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AN $O(n^3L)$ ALGORITHM USING A SEQUENCE FOR A LINEAR COMPLEMENTARITY PROBLEM

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Abstract The purpose of this paper is to present an $O(n^3L)$ algorithm for a linear complementarity problem with a positive semi-definite matrix. The algorithm is superior to other $O(n^3L)$ algorithms in the point that it is able to start from any initial feasible point whose components lie between $2^{-O(L)}$ and $2^{O(L)}$. The algorithm is based on the $O(n^{3.5}L)$ method presented by Mizuno [11]. In order to decrease the running time, we use the rank one update technique proposed by Karmarkar [5]. We evaluate the running time in an original way.

1. Introduction

The purpose of this paper is to present an $O(n^3L)$ algorithm for a linear complementarity problem with a positive semi-definite matrix under the condition that we know an interior feasible point in advance whose components lie between $2^{-O(L)}$ and $2^{O(L)}$. Here n and L denote the number of variables and the input size of the problem, respectively. For an $n \times n$ positive semi-definite matrix $\boldsymbol{M}$ and a vector $\boldsymbol{b} \in R^n$, the linear complementarity problem is expressed as follows:

LCP Find an $(\boldsymbol{x}, \boldsymbol{y}) \in R^{2n}$ such that

$$\begin{array}{lrcl} (1) & (\boldsymbol{x},\boldsymbol{y}) & \geq & \mathbf{0}, \\ (2) & \boldsymbol{y} & = & \boldsymbol{M}\boldsymbol{x}+\boldsymbol{b}, \\ & \boldsymbol{X}\boldsymbol{y} & = & \mathbf{0}. \end{array}$$

Here $\boldsymbol{X}$ =diag($\boldsymbol{x}$) denotes the diagonal matrix with diagonal entries equal to the elements of $\boldsymbol{x}$ and R^n denotes the n-dimensional Euclidean space. We call $(\boldsymbol{x}, \boldsymbol{y})$ a feasible point if (1) and (2) hold. The LCP has important applications in linear programming (LP), convex quadratic programming (QP), bimatrix games and so on (Cottle and Dantzig [1], Lemke [9], and Murty [14]).

In 1984, Karmarkar [5] presented an interior point algorithm for LP which requires at most $O(nL)$ iterations and $O(n^{3.5}L)$ arithmetic operations in total. Since then, many interior point algorithms for LP, QP, and LCP were developed by Freund [2], Gonzaga [3], Iri and Imai [4], Kojima, Mizuno, and Yoshise [8, 6, 7], Monteiro and Adler [12, 13], Renegar [15], Yamashita [17], Ye [18], Vaidya [19] and so on. See, for example, Todd [16] for the survey. The algorithms of Gonzaga [3] and Vaidya [19] attain the least running time $O(n^3L)$ for LP. (After the first presentation of this manuscript, Vaidya [20] presented a faster algorithm by using fast matrix multiplication.) The methods are based on the Renegar's path following method [15] which requires at most $O(n^{0.5}L)$ iterations and $O(n^{3.5}L)$ arithmetic operations in total. In order to decrease the running time, they used the so-called rank one update technique which was presented by Karmarkar [5]. Another $O(n^3L)$ algorithms were proposed by Kojima, Mizuno, and Yoshise [8] for LCP and Monteiro and Adler [12, 13] for LP and QP. The methods are based on the central path presented by Megiddo [10] and the primal-dual interior point algorithm of Kojima, Mizuno, and Yoshise [7]. All of these $O(n^3L)$ algorithms [3, 8, 12, 13, 19] follow the central path. So the initial point must lie in a neighborhood of the central path.

On the other hand, Ye [18] and Freund [2] proposed another type of $O(n^{0.5}L)$ iteration algorithms for LP which don't necessarily follow the central path. The methods reduce a potential function in each iteration by a constant. So if the potential at initial solution is less than $O(n^{0.5}L)$, the potential becomes less than $-O(n^{0.5}L)$ after $O(n^{0.5}L)$ iterations, and then we can obtain an approximate solution. This type of methods for LCP was presented by Kojima, Mizuno and Yoshise [6].

Mizuno [11] constructed a new $O(n^{0.5}L)$ iteration algorithm for LCP which is able to start from an initial point in wider area than [2, 6, 18]. The algorithm of Mizuno [11] requires $O(n^{3.5}L)$ arithmetic operations in total because each iteration requires $O(n^3)$ arithmetic operations. Here we reduce it to $O(n^3L)$ by using the rank one update technique. The technique is similar to that of Karmarkar [5], Kojima, Mizuno, and Yoshise [8], and others. However, we have to take another type of criteria for updating the diagonal matrices which we use to compute search directions in the algorithm. So we evaluate the running time of the method in an original way.

In section 2, we describe the algorithm and show the main result in Theorem 1 which assures the correctness of the algorithm. We prove the theorem in Section 3. In Section 4, we evaluate the running time of the method.

