\},$$

where P_A is the restriction of P to $(\Pi_{i\in A}\Omega_i, \otimes_{i\in A}\mathcal{F}_i)$.

These stochastic bounds correspond to the well-known max-min bounds in [1].

(2) Barlow-Wu system

A system $(\Pi_{i=1}^{n}\Omega_i, S, \varphi)$ is called a Barlow-Wu system if Ω_i $(i\in C)$ and S are the same finite totally ordered sets and φ is represented by using a class, $\{A_j\}_{j=1}^{p}$, of subsets of C, $A_i \setminus A_j$ $(i\neq j)$, as follows:

(3.13) $$\varphi(\underline{x}) = \max_{1\leqq j\leqq p}\min_{i\in A_j}(\underline{x})_i \text{ for any } \underline{x} \text{ of } \Pi_{i=1}^{n}\Omega_i.$$

Let $B_1, \dots, B_k$ be all the minimal elements of $\{B ; B \cap A_j \neq \phi$ for all $A_j\}$, where minimal elements are taken with respect to the inclusion relation between sets. Then φ is also represented as

(3.14) $\varphi(\underline{x}) = \min_{1 \leq j \leq k} \max_{i \in B_j} (\underline{x})_i$ for any $\underline{x}$ of $\Pi_{i=1}^n \Omega_i$.

Furthermore, $w_\gamma^S (\gamma \in \Gamma)$ and $v_\delta^S (\delta \in \Delta)$ are defined as

$$\Gamma = \{1, \dots, p\},\quad w_j^S = [s, \rightarrow)^{|A_j|} \Pi_{i \in C \setminus A_j} \Omega_i \quad (j \in \Gamma), \text{ and}$$

$$\Delta = \{1, \dots, k\},\quad v_j^S = (\leftarrow, s)^{|B_j|} \Pi_{i \in C \setminus B_j} \Omega_i \quad (j \in \Delta),$$

where $[s, \rightarrow) = \{x ; s \leq x\}$, $(\leftarrow, s) = \{x ; x < s\}$, $|A_j|$ denotes the cardinality of A_j and so is $|B_j|$, and $[s, \rightarrow)^{|A_j|}$ denotes the product set of $|A_j|$ number of $[s, \rightarrow)$ and so is $(\leftarrow, s)^{|B_j|}$.

From Theorems 3.2 and 3.3, if P is an associated probability on $(\Pi_{i=1}^n \Omega_i, \otimes_{i=1}^n \mathcal{F}_i)$, then we obtain stochastic bounds for the Barlow-Wu system.

(3.15) $$\Pi_{i \in \cup_{j=1}^p A_j} P_i\{[s, \rightarrow)\} \leq \max_{j \in \Gamma} \Pi_{i \in A_j} P_i\{[s, \rightarrow)\} \leq P\{\varphi^{-1}(s \leq)\}$$
$$\leq 1 - \max_{j \in \Delta} \Pi_{i \in B_j} P_i\{(\leftarrow, s)\} \leq 1 - \Pi_{i \in \cup_{j=1}^k B_j} P_i\{(\leftarrow, s)\},$$

(3.16) $$\Pi_{j \in \Delta} [1 - P_{B_j}\{(\leftarrow, s)^{|B_j|}\}] \leq P\{\varphi^{-1}(s \leq)\}$$
$$\leq 1 - \Pi_{j \in \Gamma} [1 - P_{A_j} \{([s, \rightarrow)^{|A_j|}\}].$$

(3) Multi-state system

A system $(\Pi_{i=1}^n \Omega_i, S, \varphi)$ is called a multi-state system if Ω_i $(i \in C)$ and S are finite totally ordered sets which are not necessarily the same. Note that $\varphi^{-1}(s \not\leq) = \varphi^{-1}(< s) = \{\underline{x} \in \Pi_{i=1}^n \Omega_i ; \varphi(\underline{x}) < s\}$ for any $s \in S$, since S is a totally ordered set. We denote all the minimal elements of $\varphi^{-1}(s \leq)$ and $\varphi^{-1}(< s)$ by $\underline{x}_1, \dots, \underline{x}_t$ and $\underline{y}_1, \dots, \underline{y}_u$ respectively.

Then $w_\gamma^S (\gamma \in \Gamma)$ and $v_\delta^S (\delta \in \Delta)$ are defined as

$$\Gamma = \{1, \dots, t\},\ w_k^S = \{\underline{x} ; \underline{x}_k \leq \underline{x}\} (k \in \Gamma),\quad \Delta = \{1, \dots, u\},\ v_k^S = \{\underline{y} ; \underline{y} \leq \underline{y}_k\} (k \in \Delta).$$

From Theorems 3.2 and 3.3, if P, a probability on $(\Pi_{i=1}^n \Omega_i, \otimes_{i=1}^n \mathcal{F}_i)$, is associated, then we obtain stochastic bounds for the multi-state system.