2. The algorithm

The algorithm based on the method of Mizuno [11] which solves the following problem:

LCP' Find an $(\boldsymbol{x}, \boldsymbol{y}) \in R^{2n}_{++}$ such that

$$\begin{aligned} \boldsymbol{y} &= M\boldsymbol{x} + \boldsymbol{b}, \\ X\boldsymbol{y} &\le 2^{-2L}\boldsymbol{e}. \end{aligned}$$

Here R^n_{++} denotes the n-dimensional positive orthant $\{\boldsymbol{x} \in R^n : \boldsymbol{x} > \boldsymbol{0}\}$ and $\boldsymbol{e}$ the vector $(1, 1, \dots, 1)^t \in R^n$. It is well known that the problem LCP' is equivalent to LCP in the sense that it requires at most $O(n^3)$ arithmetic operations to compute the solution of LCP from that of LCP'.

In this paper, the upper case letters X_k, Y_k, and V_k with (or without) subscript k designate the diagonal matrices diag($\boldsymbol{x}^k$), diag($\boldsymbol{y}^k$), and diag($\boldsymbol{v}^k$) for the vectors $\boldsymbol{x}^k$, $\boldsymbol{y}^k$, and $\boldsymbol{v}^k$ with (or without) superscript k, respectively. We denote the minimum component of the vector $\boldsymbol{v}^k$ by v^k_{min}.

Assume that a feasible point $(\boldsymbol{x}^0, \boldsymbol{y}^0) \in R^{2n}_{++}$ of LCP is known in advance and that

$$2^{-O(L)}\boldsymbol{e} \le X_0\boldsymbol{y}^0 \le 2^{O(L)}\boldsymbol{e}.$$

Let α and σ be positive constants. Let $\{\boldsymbol{v}^k : k = 0, 1, \dots, m\}$ be a sequence such that

$$\begin{aligned}
&(3) \quad & \boldsymbol{v}^0 &= X_0\boldsymbol{y}^0, \\
&(4) \quad & \boldsymbol{v}^m &\le 2^{-(2L+1)}\boldsymbol{e}, \\
&(5) \quad & \|V_k^{-0.5}(\boldsymbol{v}^k - \boldsymbol{v}^{k+1})\| &\le \alpha\sqrt{v^k_{min}} \quad \text{for } k = 0, 1, \dots, m-1, \\
&(6) \quad & v^k_{min} &\le \left(1 + \frac{\sigma}{\sqrt{n}}\right) v^{k+1}_{min} \quad \text{for } k = 0, 1, \dots, m-1.
\end{aligned}$$

An example of the sequence which satisfies the conditions above is given in Section 3 of [11] for $m = O(n^{0.5}L)$. Although the condition (6) is not imposed on the sequence in the paper [11], it will be easy to verify that the example satisfies it.

Let β and δ be positive constants. The method generates a sequence $\{(\boldsymbol{x}^k, \boldsymbol{y}^k) \in R^{2n}_{++}\}$ such that, for each $k = 0, 1, \ldots, m$,

$$(7) \quad \boldsymbol{y}^k = \boldsymbol{M}\boldsymbol{x}^k + \boldsymbol{b},$$
$$(8) \quad \|\boldsymbol{X}_k \boldsymbol{y}^k - \boldsymbol{v}^k\| \leq \beta v^k_{min}.$$

If the conditions above and (4) hold and $\beta < 1$, the point $(\boldsymbol{x}^m, \boldsymbol{y}^m)$ is a solution of LCP'. In order to decrease the running time, we also generate a sequence $\{(\bar{\boldsymbol{x}}^k, \bar{\boldsymbol{y}}^k) \in R^{2n}_{++}\}$ such that, for each $k = 0, 1, \ldots, m$ and $i = 1, 2, \ldots, n$,

$$(9) \quad |x^k_i - \bar{x}^k_i|/\bar{x}^k_i + |y^k_i - \bar{y}^k_i|/\bar{y}^k_i \leq \delta,$$
$$(10) \quad \sqrt{\bar{x}^k_i \bar{y}^k_i}\left(|x^k_i - \bar{x}^k_i|/\bar{x}^k_i + |y^k_i - \bar{y}^k_i|/\bar{y}^k_i\right) \leq \delta\sqrt{v^k_{min}}.$$

Assume that $(\boldsymbol{x}^k, \boldsymbol{y}^k) \in R^{2n}_{++}$ and $(\bar{\boldsymbol{x}}^k, \bar{\boldsymbol{y}}^k) \in R^{2n}_{++}$ satisfy (7), (8), (9), and (10) at the k-th iteration. In order to find the next point $(\boldsymbol{x}^{k+1}, \boldsymbol{y}^{k+1})$ which satisfies (7) and (8), we consider the system of equations

$$\boldsymbol{y} = \boldsymbol{M}\boldsymbol{x} + \boldsymbol{b},$$
$$\boldsymbol{X}\boldsymbol{y} = \boldsymbol{v}^{k+1}.$$