(3.17) $$\Pi_{i=1}^n P_i\{\text{pro.}_{\Omega_i}(\cap_{j \in \Gamma} w_j^S)\} \leq \max_{j \in \Gamma} \Pi_{i=1}^n P_i\{\text{pro.}_{\Omega_i}(w_j^S)\} \leq P\{\varphi^{-1}(s \leq)\}$$
$$\leq 1 - \max_{j \in \Delta} \Pi_{i=1}^n P_i\{\text{pro.}_{\Omega_i}(v_j^S)\} \leq 1 - \Pi_{i=1}^n P_i\{\text{pro.}_{\Omega_i}(\cap_{j \in \Delta} v_j^S)\},$$

(3.18) $$\Pi_{j \in \Delta}\{1 - P(v_j^S)\} \leq P\{\varphi^{-1}(s \leq)\} \leq 1 - \Pi_{j \in \Gamma}\{1 - P(w_j^S)\}.$$

If the multi-state system $(\Pi_{i=1}^n \Omega_i, S, \varphi)$ satisfies the additional condition $\Omega_i = S$ $(i \in C)$, these stochastic bounds are identical with the bounds established in [4].

(4) Numerical example of generalized system

We consider the simple generalized system $(\Omega_1 \times \Omega_2, S, \varphi)$ composed of independent two components such that $\Omega_1 = \Omega_2 = S = \{a, b, c, d\}$, $d < b < a$, $d < c < a$, $b \not\leq c$ and $c \not\leq b$, where $b < a$ denotes $b \leq a$ and $b \neq a$. The elements a and d mean the perfect functioning state and the complete failed state, respectively. The other elements b and c are the distinct intermediate states between a and d. Furthermore, the structure of system φ is given as follows:

$N[\varphi^{-1}(a)] = \{(a,b),(a,c),(b,a)\}$, $N[\varphi^{-1}(b)] = \{(b,b),(b,c)\}$ and
$N[\varphi^{-1}(c)] = \{(c,b),(c,c)\}$.

Further, for the sake of brevity, we assume that the reliabilities of both components are identical and

$p_a = 0.4$, $p_b = 0.3$, $p_c = 0.2$ and $p_d = 0.1$,

where p_i ($i \in \{a,b,c,d\}$) denotes the probability that the component takes the state i. Then the system reliability and its stochastic bounds are calculated from Theorems 3.1 and 3.3 as follows:

	Lower bound	System reliability	Upper bound
$P\{\varphi^{-1}(a\leq)\}$	0.466	0.480	0.606
$P\{\varphi^{-1}(b\leq)\}$	0.630	0.630	0.704
$P\{\varphi^{-1}(c\leq)\}$	0.599	0.660	0.733

4. Stochastic bounds using modular decompositions.

In this section we consider the generalized system composed entirely of modules. A module is a subset of components which constitutes a sub-system, and the module itself can be treated as one component of the system (see [12]). we exploit stochastic bounds in case that the generalized system is decomposed into a number of modules. We show that these stochastic bounds are always tighter than the stochastic bounds derived in section 3, if all the modules which compose a system satisfy a condition given later on. First of all, we present the definition of the modular decomposition.

Definition 4.1. A partition $\{A_j\}_{j=1}^{r}$ of C is called a modular decomposition of a generalized system $(\Pi_{i\in C}\Omega_i, S, \varphi)$ if and only if there exist generalized systems $(\Pi_{i\in A_j}\Omega_i, S_j, \chi_j)$, $j=1,\ldots,r$, and $(\Pi_{j=1}^{r} S_j, S, \phi)$ satisfying that

(4.1) $\varphi(\underline{x}) = \phi\{\chi_1(\underline{x}^{A_1}),\ldots,\chi_r(\underline{x}^{A_r})\}$ for any $\underline{x}$ of $\Pi_{i\in C}\Omega_i$.

For the sake of brevity, throughout the rest of this section we consider the generalized systems (X, Z, φ), (X_j, Y_j, χ_j), $j=1,\ldots,r$, (X, Y, χ) and (Y, Z, ϕ) satisfying the following conditions:

(i) X_j, Y_j, $(j=1,\ldots,r)$ and Z are finite partially ordered sets.

(ii) $X = \Pi_{j=1}^{r} X_j$ and $Y = \Pi_{j=1}^{r} Y_j$.

(iii) $\varphi(\underline{x}) = \phi(\chi(\underline{x}))$ for any $\underline{x}$ of X, where $\chi(\underline{x}) = (\chi_1(x_1),\ldots,\chi_r(x_r))$.

We will exploit new stochastic bounds for the above system (X, Z, φ) composed of r modules. Since the lower and upper bounds on system reliability are derived dually as shown in section 3, we are devoted to the lower bound in the following. For the upper bound the only results are given. In developing the lower bound for the system composed of a number of modules, the following condition, which will be called MC (Maximal Coincidence) condition, plays an important role. In particular, it is notable to inspect whether the generalized system (X, Y, χ) satisfies MC condition.

Definition 4.2. The generalized system (X, Y, χ) satisfies MC condition if and only if

$\chi(\underline{\mathbf{x}}) = \underline{\mathbf{y}}$ for any $\underline{\mathbf{y}}$ of Y and any $\underline{\mathbf{x}}$ of $M[\chi^{-1}(\leq \underline{\mathbf{y}})]$.

All the elements of the set $M[\chi^{-1}(\leq \underline{\mathbf{y}})]$ are the maximal vectors such that the performance of the generalized system χ is less than or equal to the state $\underline{\mathbf{y}}$. Physically, the MC property of the generalized system warrants that the system performance for any maximal vectors of $M[\chi^{-1}(\leq \underline{\mathbf{y}})]$ invariably occupies the state $\underline{\mathbf{y}}$. Clearly, the MC condition does not necessarily hold for any generalized systems.

With respect to MC condition, the next property and lemma hold.

Property 4.1. The following four conditions are equivalent:

(1) The generalized system (X, Y, χ) satisfies MC condition,

(2) $M[\chi^{-1}(\leq \underline{\mathbf{y}})] = M[\chi^{-1}(\underline{\mathbf{y}})]$ for any $\underline{\mathbf{y}}$ of Y,

(3) $\underline{\mathbf{a}}, \underline{\mathbf{b}} \in Y$ and $\underline{\mathbf{a}} < \underline{\mathbf{b}}$ imply that for any $\underline{\mathbf{u}}$ of $M[\chi^{-1}(\underline{\mathbf{a}})]$ there exists a $\underline{\mathbf{v}}$ of $M[\chi^{-1}(\underline{\mathbf{b}})]$ such that $\underline{\mathbf{u}} < \underline{\mathbf{v}}$,

(4) $\underline{\mathbf{a}}, \underline{\mathbf{b}} \in Y$ and $\underline{\mathbf{a}} < \underline{\mathbf{b}}$ imply that for any $\underline{\mathbf{u}}$ of $\chi^{-1}(\underline{\mathbf{a}})$ there exists a $\underline{\mathbf{v}}$ of $\chi^{-1}(\underline{\mathbf{b}})$ such that $\underline{\mathbf{u}} < \underline{\mathbf{v}}$,

where $\underline{\mathbf{a}} < \underline{\mathbf{b}}$ $(\underline{\mathbf{a}}, \underline{\mathbf{b}} \in Y)$ denotes $\underline{\mathbf{a}} \leq \underline{\mathbf{b}}$ and $\underline{\mathbf{a}} \neq \underline{\mathbf{b}}$.

Since it is immediate from Definition 4.2, the proof is omitted.

Lemma 4.1. The generalized system (X, Y, χ) satisfies MC condition if and only if the generalized systems (X_j, Y_j, χ_j), $j=1,\ldots,r$, satisfy MC condition.

Proof: It is proved by using Property 4.1 and the definition of (X, Y, χ) and (X_j, Y_j, χ_j), $j=1,\ldots,r$. Q.E.D.

From Lemma 4.1, if the generalized system (X, Y, χ) satisfies MC condition, it follows that for any $\underline{\mathbf{b}}$ of Y,

(4.2) $\quad M[\chi^{-1}(\underline{\mathbf{b}})] = \{(a_1,\ldots,a_r)\ ;\ a_i \in M[\chi_i^{-1}(b_i)],\ i=1,\ldots,r\}$.

Further arbitrary maximal element of $\varphi^{-1}(z \nleq)$ $(z \in Z)$ is represented by both a maximal element of $\phi^{-1}(z \nleq)$ and a maximal element of $\chi^{-1}(\leq \underline{\mathbf{y}})$, where $\underline{\mathbf{y}}$ belongs to the set of all the maximal elements of $\phi^{-1}(z \nleq)$, and the converse holds, that is, the following lemma holds.

Lemma 4.2. Suppose that the generalized system (X, Y, χ) satisfies MC condition, then we have for any z of Z,

(4.3) $\quad M[\varphi^{-1}(z \nleq)] = \cup_{\underline{\mathbf{y}} \in M[\psi^{-1}(z \nleq)]} M[\chi^{-1}(\leq \underline{\mathbf{y}})]$.

Proof: Since it is immediate that $M[\varphi^{-1}(z \nleq)] \subset \cup_{\underline{\mathbf{y}} \in M[\psi^{-1}(z \nleq)]} M[\chi^{-1}(\leq \underline{\mathbf{y}})]$ for any z of Z, we show the converse of the above inclusion relationship; Note that

(4.4) $\quad \underline{\mathbf{a}} \in \cup_{\underline{\mathbf{y}} \in M[\psi^{-1}(z \nleq)]} M[\chi^{-1}(\leq \underline{\mathbf{y}})] \quad \Leftrightarrow \quad \exists\, \underline{\mathbf{y}} \in M[\psi^{-1}(z \nleq)] : \underline{\mathbf{a}} \in M[\chi^{-1}(\leq \underline{\mathbf{y}})]$.