Let $(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{x}^k - \Delta\boldsymbol{x}^k, \boldsymbol{y}^k - \Delta\boldsymbol{y}^k)$. Then the system above is linearly approximated at $(\boldsymbol{x}^k, \boldsymbol{y}^k)$ by

$$(11) \quad \Delta\boldsymbol{y}^k = \boldsymbol{M}\Delta\boldsymbol{x}^k,$$
$$(12) \quad \boldsymbol{X}_k\Delta\boldsymbol{y}^k + \boldsymbol{Y}_k\Delta\boldsymbol{x}^k = \boldsymbol{X}_k\boldsymbol{y}^k - \boldsymbol{v}^{k+1}.$$

Using $(\bar{\boldsymbol{x}}^k, \bar{\boldsymbol{y}}^k)$ instead of $(\boldsymbol{x}^k, \boldsymbol{y}^k)$, we approximate the system (12) by

$$(13) \quad \bar{\boldsymbol{X}}_k\Delta\boldsymbol{y}^k + \bar{\boldsymbol{Y}}_k\Delta\boldsymbol{x}^k = \boldsymbol{X}_k\boldsymbol{y}^k - \boldsymbol{v}^{k+1},$$

and compute the solution of the system (11) and (13) as follows:

$$(14) \quad \Delta\boldsymbol{x}^k = (\boldsymbol{M} + \bar{\boldsymbol{X}}_k^{-1}\bar{\boldsymbol{Y}}_k)^{-1}\bar{\boldsymbol{X}}_k^{-1}(\boldsymbol{X}_k\boldsymbol{y}^k - \boldsymbol{v}^{k+1}),$$
$$(15) \quad \Delta\boldsymbol{y}^k = \boldsymbol{M}\Delta\boldsymbol{x}^k.$$

We shall prove the following theorem in the next section.

Theorem 1. *Let $\alpha = 0.2$, $\beta = 0.2$, and $\delta = 0.1$. Suppose that $\boldsymbol{v}^k \in R^n_{++}$, $\boldsymbol{v}^{k+1} \in R^n_{++}$, $(\boldsymbol{x}^k, \boldsymbol{y}^k) \in R^{2n}_{++}$, and $(\bar{\boldsymbol{x}}^k, \bar{\boldsymbol{y}}^k) \in R^{2n}_{++}$ satisfy (5), (7), (8), (9), and (10). Let*

$$(16) \quad (\boldsymbol{x}^{k+1}, \boldsymbol{y}^{k+1}) = (\boldsymbol{x}^k - \Delta\boldsymbol{x}^k, \boldsymbol{y}^k - \Delta\boldsymbol{y}^k)$$

for $(\Delta\boldsymbol{x}^k, \Delta\boldsymbol{y}^k)$ computed by (14) and (15), then we have (7) and (8) with k replaced by $k+1$ and $(\boldsymbol{x}^{k+1}, \boldsymbol{y}^{k+1}) > \boldsymbol{0}$.

Now we show how to update $(\bar{\boldsymbol{x}}^k, \bar{\boldsymbol{y}}^k)$. For the points $(\bar{\boldsymbol{x}}^k, \bar{\boldsymbol{y}}^k)$ and $(\boldsymbol{x}^{k+1}, \boldsymbol{y}^{k+1})$, we define

$$(17) \quad I^k = \left\{ i : \begin{array}{rcl} |x^{k+1}_i - \bar{x}^k_i|/\bar{x}^k_i + |y^{k+1}_i - \bar{y}^k_i|/\bar{y}^k_i & > & \delta \text{ or} \\ \sqrt{\bar{x}^k_i\bar{y}^k_i}\left(|x^{k+1}_i - \bar{x}^k_i|/\bar{x}^k_i + |y^{k+1}_i - \bar{y}^k_i|/\bar{y}^k_i\right) & > & \delta\sqrt{v^{k+1}_{min}} \end{array} \right\},$$

and compute $(\bar{\boldsymbol{x}}^{k+1}, \bar{\boldsymbol{y}}^{k+1})$ by

$$(18)\quad (\bar{x}_i^{k+1}, \bar{y}_i^{k+1}) = \begin{cases} (x_i^{k+1}, y_i^{k+1}) & \text{if } i \in I^k, \\ (\bar{x}_i^k, \bar{y}_i^k) & \text{if } i \notin I^k. \end{cases}$$

It will be easy to see that $(\boldsymbol{x}^{k+1}, \boldsymbol{y}^{k+1})$ and $(\bar{\boldsymbol{x}}^{k+1}, \bar{\boldsymbol{y}}^{k+1})$ satisfy (9) and (10) for $k+1$.