Let $\underline{\mathbf{a}} \in \cup_{\underline{\mathbf{y}} \in M[\psi^{-1}(z \nleq)]} M[\chi^{-1}(\leq \underline{\mathbf{y}})]$. From the hypothesis and Property 4.1, we have $\chi(\underline{\mathbf{a}}) = \underline{\mathbf{y}}$ and $\underline{\mathbf{a}} \in M[\chi^{-1}(\underline{\mathbf{y}})]$. Since $z \nleq \phi(\underline{\mathbf{y}}) = \phi(\chi(\underline{\mathbf{a}})) = \varphi(\underline{\mathbf{a}})$, it follows that $\underline{\mathbf{a}} \in \varphi^{-1}(z \nleq)$. Supposing that $\underline{\mathbf{b}} \in X$ and $\underline{\mathbf{a}} < \underline{\mathbf{b}}$. Then $\underline{\mathbf{y}} < \chi(\underline{\mathbf{b}})$ holds from the increasing property of χ and $\underline{\mathbf{a}} \in M[\chi^{-1}(\underline{\mathbf{y}})]$. Further $z \leq \phi(\chi(\underline{\mathbf{b}}))$ holds from the increasing property of ϕ and $\underline{\mathbf{y}} \in M[\psi^{-1}(\nleq z)]$. Hence it follows that $z \leq \phi(\chi(\underline{\mathbf{b}})) = \varphi(\underline{\mathbf{b}})$, so that $\underline{b} \notin \varphi^{-1}(z \nleq)$. Therefore, we obtain that $\underline{a} \in M[\varphi^{-1}(z \nleq)]$. Q.E.D.

Remark to lemma 4.2. If χ does not satisfy MC condition, we have only

(4.5) $M[\varphi^{-1}(z \nleq)] \subset \cup_{\underline{y} \in M[\psi^{-1}(z \nleq)]} M[\chi^{-1}(\leq \underline{y})]$ for any z of Z.

(See Example 4.2 later on.)

Let P_i be a probability on $(X_i, \mathcal{F}_i)$ and let $\mathbf{P}$ be the product probability of P_i (i=1,...,r), where $\mathcal{F}_i$ is the class of all the subsets of X_i. If P_i (i=1,...,r) are associated and χ satisfies MC condition, from Theorem 3.3 the lower bounds on $\mathbf{P}\{\varphi^{-1}(z \leq)\}$ $(z \in Z)$ are directly derived as follows:

(4.6) $\mathbf{P}\{\varphi^{-1}(z \leq)\} \geq \Pi_{\underline{b} \in M[\psi^{-1}(z \nleq)]} [1 - \Pi_{i=1}^{r} P_i\{x_i \in X_i ; \chi_i(x_i) \leq b_i\}]$

$\geq \Pi_{\underline{b} \in M[\psi^{-1}(z \nleq)]} \Pi_{\underline{a} \in M[\chi^{-1}(\leq \underline{b})]} [1 - \Pi_{i=1}^{r} P_i\{x_i \in X_i ; x_i \leq a_i\}]$

Furthermore, since it follows that

$$\{x_i \in X_i ; \chi_i(x_i) \leq b_i\} = X_i \setminus \cap_{a_i \in M[\chi^{-1}_i(\leq b_i)]} [X_i \setminus \{x_i \in X_i ; x_i \leq a_i\}],$$

by the association of P_i (i=1,...,r), we see that

(4.7) $P_i\{x_i \in X_i ; \chi_i(x_i) \leq b_i\} \leq 1 - \Pi_{a_i \in M[\chi^{-1}_i(\leq b_i)]} [1 - P_i\{x_i \in X_i ; x_i \leq a_i\}]$

$\leq 1 - \Pi_{a_i \in M[\chi^{-1}_i(\leq b_i)]} P_i\{x_i \in X_i ; x_i \nleq a_i\}$

Let F(z) be the expression which is substituted the third expression of (4.7) in place of $P_i\{x_i \in X_i ; \chi_i(x_i) \leq b_i\}$ of the second expression of (4.6), that is,

(4.8) $F(z) = \Pi_{\underline{b} \in M[\psi^{-1}(z \nleq)]} [1 - \Pi_{i=1}^{r} \{1 - \Pi_{a_i \in M[\chi^{-1}_i(\leq b_i)]} P_i\{x_i \in X_i ; x_i \nleq a_i\}\}]$

Denote by G(z) the modified third expression in (4.6), namely,

(4.9) $G(z) = \Pi_{\underline{b} \in M[\psi^{-1}(z \nleq)]} \Pi_{\underline{a} \in M[\chi^{-1}(\leq \underline{b})]} [1 - \Pi_{i=1}^{r} \{1 - P_i\{x_i \in X_i ; x_i \nleq a_i\}\}]$

L.D.Bodin [3] has shown that for any binary-state systems the stochastic bounds using modular decompositions are better than those directly derived. We now rewrite lemma 1 in [3] as follows:

Lemma 4.3. Let T_i be a set $\{1,\ldots,k_i\}$ for every i(=1,...,r). Let $0 \leq P_{ij_i} \leq 1$ for every i(=1,...,r) and each j_i of T_i. Then it follows that

(4.10) $1 - \Pi_{i=1}^{r} [1 - \Pi_{j_i \in T_i} P_{ij_i}] \geq \Pi_{(j_1,\ldots,j_r) \in \Pi_{i=1}^{r} T_i} [1 - \Pi_{i=1}^{r} (1 - P_{ij_i})]$

Proof: Consider the binary-state system $(\Pi_{i=1}^{r} \Pi_{j_i \in T_i} \Omega_{ij_i}, S, \varphi)$ represented by Fig. 4.1, and composed of $\Pi_{i=1}^{r}(k_i)$ independent components. Hence it follows that

$\Omega_{ij_i} = S = \{0, 1\}$ for each i(=1,...,r) and each j_i of T_i, and

$\varphi(\underline{x}) = 1 - \Pi_{i=1}^{r} \{1 - \Pi_{j_i \in T_i} (x_{ij_i})\}$ for any $\underline{x}$ of $\Pi_{i=1}^{r} \Pi_{j_i \in T_i} \Omega_{ij_i}$.

Here letting P_{ij_i} be a probability of 1 of Ω_{ij_i}, then from Theorem 3.1, we obtain that

(4.11) $\mathbf{P}\{\varphi^{-1}(1)\} = 1 - \Pi_{i=1}^{r} [1 - \Pi_{j_i \in T_i} P_{ij_i}]$.

Further let $\underline{y}_1, \ldots, \underline{y}_t$ be all the maximal elements of $\varphi^{-1}(0)$, where $t = \Pi_{i=1}^{r}(k_i)$. From the network representation of Fig. 4.1, any maximal elements are the form; for each i(=1,...,r), there exists only one s of T_i such that $y_{is} = 0$, and for any $u (\neq s)$ $y_{iu} = 1$. We have, from Theorem 3.2,

(4.12) $\mathbf{P}\{\varphi^{-1}(1)\} \geq \Pi_{1 \leq j \leq t} [1 - \mathbf{P}\{\underline{x} \in \Pi_{i=1}^{r} \Pi_{j_i \in T_i} \Omega_{ij_i} ; \underline{x} \leq \underline{y}_j\}]$

Considering the mapping $f : \Pi_{i=1}^{r} T_i \to \{1,\ldots,t\}$, $(t = \Pi_{i=1}^{r}(k_i))$ such that

$$f(j_1,\dots,j_r)=\Sigma_{i=1}^{r-1}[(j_i-1)\times\Pi_{j=i+1}^{r}(k_j)] + j_r \text{ for any } (j_1,\dots,j_r) \text{ of } \Pi_{i=1}^{r} T_i,$$

then f is an increasing bijection.

Since $f^{-1}(j)=(j_1,\dots,j_r)$ implies $P\{\underline{x};\underline{x}\leq\underline{y}_j\} = \Pi_{i=1}^{r}(1-P_{ij_i})$,

(4.13) $\Pi_{1\leq j\leq t}[1-P\{\underline{x};\underline{x}\leq\underline{y}_j\}]=\Pi_{(j_1,\dots,j_r)\in\Pi_{i=1}^{r}T_i}[1-\Pi_{i=1}^{r}(1-P_{ij_i})]$

Therefore, from (4.11), (4.12) and (4.13), the proof is complete. Q.E.D.

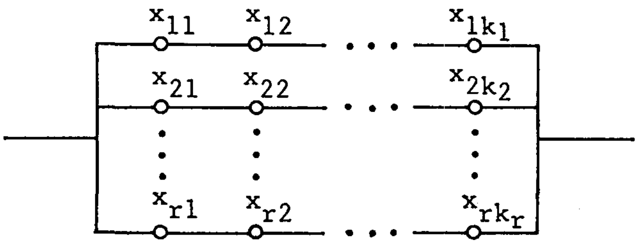

Fig. 4.1 Network representation of a binary-state system.

Theorem 4.1. Suppose that χ satisfies MC condition and P_i, a probability, on $(X_i,\mathcal{F}_i)$, $i=1,\dots,r$, are associated. Then we have for any z of Z,

(4.14)
$$\begin{aligned} P\{\varphi^{-1}(z\leq)\} &\geq \Pi_{\underline{b}\in M[\psi^{-1}(z\nleq)]}[1-\Pi_{i=1}^{r}\{1-\Pi_{a_i\in M[\chi^{-1}_i(\leq b_i)]}P_i\{x_i\in X_i;x_i\nleq a_i\}\}] \\ &\geq \Pi_{\underline{b}\in M[\psi^{-1}(z\nleq)]}\Pi_{\underline{a}\in M[\chi^{-1}(\leq b)]}[1-\Pi_{i=1}^{r}\{1-P_i\{x_i\in X_i;x_i\nleq a_i\}\}] \\ &= \Pi_{\underline{a}\in M[\varphi^{-1}(z\nleq)]}[1-\Pi_{i=1}^{r}\{1-P_i\{x_i\in X_i;x_i\nleq a_i\}\}] \end{aligned}$$

Proof: The first inequality is obvious from (4.6), (4.7) and (4.8). For arbitrarily fixed $\underline{b}$ of $M[\psi^{-1}(z\nleq)]$, letting

$M[\chi^{-1}_i(\leq b_i)]=\{a_{ij_i}\in X_i;\ j_i\in T_i\}$, $(i=1,\dots,r)$ and

$P_{ij_i}= P_i\{x_i\in X_i;x_i\nleq a_{ij_i}\}$, $(i=1,\dots,r,\ j_i\in T_i)$.