From the discussions above, we can compute the solution $(\boldsymbol{x}^m, \boldsymbol{y}^m)$ of LCP' by the following algorithm.

Step 1: Let $k = 0$ and $(\bar{\boldsymbol{x}}^0, \bar{\boldsymbol{y}}^0) = (\boldsymbol{x}^0, \boldsymbol{y}^0)$. Compute $(\boldsymbol{M} + \bar{\boldsymbol{X}}_0^{-1}\bar{\boldsymbol{Y}}_0)^{-1}$.

Step 2: Compute a sequence $\{\boldsymbol{v}^k : k = 0, 1, \ldots, m\}$ which satisfies (3), (4), (5), and (6).

Step 3: Compute $(\boldsymbol{\Delta x}^k, \boldsymbol{\Delta y}^k)$ and $(\boldsymbol{x}^{k+1}, \boldsymbol{y}^{k+1})$ by (14), (15), and (16).

Step 4: Compute $(\bar{\boldsymbol{x}}^{k+1}, \bar{\boldsymbol{y}}^{k+1})$ by (18).

Step 5: Update the matrix $(\boldsymbol{M} + \bar{\boldsymbol{X}}_{k+1}^{-1}\bar{\boldsymbol{Y}}_{k+1})^{-1}$ from $(\boldsymbol{M} + \bar{\boldsymbol{X}}_k^{-1}\bar{\boldsymbol{Y}}_k)^{-1}$ by using $|I^k|$ times of the rank one update.

Step 6: If $k + 1 = m$ then stop. Otherwise increase k by 1 and go to Step 3.

3. Proof of Theorem 1

In this section, we assume that $\boldsymbol{v}^k \in R^n_{++}$, $\boldsymbol{v}^{k+1} \in R^n_{++}$, $(\boldsymbol{x}^k, \boldsymbol{y}^k) \in R^{2n}_{++}$, and $(\bar{\boldsymbol{x}}^k, \bar{\boldsymbol{y}}^k) \in R^{2n}_{++}$ satisfy (5), (7), (8), (9), and (10). Then we easily see that, for each i,

$$\begin{array}{lrcccl}
(19) & (1-\alpha)v_i^k & \le & v_i^{k+1} & \le & (1+\alpha)v_i^k \text{ from (5)}, \\
(20) & (1-\beta)v_i^k & \le & x_i^k y_i^k & \le & (1+\beta)v_i^k \text{ from (8)}, \\
(21) & (1-\delta)\bar{x}_i^k & \le & x_i^k & \le & (1+\delta)\bar{x}_i^k \text{ from (9)}, \\
(22) & (1-\delta)\bar{y}_i^k & \le & y_i^k & \le & (1+\delta)\bar{y}_i^k \text{ from (9)}, \\
(23) & \sqrt{\bar{y}_i^k/\bar{x}_i^k}|x_i^k - \bar{x}_i^k| & \le & \delta\sqrt{v_{min}^k} & & \text{from (10)}, \\
(24) & \sqrt{\bar{x}_i^k/\bar{y}_i^k}|y_i^k - \bar{y}_i^k| & \le & \delta\sqrt{v_{min}^k} & & \text{from (10)}.
\end{array}$$

Multiplying both sides of (13) by $\bar{\boldsymbol{X}}_k^{-0.5}\bar{\boldsymbol{Y}}_k^{-0.5}$, we have

$$(25)\quad \boldsymbol{p}^k + \boldsymbol{q}^k = \boldsymbol{r}^k,$$

where

$$\begin{array}{lrcl}
(26) & \boldsymbol{p}^k & = & \bar{\boldsymbol{X}}_k^{-0.5}\bar{\boldsymbol{Y}}_k^{0.5}\boldsymbol{\Delta x}^k, \\
(27) & \boldsymbol{q}^k & = & \bar{\boldsymbol{X}}_k^{0.5}\bar{\boldsymbol{Y}}_k^{-0.5}\boldsymbol{\Delta y}^k, \\
(28) & \boldsymbol{r}^k & = & \bar{\boldsymbol{X}}_k^{-0.5}\bar{\boldsymbol{Y}}_k^{-0.5}(\boldsymbol{X}_k\boldsymbol{y}^k - \boldsymbol{v}^{k+1}).
\end{array}$$

The next results are also shown in Kojima, Mizuno, and Yoshise [8] and Mizuno [11].