Then from Lemma 4.3 and (4.2), we see that

(4.15)
$$\begin{aligned} &1-\Pi_{i=1}^{r}[1-\Pi_{a_{ij_i}\in M[\chi^{-1}_i(\leq b_i)]}P_i\{x_i\in X_i;x_i\nleq a_{ij_i}\}] \\ &\geq \Pi_{(a_{1j_1},\dots,a_{rj_r})\in M[\chi^{-1}(\leq\underline{b})]}[1-\Pi_{i=1}^{r}\{1-P_i\{x_i\in X_i;x_i\nleq a_{ij_i}\}\}] \end{aligned}$$

Therefore, it follows that $F(z)\geq G(z)$ for any z of Z. Finally, the third inequality holds from lemma 4.2. Q.E.D.

Remark to Theorem 4.1. If χ does not satisfy MC condition, the second inequality, i.e., $F(z)\geq G(z)$, does not necessarily hold. See Example 4.2. On the other hand, note that MC condition of χ is invariably satisfied for any binary systems since the state spaces of the system and all the components are two-element totally ordered sets.

Theorem 4.1 shows that $F(z)$ is always the better bound than $G(z)$ if χ satisfies MC condition and P_i is associated. From a physical point of view, $F(z)$ is the lower bound for $P\{\psi^{-1}(z\leq)\}$ when the lower bound of all the modules is regarded as the precise reliability of them. Further, $G(z)$ is the lower bound for $P\{\varphi^{-1}(z\le)\}$ which is directly derived from Theorem 3.3 and has no connection with the existence of the modules.

From (4.7), we see that

(4.16) $P_i\{x_i\in X_i;\chi_i(x_i)\gneq b_i\} \ge \Pi_{a_i\in M[\chi^{-1}_i(\le b_i)]} P_i\{x_i\in X_i;x_i\gneq a_i\}$,

so that we define $Q_i\{x_i\in X_i;\chi_i(x_i)\gneq b_i\}$ as

(4.17) $Q_i\{x_i\in X_i;\chi_i(x_i)\gneq b_i\} = \max_{b_i\le c_i} \Pi_{a_i\in M[\chi^{-1}_i(\le c_i)]} P_i\{x_i\in X_i;x_i\gneq a_i\}$.

Then the next Theorem holds.

Theorem 4.2. Under the same hypotheses of Theorem 4.1, it follows that for any z of Z,

(4.18)
$$\begin{aligned} P\{\varphi^{-1}(z\le)\} &\ge \Pi_{\underline{b}\in M[\psi^{-1}(z\gneq)]}[1-\Pi_{i=1}^{r}\{1-P_i\{x_i\in X_i;\chi_i(x_i)\gneq b_i\}\}] \\ &\ge \Pi_{\underline{b}\in M[\psi^{-1}(z\gneq)]}[1-\Pi_{i=1}^{r}\{1-Q_i\{x_i\in X_i;\chi_i(x_i)\gneq b_i\}\}] \\ &\ge F(z) \ge G(z). \end{aligned}$$

Proof: The first and fourth inequalities are obvious from Theorems 3.3 and 4.1 respectively. Since $b_i, c_i\in Y_i$ and $b_i\le c_i$ imply

$P_i\{x_i\in X_i;\chi_i(x_i)\gneq c_i\} \le P_i\{x_i\in X_i;\chi_i(x_i)\gneq b_i\}$,

we obtain that

$P_i\{x_i\in X_i;\chi_i(x_i)\gneq b_i\} \ge Q_i\{x_i\in X_i;\chi_i(x_i)\gneq b_i\}$ for any b_i of Y_i.

Thus, the second inequality follows.

The third inequality holds since we see that, from (4.17), for any b_i of Y_i.

$Q_i\{x_i\in X_i;\chi_i(x_i)\gneq b_i\} \ge \Pi_{a_i\in M[\chi^{-1}_i(\le b_i)]} P_i\{x_i\in X_i;x_i\gneq a_i\}$ Q.E.D.

Now we will establish upper bounds on system reliability based on modular decompositions. Since all results and their proof with respect to upper bounds are similarly given, we present the Theorems corresponding to Theorems 4.1 and 4.2.

Theorem 4.3. Under the same hypotheses of Theorem 4.1, it follows that for any z of Z,

(4.19)
$$\begin{aligned} P\{\varphi^{-1}(z\le)\} &\le 1-\Pi_{\underline{b}\in N[\psi^{-1}(z\le)]}[1-\Pi_{i=1}^{r}\{1-\Pi_{a_i\in N[\chi^{-1}_i(\le b_i)]}(1-P_i\{x_i;a_i\le x_i\})\}]\ (\triangleq H(z)) \\ &\le 1-\Pi_{\underline{b}\in N[\psi^{-1}(z\le)]}\Pi_{\underline{a}\in N[\chi^{-1}(b\le)]}[1-\Pi_{i=1}^{r}P_i\{x_i\in X_i;a_i\le x_i\}] \\ &= 1-\Pi_{\underline{a}\in N[\varphi^{-1}(z\le)]}[1-\Pi_{i=1}^{r}P_i\{x_i\in X_i;a_i\le x_i\}]\ (\triangleq I(z)) \end{aligned}$$