Lemma 2. *Let $\boldsymbol{p}^k$, $\boldsymbol{q}^k$, and $\boldsymbol{r}^k$ be defined by (26), (27), and (28). Then we have*

$$\begin{array}{rcl}
\|\boldsymbol{p}^k\| & \le & \|\boldsymbol{r}^k\|, \quad \|\boldsymbol{q}^k\| \le \|\boldsymbol{r}^k\|, \\
\|\boldsymbol{p}^k\| + \|\boldsymbol{q}^k\| & \le & \sqrt{2}\|\boldsymbol{r}^k\|, \\
\|\boldsymbol{P}_k\boldsymbol{q}^k\| & = & \|\boldsymbol{\Delta X}_k\boldsymbol{\Delta y}^k\| \le \frac{\sqrt{2}}{4}\|\boldsymbol{r}^k\|^2.
\end{array}$$

Here $\boldsymbol{P}_k = diag(\boldsymbol{p}^k)$ and $\boldsymbol{\Delta X}_k = diag(\boldsymbol{\Delta x}^k)$.

Proof: From (26) and (27), we see

$$P_k q^k = \Delta X_k \Delta y^k.$$

Since $\Delta y^k = M\Delta x^k$ and M is positive semi-definite, we have

$$(p^k)^t q^k = (\Delta x^k)^t \Delta y^k = (\Delta x^k)^t M \Delta x^k \geq 0.$$

From (25) and the inequality above, we have

$$\begin{aligned}
\|p^k\|^2 + \|q^k\|^2 &\leq \|r^k\|^2, \\
(\|p^k\| + \|q^k\|)^2 &\leq 2(\|p^k\|^2 + \|q^k\|^2) \\
&\leq 2\|r^k\|^2, \\
\|P_k q^k\|^2 &\leq (\sum_{p_i^k q_i^k > 0} p_i^k q_i^k)^2 + (\sum_{p_i^k q_i^k < 0} p_i^k q_i^k)^2 \\
&\leq 2(\sum_{p_i^k q_i^k > 0} p_i^k q_i^k)^2 \quad (\text{ since } (p^k)^t q^k \geq 0) \\
&\leq 2\left(\sum_{p_i^k q_i^k > 0} \frac{(r_i^k)^2}{4}\right)^2 \\
&\leq 2(\|r^k\|^2/4)^2.
\end{aligned}$$

Hence we obtain the results. ∎

Using Lemma 2, we evaluate $\|X_{k+1}y^{k+1} - v^{k+1}\|$ by $\|r^k\|$ in the next lemma.

Lemma 3. *If* $\|r^k\| \leq \gamma\sqrt{v_{min}^k}$, *then*

$$\|X_{k+1}y^{k+1} - v^{k+1}\| \leq \left(\sqrt{2}\delta\gamma + \frac{\sqrt{2}}{4}\gamma^2\right) v_{min}^k.$$

Proof: From (16), we see

$$\begin{aligned}
X_{k+1}y^{k+1} &= (X_k - \Delta X_k)(y^k - \Delta y^k) \\
&= X_k y^k - (\bar{X}_k \Delta y^k + \bar{Y}_k \Delta x^k) \\
&\quad + (\bar{X}_k - X_k)\Delta y^k + (\bar{Y}_k - Y_k)\Delta x^k + \Delta X_k \Delta y^k \\
&= v^{k+1} + (\bar{X}_k - X_k)\Delta y^k + (\bar{Y}_k - Y_k)\Delta x^k + \Delta X_k \Delta y^k \quad (\text{by } (13)).
\end{aligned}$$

Thus we have

$$\begin{aligned}
\|X_{k+1}y^{k+1} - v^{k+1}\| &\leq \|(\bar{X}_k - X_k)\Delta y^k\| + \|(\bar{Y}_k - Y_k)\Delta x^k\| + \|\Delta X_k \Delta y^k\| \\
&\leq \|\bar{X}_k^{-0.5}\bar{Y}_k^{0.5}(\bar{X}_k - X_k)\|\|\bar{X}_k^{0.5}\bar{Y}_k^{-0.5}\Delta y^k\| \\
&\quad + \|\bar{X}_k^{0.5}\bar{Y}_k^{-0.5}(\bar{Y}_k - Y_k)\|\|\bar{X}_k^{-0.5}\bar{Y}_k^{0.5}\Delta x^k\| + \|\Delta X_k \Delta y^k\| \\
&\leq \delta\sqrt{v_{min}^k}(\|q^k\| + \|p^k\|) + \|\Delta X_k \Delta y^k\| \quad (\text{by } (23) \text{ and } (24)) \\
&\leq \delta\sqrt{v_{min}^k}\sqrt{2}\|r^k\| + \frac{\sqrt{2}}{4}\|r^k\|^2 \quad (\text{from Lemma 2}).
\end{aligned}$$

Hence we obtain the result. ∎

Next we evaluate the size of $\|r^k\|$.

Lemma 4. *We have* $\|\boldsymbol{r}^k\| \leq \gamma\sqrt{v^k_{min}}$ *for* $\gamma = (1+\delta)(1-\beta)^{-0.5}(\alpha+\beta)$.