Theorem 4.4. Under the same hypotheses of Theorem 4.1, it follows that for any z of Z,

(4.20)
$$\begin{aligned} P\{\varphi^{-1}(z\le)\} &\le 1-\Pi_{b\in N[\psi^{-1}(z\le)]}[1-\Pi_{i=1}^{r}\{1-P_i\{x_i\in X_i;b_i\gneq\chi_i(x_i)\}\}] \\ &\le 1-\Pi_{b\in N[\psi^{-1}(z\le)]}[1-\Pi_{i=1}^{r}\{1-Q'_i\{x_i\in X_i;b_i\gneq\chi_i(x_i)\}\}] \\ &\le H(z) \le I(z), \end{aligned}$$

where $Q'_i\{x_i\in X_i;b_i\gneq\chi_i(x_i)\} = \min_{c_i\le b_i}\Pi_{a_i\in N[\chi^{-1}_i(c_i\le)]}P_i\{x_i;a_i\gneq x_i\}$.

In order to explain the behavior of the stochastic bounds mentioned above, we present two simple examples by using multi-state systems, which defined in 3.2 (3), in the rest of this section. For convenience, both examples of multi-state systems $(\Pi_{i\in C}\Omega_i, S, \varphi)$ are assumed as follows:

(1) The system is composed of four independent components, that is, $C=\{1, 2, 3, 4\}$

(2) All the state spaces of the system and their components are restricted such as

$\Omega_i = S = \{0, 1, 2\}$ $(i\in C)$.

(3) The modular decomposition $\{A_1, A_2\}$ of the system is such that $A_1=\{1, 2\}$ and $A_2=\{3, 4\}$. From Definition 4.1, three multistate systems

$(\Pi_{i\in A_1}\Omega_i, S_1, \chi_1)$, $(\Pi_{i\in A_2}\Omega_i, S_2, \chi_2)$, and $(\Pi_{j=1}^{2}S_j, S, \phi)$ are defined as

$$\varphi(\underline{x}) = \phi\{\chi_1(\underline{x}^{A_1}), \chi_2(\underline{x}^{A_2})\} \quad \text{for any } \underline{x} \text{ of } \Pi_{i\in C}\Omega_i.$$

(4) To simplify the calculation, the reliabilities of all the four components are identical and let p_i be the probability that the component takes the state i (i=0, 1, 2).

In the following, R(i) denotes the system reliability $P\{\varphi^{-1}(i\leq)\}$ obtained by Theorem 3.1, and F(i) and G(i) denote the lower bounds of $P\{\varphi^{-1}(i\leq)\}$ by (4.8) and (4.9) respectively, where i=1, 2.

Example 4.1. φ, χ_1, χ_2 and ϕ are given as follows:

$M[\varphi^{-1}(\leq 0)]=$ {(0012), (0021), (0202), (1102), (2002), (0220), (1120), (2020), (0211), (1111), (2011)},

$M[\varphi^{-1}(\leq 1)]=$ {(0022), (0212), (1112), (2012), (0221), (1121), (2021), (2202), (2220), (2211)}.

$M[\chi_1^{-1}(\leq 0)]=$ {(00)}, $M[\chi_1^{-1}(\leq 1)]=$ {(02), (11), (20)},

$M[\chi_2^{-1}(\leq 0)]=$ {(02), (11), (20)}, $M[\chi_2^{-1}(\leq 1)]=$ {(12), (21)}.

$M[\phi^{-1}(\leq 0)]=$ {(01), (10)}, $M[\phi^{-1}(\leq 1)]=$ {(02), (11), (20)}.

Note that χ_1 and χ_2 satisfy MC condition. The calculated results are as follows:

p_0	p_1	p_2	R(1)	F(1)	G(1)	R(2)	F(2)	G(2)
0.10	0.45	0.45	0.84189	0.80429	0.45485	0.44651	0.36528	0.29589
0.45	0.10	0.45	0.48122	0.31651	0.02582	0.18782	0.06239	0.03538
0.45	0.45	0.10	0.17178	0.08926	8.180e-4	0.01698	3.066e-3	3.148e-4
0.05	0.05	0.90	0.98978	0.98818	0.77371	0.88898	0.87320	0.86962
0.05	0.90	0.05	0.17622	0.16783	0.01124	0.01082	7.924e-3	1.403e-3
0.90	0.05	0.05	0.01089	3.743e-4	1.15e-10	5.125e-4	6.433e-7	1.271e-9
0.33	0.34	0.33	0.53107	0.39150	0.04320	0.17183	0.07260	0.03468

Thus we see that $G(i)\leq F(i)\leq R(i)$ for i=1, 2.

Example 4.2. φ, χ_1, χ_2 and ϕ are given as follows:

$M[\varphi^{-1}(\leq 0)]=$ {(0202), (0220), (2002), (2020), (1111), (0211), (2011)},

$M[\varphi^{-1}(\leq 1)]=$ {(0222), (2022), (1102), (1120), (2211)}.