Proof: We see that

$$\begin{aligned}
\|\boldsymbol{r}^k\| &= \|\bar{\boldsymbol{X}}_k^{-0.5}\bar{\boldsymbol{Y}}_k^{-0.5}(\boldsymbol{X}_k\boldsymbol{y}^k - \boldsymbol{v}^{k+1})\| \\
&\leq \|\bar{\boldsymbol{X}}_k^{-0.5}\bar{\boldsymbol{Y}}_k^{-0.5}\boldsymbol{V}_k^{0.5}\| \left(\|\boldsymbol{V}_k^{-0.5}\|\|\boldsymbol{X}_k\boldsymbol{y}^k - \boldsymbol{v}^k\| + \|\boldsymbol{V}_k^{-0.5}(\boldsymbol{v}^k - \boldsymbol{v}^{k+1})\|\right) \\
&\leq \|\bar{\boldsymbol{X}}_k^{-0.5}\bar{\boldsymbol{Y}}_k^{-0.5}\boldsymbol{V}_k^{0.5}\| \left((v^k_{min})^{-0.5}\beta v^k_{min} + \alpha\sqrt{v^k_{min}}\right) \quad \text{(by (5) and (8))} \\
&\leq \|(\bar{\boldsymbol{X}}_k\bar{\boldsymbol{Y}}_k)^{-0.5}(\boldsymbol{X}_k\boldsymbol{Y}_k)^{0.5}\|\|(\boldsymbol{X}_k\boldsymbol{Y}_k)^{-0.5}\boldsymbol{V}_k^{0.5}\|(\alpha+\beta)\sqrt{v^k_{min}} \\
&\leq (1+\delta)(1-\beta)^{-0.5}(\alpha+\beta)\sqrt{v^k_{min}} \quad \text{(by (20), (21), and (22))}.
\end{aligned}$$

∎

Let $\alpha = 0.2$, $\beta = 0.2$, and $\delta = 0.1$. Then, from Lemma 4, we have $\|\boldsymbol{r}^k\| \leq \gamma\sqrt{v^k_{min}}$ for

$$(29)\quad \gamma = \frac{11\sqrt{5}}{50}.$$

From Lemma 3 and (19), we see

$$\begin{aligned}
\|\boldsymbol{X}_{k+1}\boldsymbol{y}^{k+1} - \boldsymbol{v}^{k+1}\| &\leq \left(\sqrt{2}\delta\gamma + \frac{\sqrt{2}}{4}\gamma^2\right)(1-\alpha)^{-1}v^{k+1}_{min} \\
&= \left(\sqrt{2}\frac{1}{10}\frac{11\sqrt{5}}{50} + \frac{\sqrt{2}}{4}\left(\frac{11\sqrt{5}}{50}\right)^2\right)\frac{5}{4}v^{k+1}_{min} \\
&\leq \beta v^{k+1}_{min}.
\end{aligned}$$

In order to prove Theorem 1 completely, we have to show that

$$\begin{aligned}
\boldsymbol{y}^{k+1} &= \boldsymbol{M}\boldsymbol{x}^{k+1} + \boldsymbol{b}, \\
(\boldsymbol{x}^{k+1}, \boldsymbol{y}^{k+1}) &> 0.
\end{aligned}$$

The first equality is easily deduced from (7), (15), and (16). We see that, for each i,

$$\begin{aligned}
\frac{|\Delta x^k_i|}{x^k_i} &= \frac{1}{\sqrt{x^k_i y^k_i}}\sqrt{\frac{\bar{x}^k_i}{x^k_i}}\sqrt{\frac{y^k_i}{\bar{y}^k_i}}\sqrt{\frac{\bar{y}^k_i}{\bar{x}^k_i}}|\Delta x^k_i| \\
&\leq \frac{1}{\sqrt{x^k_i y^k_i}}\sqrt{\frac{\bar{x}^k_i}{x^k_i}}\sqrt{\frac{y^k_i}{\bar{y}^k_i}}\|\boldsymbol{p}^k\| \quad \text{(by (26))} \\
&\leq \frac{1}{\sqrt{(1-\beta)v^k_i}}\frac{1}{\sqrt{1-\delta}}\sqrt{1+\delta}\|\boldsymbol{r}^k\| \quad \text{(by (20), (21), and (22))} \\
&\leq \sqrt{\frac{5}{4}}\sqrt{\frac{10}{9}}\sqrt{\frac{11}{10}}\frac{11\sqrt{5}}{50}\frac{\sqrt{v^k_{min}}}{\sqrt{v^k_i}} \\
&< 1.
\end{aligned}$$

Hence we have $\boldsymbol{x}^{k+1} = \boldsymbol{x}^k - \boldsymbol{\Delta x}^k > 0$. Similarly, we can obtain $\boldsymbol{y}^{k+1} > 0$.

4. Computational complexity

In this section, we prove the following theorem.

Theorem 5. *The algorithm described in Section 2 requires at most $O(n^3+n^{2.5}m)$ arithmetic operations. In the case of $m = O(n^{0.5}L)$, it becomes $O(n^3L)$.*

It will be easy to see the following results.

(i) Step 1 requires at most $O(n^3)$ arithmetic operations.

(ii) Step 2 requires at most $O(mn)$ arithmetic operations.

(iii) Steps 3, 4, and 6 require at most $O(n^2)$ arithmetic operations in each iteration.

(iv) Step 5 requires at most $O(|I^k|n^2)$ arithmetic operations in the k-th iteration.

In the algorithm, Steps 1 and 2 are executed only once but the other steps are repeated for $k = 0$ to $m-1$. So the total number of arithmetic operations is bounded by

$$O(n^3) + O(mn) + \sum_{k=0}^{m-1} \left(O(n^2) + O(|I^k|n^2)\right) = O\left(n^3 + \sum_{k=0}^{m-1} |I^k|n^2\right).$$

Hence we obtain the results of the theorem if we show that

$$(30)\quad \sum_{k=0}^{m-1} |I^k| = O(n^{0.5}m).$$

In the remainder of this section, we prove (30). From (17), if $i \in I^k$ then

$$\begin{aligned}\sqrt{\bar{x}_i^k\bar{y}_i^k}\left(|x_i^{k+1} - \bar{x}_i^k|/\bar{x}_i^k + |y_i^{k+1} - \bar{y}_i^k|/\bar{y}_i^k\right) &> \min\{\delta\sqrt{\bar{x}_i^k\bar{y}_i^k}, \delta\sqrt{v_{min}^{k+1}}\} \\ &\geq \min\left\{\frac{\delta\sqrt{1-\beta}}{1+\delta}, \sqrt{1-\alpha\delta}\right\}\sqrt{v_{min}^k},\end{aligned}$$

where the second inequality is derived from (19), (20), (21), and (22). Let

$$\delta' = \min\left\{\frac{\delta\sqrt{1-\beta}}{1+\delta}, \sqrt{1-\alpha\delta}\right\},$$

then we have

$$(31)\quad \sqrt{\bar{x}_i^k\bar{y}_i^k}\left(|x_i^{k+1} - \bar{x}_i^k|/\bar{x}_i^k + |y_i^{k+1} - \bar{y}_i^k|/\bar{y}_i^k\right) - \delta'\sqrt{v_{min}^k} \;\geq\; 0 \quad \text{for each } i \in I^k.$$

For each $k = 0, 1, \ldots, m$, let

$$(32)\quad c_k = \sum_{i=1}^{n} \sqrt{\bar{x}_i^k\bar{y}_i^k}\left(|x_i^k - \bar{x}_i^k|/\bar{x}_i^k + |y_i^k - \bar{y}_i^k|/\bar{y}_i^k\right).$$

Then we have

$$\begin{aligned}
c_{k+1} &= \sum_{i\in I^k} \sqrt{\bar{x}_i^{k+1}\bar{y}_i^{k+1}}\left(|x_i^{k+1}-\bar{x}_i^{k+1}|/\bar{x}_i^{k+1}+|y_i^{k+1}-\bar{y}_i^{k+1}|/\bar{y}_i^{k+1}\right)\\
&\quad+\sum_{i\notin I^k} \sqrt{\bar{x}_i^{k+1}\bar{y}_i^{k+1}}\left(|x_i^{k+1}-\bar{x}_i^{k+1}|/\bar{x}_i^{k+1}+|y_i^{k+1}-\bar{y}_i^{k+1}|/\bar{y}_i^{k+1}\right)\\
&= \sum_{i\notin I^k} \sqrt{\bar{x}_i^{k}\bar{y}_i^{k}}\left(|x_i^{k+1}-\bar{x}_i^{k}|/\bar{x}_i^{k}+|y_i^{k+1}-\bar{y}_i^{k}|/\bar{y}_i^{k}\right) \quad \text{(by (18))}\\
&\le \sum_{i\in I^k}\left(\sqrt{\bar{x}_i^{k}\bar{y}_i^{k}}\left(|x_i^{k+1}-\bar{x}_i^{k}|/\bar{x}_i^{k}+|y_i^{k+1}-\bar{y}_i^{k}|/\bar{y}_i^{k}\right)-\delta'\sqrt{v_{min}^k}\right)\\
&\quad+\sum_{i\notin I^k} \sqrt{\bar{x}_i^{k}\bar{y}_i^{k}}\left(|x_i^{k+1}-\bar{x}_i^{k}|/\bar{x}_i^{k}+|y_i^{k+1}-\bar{y}_i^{k}|/\bar{y}_i^{k}\right) \quad \text{(by (31))}\\
&= \sum_{i=1}^{n} \sqrt{\bar{x}_i^{k}\bar{y}_i^{k}}\left(|x_i^{k+1}-\bar{x}_i^{k}|/\bar{x}_i^{k}+|y_i^{k+1}-\bar{y}_i^{k}|/\bar{y}_i^{k}\right)-\delta'\sqrt{v_{min}^k}|I^k|\\
&\le \sum_{i=1}^{n} \sqrt{\bar{x}_i^{k}\bar{y}_i^{k}}\left(|x_i^{k+1}-x_i^{k}|/\bar{x}_i^{k}+|y_i^{k+1}-y_i^{k}|/\bar{y}_i^{k}\right)\\
&\quad+\sum_{i=1}^{n} \sqrt{\bar{x}_i^{k}\bar{y}_i^{k}}\left(|x_i^{k}-\bar{x}_i^{k}|/\bar{x}_i^{k}+|y_i^{k}-\bar{y}_i^{k}|/\bar{y}_i^{k}\right)-\delta'\sqrt{v_{min}^k}|I^k|\\
&= \sum_{i=1}^{n}\left(|p_i^k|+|q_i^k|\right)+c_k-\delta'\sqrt{v_{min}^k}|I^k| \quad \text{(by (26), (27), and (32))}\\
&\le \sqrt{n}\left(\|\boldsymbol{p}^k\|+\|\boldsymbol{q}^k\|\right)+c_k-\delta'\sqrt{v_{min}^k}|I^k|\\
&\le \sqrt{2n}\|\boldsymbol{r}^k\|+c_k-\delta'\sqrt{v_{min}^k}|I^k| \quad \text{(from Lemma 2).}
\end{aligned}$$