$M[\chi_1^{-1}(\leq 0)]=$ {(02), (20)}, $M[\chi_1^{-1}(\leq 1)]=$ {(02), (11), (20)},

$M[\chi_2^{-1}(\leq 0)]=$ {(11)}, $M[\chi_2^{-1}(\leq 1)]=$ {(02), (11), (20)}.

$M[\phi^{-1}(\leq 0)]=$ {(01), (10)}, $M[\phi^{-1}(\leq 1)]=$ {(02), (11), (20)}.

Note that χ_1 and χ_2 do not satisfy MC condition. The calculated results are as follows:

p_0	p_1	p_2	R(1)	F(1)	G(1)	R(2)	F(2)	G(2)
0.10	0.45	0.45	0.86417	0.79663	0.82070	0.54675	0.45806	0.53131
0.45	0.10	0.45	0.50349	0.34234	0.27425	0.20694	0.07964	0.15746
0.45	0.45	0.10	0.20823	0.08104	0.05618	0.03925	6.417e-3	0.02321
0.05	0.05	0.90	0.99023	0.98856	0.98895	0.89325	0.88334	0.89258
0.05	0.90	0.05	0.18049	0.16118	0.16745	0.08394	0.01480	0.08023
0.90	0.05	0.05	0.01517	1.079e-3	8.521e-6	7.500e-4	1.900e-6	3.437e-5
0.33	0.34	0.33	0.58069	0.38752	0.36535	0.22221	0.10726	0.17952

Thus since χ_1 and χ_2 do not satisfy MC condition in this example, we see that $G(i) \leqq F(i)$ does not necessarily hold.

5. Comparison of Computational Complexity

We will present a rough estimation of the computational complexity for the system reliability and the stochastic bounds mentioned above. In order to avoid the complicated discussion, we treat the multistate system as an object of our investigation in this section. We assume the multi-state system $(\Pi_{i \in C}\Omega_i, S, \varphi)$ satisfying that n is the number of all the components, m is the cardinality of the state space and r is the number of all the modules.

In the following, R_c, G_c and F_c denote the computational complexity needed for calculating the precise reliability $\mathbf{P}\{\varphi^{-1}(s\leq)\}$, the lower bounds $G(s)$ and $F(s)$ for any $s(=0,\ldots,m-1)$, respectively, where the computational complexity indicates the time complexity which counts an elementary operation, i.e., an arithmetical operation, as one step. Note that the precise reliability $\mathbf{P}\{\varphi^{-1}(s\leq)\}$ is obtained by (3.5), and the lower bounds $G(s)$ and $F(s)$ are done by (3.9) and (4.8) respectively.

Equation (3.5) means that all of the elements of $\varphi^{-1}(s\leq)$ should be ascertained without omission so as to calculate the precise reliability. Noticing that the number of all the elements of $\Pi_{i \in C}\Omega_i$ is m^n, we conclude that the computational complexity R_c to the precise reliability is $O(m^n)$. On the other hand, in order to calculate the lower bound $G(s)$, all the maximal elements of $\varphi^{-1}(<s)$ are necessary. Letting k_s to be the number of all the maximal elements of $\varphi^{-1}(<s)$ for $s=1,\ldots,m-1$, from (3.9), the computational complexity G_c is $O(n\sum_{s=1}^{m-1}k_s)$. Supposing $k=k_s$ for any $s(=1,\ldots,m-1)$, it follows that G_c is $O(nmk)$. Let t_i be the number of all the components of the i-th module for $i=1,\ldots,r$. Let u_{is} and v_s be the number of all the maximal elements of $\chi_i^{-1}(s\leq)$ and $\psi^{-1}(<s)$, respectively. Similarly, supposing that $t=t_i$, $u=u_{is}$ and $v=v_s$ for each $i(=1,\ldots,r)$ and $s(=1,\ldots,m-1)$, from (4.8), we see that the computational complexity F_c is $O(m\cdot\max\{nu, rv\})$.

Consequently, under several above assumptions, we have a rough estimation that the computational complexity as follows:

$R_c = O(m^n)$, $G_c = O(nmk)$ and $F_c = O(m \cdot \max\{nu, rv\})$.

Thus, in general, it is not feasible to calculate the precise reliability of a large and complex system. With respect to the computational complexity of stochastic bounds, we conclude that the method using the modular decomposition is more useful than that of no use.

Finally, we demonstrate the two following case;

<u>Case 1</u> : $n=100$, $m=5$, $r=10$, $t=10$, $k=50$, $u=10$, $v=5$.

$R_c = 5^{100} = 7.9 \times 10^{69}$

$G_c = 100 \times 5 \times 50 = 2.5 \times 10^4$

$F_c = 5 \times \max\{100 \times 10,\ 10 \times 5\} = 5.0 \times 10^3$

<u>Case 2</u> : $n=10000$, $m=10$, $r=100$, $t=100$, $k=10000$, $u=200$, $v=50$.

$R_c = 10^{10000}$

$G_c = 10000 \times 10 \times 10000 = 1.0 \times 10^9$

$F_c = 10 \times \max\{10000 \times 200,\ 100 \times 50\} = 2.0 \times 10^7$

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