Since $\|\boldsymbol{r}^k\|\le\gamma\sqrt{v_{min}^k}$ from Lemma 4, the last inequality above implies

$$\begin{aligned}
\sum_{k=0}^{m-1}\delta'|I^k| &\le \sum_{k=0}^{m-1}\left(\frac{c_k}{\sqrt{v_{min}^k}}-\frac{c_{k+1}}{\sqrt{v_{min}^k}}+\gamma\sqrt{2n}\right)\\
&= \frac{c_0}{\sqrt{v_{min}^0}}+\sum_{k=1}^{m-1}c_k\left(\frac{1}{\sqrt{v_{min}^k}}-\frac{1}{\sqrt{v_{min}^{k-1}}}\right)-\frac{c_m}{\sqrt{v_{min}^{m-1}}}+\gamma\sqrt{2n}m\\
&\le \sum_{k=1}^{m-1}\frac{c_k}{\sqrt{v_{min}^k}}\left(1-\frac{1}{\sqrt{1+\sigma/\sqrt{n}}}\right)+\gamma\sqrt{2n}m,
\end{aligned}$$

where the last inequality is derived from (6), $c_0=0$, and $c_m\ge 0$. Since we see $c_k\le n\delta\sqrt{v_{min}^k}$ from (10) and (32), we obtain

$$\begin{aligned}
\sum_{k=0}^{m-1}\delta'|I^k| &\le \sum_{k=1}^{m-1}n\delta\left(\frac{\sqrt{1+\sigma/\sqrt{n}}-1}{\sqrt{1+\sigma/\sqrt{n}}}\right)+\gamma\sqrt{2n}m\\
&= \frac{\delta\sigma\sqrt{n}(m-1)}{(\sqrt{1+\sigma/\sqrt{n}}+1)\sqrt{1+\sigma/\sqrt{n}}}+\gamma\sqrt{2n}m.
\end{aligned}$$

Since σ, δ, δ', and γ are positive constants, this inequality means (30).

5. Conclusions

In this paper, we propose an $O(n^3L)$ algorithm for LCP which is based on the $O(n^{3.5}L)$ algorithm presented by Mizuno [11]. The algorithm has advantages that the running time is obtained for the initial interior point in a wider area than the other $O(n^3L)$ algorithms [3, 8, 12, 13, 19] and it solves linear complementarity problems which include linear and quadratic programming problems.

In order to decrease the running time, we use the rank one update technique presented by Karmarkar [5], Kojima, Mizuno, and Yoshise [8] and so on. Since we design the criteria (9) and (10) for $(\bar{\boldsymbol{x}}^k, \bar{\boldsymbol{y}}^k)$ which are different from that of the other algorithms [5, 8], we evaluate the running time in an original way.

The algorithm follows the sequence $\{\boldsymbol{v}^k\}$ which satisfies (5). Although the condition (5) theoretically assures us that the sequence $\{(\boldsymbol{x}^k, \boldsymbol{y}^k)\}$ satisfies (8), it causes a short step of the algorithm. In a practical computation, the condition (8) may hold even if the sequence $\{\boldsymbol{v}^k\}$ doesn't satisfy (5). In order to get an efficient algorithm from a practical point of view, we can represent $\boldsymbol{v}^{k+1}$ by a parameter and $\boldsymbol{v}^k$ and use a line search technique in each iteration.

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